Multiple Objectives & Goal Programming

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Deviational Variables

Recall that to convert an inequality to an equality constraint, we introduce slack or surplus variables:

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n \le b_i$$

becomes $a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n + S_i^- = b_i$

$S_i^- \ge 0$ (S_i^- is a **slack** variable)

while $a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n \ge b_i$

becomes $a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n - S_i^+ = b_i$

$S_i^+ \ge 0$ (S_i^+ is a **surplus** variable)

Suppose that we have as a goal the satisfying of a linear constraint

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n \cong b_i$$

We can introduce **deviational** variables (essentially slack & surplus variables):

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n + u_i - v_i = b_i$$

$u_i \ge 0$ (underachievement)

$v_i \ge 0$ (overachievement)

and use as an objective: Minimize $u_i + v_i$

As alternatives, we could

minimize underachievement alone, i.e. Min u_i

or

minimize overachievement alone, i.e., Min v_i

If we have several such goals, we can minimize the sum of **all** the deviational variables:

Minimize $u_1 + v_1 + \dots + u_m + v_m$

s.t.

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n + u_1 - v_1 = b_1$$
$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2n}X_n + u_2 - v_2 = b_2$$
$$\vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{1n}X_n + u_m - v_m = b_m$$

$X_j \ge 0 \ \forall j$, $u_i \ge 0$ & $v_i \ge 0 \ \forall i$

At the optimal solution, at most one of each pair of deviational variables (u_i , v_i) will be basic (i.e., nonzero).

For example, although u_i=5, v_i=2 will give a deviation of $u_i - v_i$ =3 (a net underachievement), the associated cost is $u_i + v_i$ = 7 which is more than the cost (namely, 3) of the equivalent choice u_i =3, v_i =0.

Example: designing an educational program

Decision variables: X_1 = hours of classroom work
X_2 = hours of laboratory work

Suppose that each hour of work involves the following small--group experience and individual problem-solving experience

	classroom	laboratory
small-group	12 minutes	29 minutes
individual	19 minutes	11 minutes

System constraint ("hard constraint"):

total program hours limited to 100

$$X_1 + X_2 \le 100$$

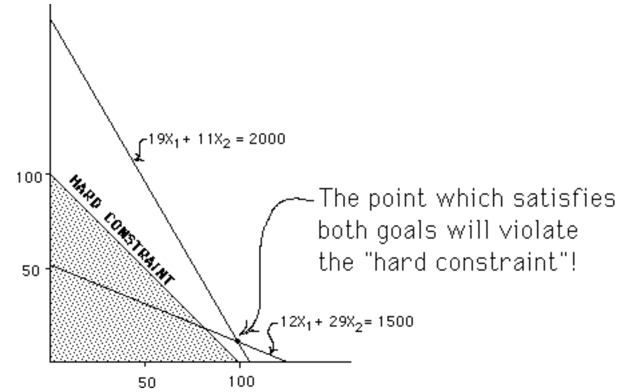
Goal constraints ("soft constraints"):

- each student should spend as close as possible to 25% of the maximum program time working in small groups

$$12 X_1 + 29 X_2 \cong 1500 \text{ (minutes)}$$

- each student should, if possible, spend one-third of the time on individual problem-solving activities

$$19 X_1 + 11 X_2 \cong 2000 \text{ (minutes)}$$



The **goal programming** model:

$$\text{Minimize} \quad U_1 + V_1 + U_2 + V_2$$

s.t.

$X_1 + X_2 \le 100$		total hrs
$12 X_1 + 29 X_2 + U_1 - V_1 = 1500$		small-gp
$19 X_1 + 11 X_2 + U_2 - V_2 = 2000$		individual

$$X_j \ge 0,\ j=1,2;\ U_i \ge 0,\ i=1,2;\ V_i \ge 0,\ i=1,2$$

LINDO output

OBJECTIVE FUNCTION VALUE

1) 241.176500

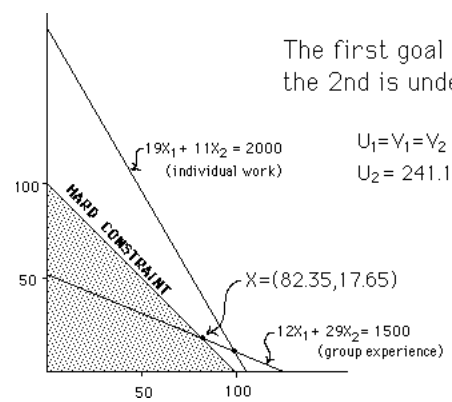
VARIABLE	VALUE	REDUCED COST
U1	.000000	.529412
V1	.000000	1.470588
U2	241.176500	.000000
V2	.000000	2.000000
X1	82.352940	.000000
X2	17.647060	.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
U1	1.000000	INFINITY	.529412
V1	1.000000	INFINITY	1.470588
U2	1.000000	1.125000	1.000000
V2	1.000000	INFINITY	2.000000
X1	.000000	14.448280	9.000000
X2	.000000	9.000000	25.000000

The first goal is satisfied, but the 2nd is under-achieved



$U_1 = V_1 = V_2 = 0$

$U_2 = 241.176$

Weighting the Deviation Variables

To reflect a preference for under- &/or over-achievement of the various goals, one may weight the deviation variables accordingly.

Weighting the Deviation Variables

For example,

if the individual problem-solving experience in the educational program design problem was considered to be 3 times as important as the small-group experience,

and

an underachievement was to be penalized by 5 times the penalty for overachievement,

then the objective would become

$$\text{Minimize} \quad 5U_1 + V_1 + 15U_2 + 3V_2$$

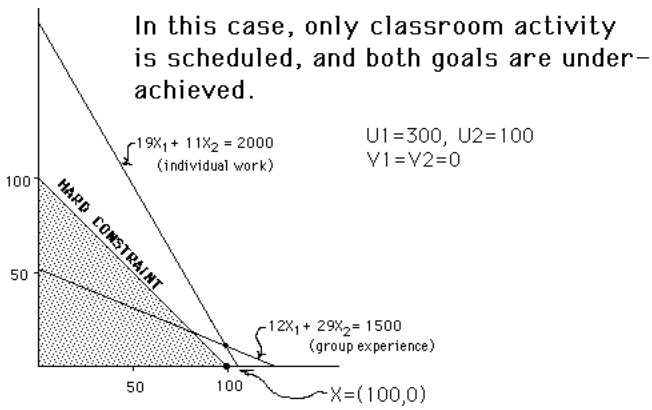
LINDO output

```
MIN     5 U1 + V1 + 15 U2 + 3 V2
SUBJECT TO
      2)  X1 + X2 <=   100
      3)  U1 - V1 + 12 X1 + 29 X2 =    1500
      4)  U2 - V2 + 19 X1 + 11 X2 =    2000
END
```

OBJECTIVE FUNCTION VALUE

1) 3000.00000

VARIABLE	VALUE	REDUCED COST
U1	300.000000	.000000
V1	.000000	6.000000
U2	100.000000	.000000
V2	.000000	18.000000
X1	100.000000	.000000
X2	.000000	35.000000



Constructing a Trade-Off Curve

In the educational design problem, with only 2 goal constraints, one may use the ***parametric programming*** facility of LINDO to construct a trade-off curve:

Maximize small-group experience (T_g)
subject to total program time ≤ 100 hrs
individual problem-solving time $\geq T_i$
where T_i varies from 0 to 2000

```
MAX     12 X1 + 29 X2
SUBJECT TO
      2)  X1 + X2 <=   100
      3)  19 X1 + 11 X2 >=   0
END
```

OBJECTIVE FUNCTION VALUE

1) 2900.00000

VARIABLE	VALUE	REDUCED COST
X1	.000000	17.000000
X2	100.000000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	29.000000
3)	1100.000000	.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	100.000000	INFINITY	100.000000
3	.000000	1100.000000	INFINITY

VAR OUT	VAR IN	PIVOT ROW	RHS VAL	DUAL PRICE BEFORE PIVOT	OBJ VAL
			.00	.00000	2900.00
SLK 3	X1	3	1100.00	.00000	2900.00
X2	ART	2	1900.00	-2.12500	1200.00
			2000.00	-INFINITY	INFEASIBLE