Example

A company must complete 3 jobs on 4 machines, requiring the following processing times:

	Machine			
Job	1	2	3	4
1	20	--	25	30
2	15	20	--	18
3	--	35	28	--

A job cannot be processed on machine j unless for all i<j, the job has completed processing on machine i.

The "flow time" of a job is the difference between the completion time and the time it begins its first stage of processing.

The company wishes to minimize the average flow time of the three jobs.

Model

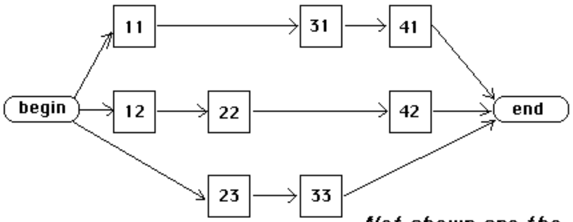
This is a project scheduling problem, with some added restrictions and a different objective.

There are 8 tasks to be performed (processing of jobs on machines) with precedence restrictions.

Label the tasks

ij ≈ processing of job j on machine i

AON (arrow-on-node) Network

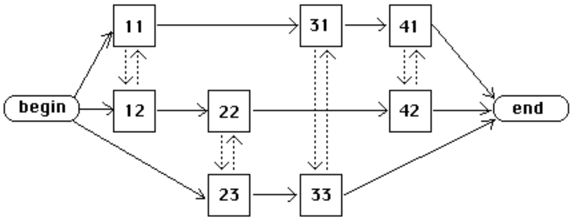


The arrows shown are the precedence restrictions within each job.

Not shown are the restrictions that 2 jobs cannot be processed on one machine simultaneously

For example, tasks 11 and 12 cannot be in progress simultaneously; one of them must precede the other. *But which?*

AON (arrow-on-node) Network



Exactly ***one*** *arrow of each pair (with dotted lines) is to be selected!*

Decision Variables

We will define binary variables to represent this decision:

$$X_{ij} = \begin{cases} 1 & \text{if job j is the first to be processed on machine i} \\ 0 & \text{otherwise} \end{cases}$$

Decision Variables

In addition to the binary variables, we need to define variables as in the LP formulation of the critical path problem:

t_{ij} = starting time of task ij

Objective

Flow time of a job is the difference between the completion time of the last task of the job, and the start time of the first task of the job.
For example, for job #1,

$t_{41} + 30$ = completion time of task 41
t_{11} = start time of task 11
$(t_{41} + 30) - t_{11}$ = Flow time for job #1

Objective

Mininimize average flow time:

$$\text{Minimize} \quad \frac{[t_{41}-t_{11}+30]+[t_{42}-t_{12}+18]+[t_{33}-t_{23}+28]}{3}$$

This is equivalent to minimizing the sum of the flow times, which (omitting constants) is

$$\text{Minimize} \quad t_{41}-t_{11}+t_{42}-t_{12}+t_{33}-t_{23}$$

Constraints

One precedence between jobs on each machine must be selected:
e.g.,

$X_{11} + X_{12} = 1$, i.e., either job 1 or job 2 must be first to be processed on machine 1

Constraints

There are the within-job precedence constraints:
for example,

$t_{13} \geq t_{11} + 20$ i.e., start time of task 13 must be later (or equal) to completion time of task 11

Constraints

We must also include the within-machine precedence constraints:
for example,

$$\begin{cases} t_{11} \geq t_{12} + 15 - M X_{11} \\ t_{12} \geq t_{11} + 20 - M X_{12} \end{cases}$$

(If job 1 is NOT first on machine 1, then it must start AFTER job 2 is completed.)

where "M" is a sufficiently big number.

Example

Four trucks are available to deliver milk to 5 groceries. Capacities & daily operating costs vary among the trucks. Demand of each grocery may be supplied by only one truck, but a truck may deliver to more than one grocery.

Formulate an ILP to minimize the daily cost of meeting the demands of the 5 groceries.

(data on next card)

Truck #	Capacity (gal.)	Daily operating cost ($)
1	400	45
2	500	50
3	600	55
4	1100	60

Grocery #	Daily demand (gal.)
1	100
2	200
3	300
4	500
5	800

Decision Variables

Define

$$Y_i = \begin{cases} 1 & \text{if truck } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$$X_{ij} = \begin{cases} 1 & \text{if truck } i \text{ delivers to grocery } j \\ 0 & \text{otherwise} \end{cases}$$

Model

Objective

Minimize the daily operating costs

$$\text{Minimize} \sum_{i=1}^{4} C_i Y_i$$

where C_i = daily operating cost of truck i

i.e., Minimize $45Y_1 + 50Y_2 + 55Y_3 + 60Y_4$

Constraints

Each grocery must be on a delivery route:

$$\sum_{i=1}^{4} X_{ij} = 1, \text{ for } j=1,\ldots 5$$

e.g., $X_{11} + X_{21} + X_{31} + X_{41} = 1$

Constraints

The deliveries made by a truck i should not exceed its capacity K_i:

$$\sum_{j=1}^{5} D_j X_{ij} \le K_i, \text{ for } i=1,\ldots 4$$

where D_j = demand of grocery j

e.g., for truck #1:

$$100X_{11} + 200X_{12} + 300X_{13} + 500X_{14} + 800X_{15} \le 400$$

Constraints

We need constraints which force $X_{ij}=0$ if $Y_i=0$, i.e., if truck i is not used, it cannot deliver to a grocery.

One way to do this is to include constraints

$X_{ij} \le Y_i$ for all 20 combinations of i & j

Constraints

Another way to force $X_{ij}=0$ if $Y_i=0$ is to modify the earlier truck capacity constraints, adding a factor Y to the RHS:

$$\sum_{j=1}^{5} D_j X_{ij} \le K_i Y_i, \text{ for } i=1,\ldots 4$$

e.g.,

$$100X_{11}+200X_{12}+300X_{13}+500X_{14}+800X_{15} \le 400Y_i$$

Example

Governor Blue of the State of Berry is attempting to get the state legislature to "gerrymander" Berry's 5 congressional districts.

The state consists of 10 cities. To form districts, cities must be grouped according to the following restrictions:

- All voters in a city must be in the same district.
- Each district must contain between 150,000 and 250,000 voters.

Registered Voters (thousands)		
City	Republicans	Democrats
1	80	30
2	60	40
3	40	40
4	20	20
5	40	110
6	40	60
7	70	20
8	50	40
9	70	50
10	70	60

(assume no independent voters!)

Gov. Blue is a Democrat. Formulate an ILP to maximize the number of Democratic congressmen, assuming voters vote a straight party ticket.

Model

Decision Variables

$$X_{ij} = \begin{cases} 1 & \text{if district i includes city j} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_i = \begin{cases} 1 & \text{if district i has a Democratic majority} \\ 0 & \text{otherwise} \end{cases}$$

Objective

Maximize the number of districts with Democratic majorities:

Maximize $Y_1 + Y_2 + Y_3 + Y_4 + Y_5$

Constraints

Every city must be assigned to a district:

$$\sum_{i=1}^{n} X_{ij} = 1 \quad \forall\ j=1,\ldots 10$$

For example, in the case of city 1 (j=1):

$$X_{11}+X_{21}+X_{31}+X_{41}+X_{51} = 1$$

Constraints

The population of a district must be in the range from 150 thousand to 250 thousand:

$$150 \le \sum_{j=1}^{10} P_j X_{ij} \le 250 \quad \forall\ i=1,2,3,4,5$$

where P_j = population of city j (in thousands)

Constraints

$Y_i = 1$ only if there is a Democratic majority in district i, i.e., only if $\dfrac{\sum_{j=1}^{10} D_j X_{ij}}{\sum_{j=1}^{10} P_j X_{ij}} \ge \dfrac{1}{2}$

$$\Longrightarrow \sum_{j=1}^{10} D_j X_{ij} \ge \frac{1}{2} \sum_{j=1}^{10} P_j X_{ij} \Longrightarrow \sum_{j=1}^{10} \left(D_j - \tfrac{1}{2}P_j\right) X_{ij} \ge 0$$

Constraints

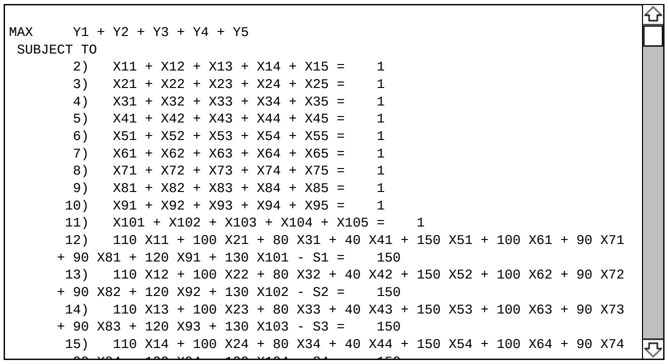
We wish to impose the constraints:

$$\begin{cases} \sum_{j=1}^{10} \left(D_j - \tfrac{1}{2}P_j\right) X_{ij} \ge 0 & \text{if } Y_i = 1 \\ \sum_{j=1}^{10} \left(D_j - \tfrac{1}{2}P_j\right) X_{ij} \ge -\infty & \text{if } Y_i = 0 \end{cases}$$

i.e., $\sum_{j=1}^{10} \left(D_j - \tfrac{1}{2}P_j\right) X_{ij} \ge -M(1-Y_i)$ for "M" sufficiently large

Note that there is lacking in this model any consideration of the geographical location of the cities, so that the districts which are formed may not be "nicely" shaped, and in fact may not even be connected!

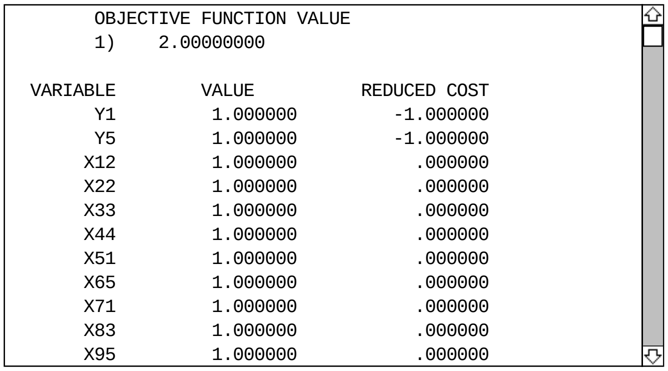
Actual computer models for this problem should contain constraints to ensure that the districts are connected and "compact", i.e., the ratio of length to width should be "close" to 1.



```
MAX     Y1 + Y2 + Y3 + Y4 + Y5
 SUBJECT TO
        2)   X11 + X12 + X13 + X14 + X15 =    1
        3)   X21 + X22 + X23 + X24 + X25 =    1
        4)   X31 + X32 + X33 + X34 + X35 =    1
        5)   X41 + X42 + X43 + X44 + X45 =    1
        6)   X51 + X52 + X53 + X54 + X55 =    1
        7)   X61 + X62 + X63 + X64 + X65 =    1
        8)   X71 + X72 + X73 + X74 + X75 =    1
        9)   X81 + X82 + X83 + X84 + X85 =    1
       10)   X91 + X92 + X93 + X94 + X95 =    1
       11)   X101 + X102 + X103 + X104 + X105 =    1
       12)   110 X11 + 100 X21 + 80 X31 + 40 X41 + 150 X51 + 100 X61 + 90 X71
      + 90 X81 + 120 X91 + 130 X101 - S1 =    150
       13)   110 X12 + 100 X22 + 80 X32 + 40 X42 + 150 X52 + 100 X62 + 90 X72
      + 90 X82 + 120 X92 + 130 X102 - S2 =    150
       14)   110 X13 + 100 X23 + 80 X33 + 40 X43 + 150 X53 + 100 X63 + 90 X73
      + 90 X83 + 120 X93 + 130 X103 - S3 =    150
       15)   110 X14 + 100 X24 + 80 X34 + 40 X44 + 150 X54 + 100 X64 + 90 X74
```

LINDO model

```
LP OPTIMUM FOUND AT STEP     34
OBJECTIVE VALUE =   4.46153830
          ⋮
NO. ITERATIONS=   10436
BRANCHES=  632 DETERM.=  1.000E  0
```



```
        OBJECTIVE FUNCTION VALUE
        1)      2.00000000

  VARIABLE          VALUE          REDUCED COST
        Y1          1.000000          -1.000000
        Y5          1.000000          -1.000000
       X12          1.000000            .000000
       X22          1.000000            .000000
       X33          1.000000            .000000
       X44          1.000000            .000000
       X51          1.000000            .000000
       X65          1.000000            .000000
       X71          1.000000            .000000
       X83          1.000000            .000000
       X95          1.000000            .000000
```

Optimal Solution

Example

A Sunco oil delivery truck contains 5 compartments, holding up to 2700, 2800,1100, 1800, and 3400 gallons of fuel, respectively.
The company must deliver 3 types of fuel (super, regular, and unleaded) to a customer. Each compartment can carry only one type of fuel.

Fuel type	Demand (gal.)	Cost per gal. short	Max allowed shortage
super	2900	$10	500
regular	4000	$8	500
unleaded	4900	$6	500

Formulate an ILP model to find the loading of the truck which minimizes shortage costs.

Model

Decision Variables

Objective

Constraints