Example:

Because of excessive pollution on the Momiss River, the state is going to build some pollution control stations.
Three sites are under consideration.
The state is interested in controlling the pollution levels of two pollutants: #1 & #2

The legislature requires that at least 80,000 lbs. of pollutant #1 and at least 50,000 lbs. of pollutant #2 be removed from the river annually.

author

Data:

(Cost of building station: annual cost over lifetime of station)

Site #	Cost of building station	Cost of treating 1 ton H_2O	Amt. removed per ton of water: Pollutant 1	Pollutant 2
1	\$100,000	\$20	0.4 lb.	0.3 lb.
2	\$ 60,000	\$30	0.25 lb.	0.2 lb.
3	\$ 40,000	\$40	0.2 lb.	0.25 lb.

Formulate an ILP to minimize the cost of meeting the state legislature's goals.

Model

Decision Variables

continuous:

X_i = tons of H_2O treated at station #i annually

binary:

$$Y_i = \begin{cases} 1 & \text{if a station is built at site \#i} \\ 0 & \text{otherwise} \end{cases}$$

Objective

$$\text{Minimize } \underbrace{100000Y_1 + 60000Y_2 + 40000Y_3}_{\text{construction cost of stations}}$$

$$+ \underbrace{20X_1 + 30X_2 + 40X_3}_{\text{water treatment costs}}$$

Constraints

The required amounts of pollutants are removed:

$$\begin{cases} 0.4X_1 + 0.25X_2 + 0.2X_3 \geq 80{,}000 & \textit{pollutant 1} \\ 0.3X_1 + 0.2X_2 + 0.25X_3 \geq 50{,}000 & \textit{pollutant 2} \end{cases}$$

Water cannot be treated at a station unless it has been built:

$$X_i \leq M Y_i \quad \forall i$$

where M is a suitably "big" number

(If Y_i =0, this forces X_i =0)

Example:

Glueco produces 3 types of glue on 2 different production lines.

Each line can be utilized by up to 7 workers at a time. Workers on production line #1 are paid \$500/week, and workers on line #2 are paid \$900/week.

It costs \$1000 to set up production line #1 for a week of production, and \$2000 for line #2.

Data:

Line #	Weekly output per worker (Barrels): Glue #1	Glue #2	Glue #3
1	20	30	40
2	50	35	45
Weekly Rqmt:	120	150	200

Formulate an ILP to minimize the total cost of meeting weekly requirements.

Model

Decision Variables

continuous:

X_{ij} = man-weeks of time on line i producing glue j

integer:

Y_i = number of persons assigned to line i

binary:

$$Z_i = \begin{cases} 1 & \text{if line i is set up for production} \\ 0 & \text{otherwise} \end{cases}$$

Objective

Minimize $\underbrace{1000 Z_1 + 2000 Z_2}_{\text{set-up costs}} + \underbrace{500 Y_1 + 900 Y_2}_{\text{labor costs}}$

Constraints

Requirements for glue are met:

$20X_{11} + 50X_{21} \geq 120$ *glue #1*

$30X_{12} + 35X_{22} \geq 150$ *glue #2*

$40X_{13} + 45X_{23} \geq 200$ *glue #3*

Capacity of line:

$X_{11} + X_{12} + X_{13} \leq Y_1$ *line #1*

$X_{21} + X_{22} + X_{23} \leq Y_2$ *line #2*

Constraints

We cannot assign workers to a line unless it has been set up:

$Y_i \leq 7 Z_i$, i=1,2

Example:

The manager of the University's computer system wants to be able to access 5 different files. There are multiple copies of each file, scattered among 10 disks, as shown below:

File #	Disk 1	2	3	4	5	6	7	8	9	10
1	X	X		X	X			X	X	
2	X		X							
3		X			X		X			X
4			X			X		X		
5	X	X		X		X	X		X	X

If disk 3 or disk 5 is used, then disk 2 must be used.

Formulate an ILP to select the smallest set of disks which contain at least once copy of each file.

model

Decision Variables

$$X_i = \begin{cases} 1 & \text{if disk \#i is selected} \\ 0 & \text{otherwise} \end{cases}$$

Objective

Minimize $X_1 + X_2 + \ldots + X_{10}$

Constraints

At least one copy of each file must be found among the selected disks:

$$X_1 + X_2 + X_4 + X_5 + X_8 + X_9 \geq 1$$
$$X_1 + X_3 \geq 1$$
$$X_2 + X_5 + X_7 + X_{10} \geq 1$$
$$X_3 + X_6 + X_8 \geq 1$$
$$X_1 + X_2 + X_4 + X_6 + X_7 + X_9 + X_{10} \geq 1$$

Constraints

If disk 3 or disk 5 is used, then disk 2 must be used.

$$\begin{cases} X_3 \leq X_2 \\ X_5 \leq X_2 \end{cases}$$

or

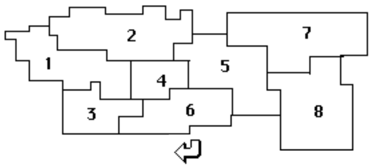
$$X_3 + X_5 \leq 2X_2$$

(The pair of constraints is preferred from a computational point of view.)

Example

Gotham City has been divided into 8 districts.

The city wishes to station 2 ambulances so as to maximize the number of people who live within 3 minutes of an ambulance.



from district \ to district	1	2	3	4	5	6	7	8	Population (thousands)
1	0	3	4	6	8	9	8	10	40
2	3	0	5	4	8	6	12	9	30
3	4	8	0	2	2	3	5	7	35
4	6	4	2	0	3	2	5	4	20
5	8	8	2	3	0	2	2	4	15
6	9	6	3	2	2	0	3	2	50
7	8	12	5	5	2	3	0	2	45
8	10	9	7	4	4	2	2	0	60

Model

Decision Variables

The primary decision is whether to locate an ambulance in district i for i=1,2,...8:

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district } i \\ 0 & \text{otherwise} \end{cases}$$

Another set of decisions is whether to assign for service a district j to an ambulance location i

$$X_{ij} = \begin{cases} 1 & \text{if district } j \text{ is to be served from district } i \\ 0 & \text{otherwise} \end{cases}$$

Objective

We wish to maximize the number of persons within 3 minutes' travel time of an ambulance

For each district j, we need to include in the objective an expression of the form

$$P_j \sum \{X_{ij} : t_{ij} \le 3\}$$

e.g., for j=3:

$$35(X_{33}+X_{43}+X_{53}+X_{63})$$

since districts 3,4,5,&6 are within 3 minutes' travel time of district 3.

Objective

$$\begin{aligned} \text{Maximize } & 40(X_{11}+X_{21}) + 30(X_{12}+X_{22}) \\ & + 35(X_{33}+X_{43}+X_{53}+X_{63}) + 20(X_{34}+X_{44}+X_{54}+X_{64}) \\ & + 15(X_{35}+X_{45}+X_{55}+X_{65}+X_{75}) \\ & + 50(X_{36}+X_{46}+X_{56}+X_{66}+X_{76}+X_{86}) \\ & + 45(X_{57}+X_{67}+X_{77}+X_{87}) + 60(X_{68}+X_{78}+X_{88}) \end{aligned}$$

Constraints

Every district must be assigned to an ambulance:

$$\sum_{i=1}^{8} X_{ij} = 1 \quad \forall\ j=1,2,\ldots 8$$

e.g.,

$$X_{11}+X_{21}+X_{31}+X_{41}+X_{51}+X_{61}+X_{71}+X_{81} = 1 \quad (j=1)$$

Constraints

District j can be served from district i only if there is an ambulance in district i:

$$\sum_{j=1}^{8} X_{ij} \le 8Y_i \quad \text{for } i=1,2,\ldots 8$$

e.g.,

$$X_{11}+X_{12}+X_{13}+X_{14}+X_{15}+X_{16}+X_{17}+X_{18} \le 8Y_1$$

Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^{8} Y_i = 2,$$

i.e.,

$$Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8=2$$

Model Statistics

of binary variables:

Y: 8
X: 64
total: 72

# of continuous variables: 0

# of constraints: 17

Another Formulation

Define as before:

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district i} \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_j = \begin{cases} 1 & \text{if district j is within 3 minutes of an ambulance} \\ 0 & \text{otherwise} \end{cases}$$

Objective

Maximize $40X_1+30X_2+35X_3+20X_4+15X_5+50X_6 +45X_7+60X_8$

Constraints

X_j must be zero unless at least one ambulance is located within 3 minutes of district j

$X_1 \leq Y_1 + Y_2$
$X_2 \leq Y_1 + Y_2$
$X_3 \leq Y_3 + Y_4 + Y_5 + Y_6$
⋮
$X_8 \leq Y_6 + Y_7 + Y_8$

Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^{8} Y_i = 2,$$

i.e.,

$$Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8=2$$

Model Statistics

of binary variables:

Y: 8
X: 8
total: 16

# of continuous variables: 0

# of constraints: 9


```
:MAX    40 X1 + 30 X2 + 35 X3 + 20 X4 + 15 X5 + 50 X6 +
45 X7 + 60 X8
SUBJECT TO
    2)  X1 - Y1 - Y2  <=  0
    3)  X2 - Y1 - Y2  <=  0
    4)  X3 - Y3 - Y4 - Y5 - Y6  <=  0
    5)  X4 - Y3 - Y4 - Y5 - Y6 - Y7  <=  0
    6)  X5 - Y3 - Y4 - Y5 - Y6 - Y7  <=  0
    7)  X6 - Y3 - Y4 - Y5 - Y6 - Y7 - Y8 <=  0
    8)  X7 - Y5 - Y6 - Y7 - Y8  <=  0
    9)  X8 - Y6 - Y7 - Y8  <=  0
   10)  Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 =   2
END
```



```
INTE    8

INTE    Y1
INTE    Y2
INTE    Y3
INTE    Y4
INTE    Y5
INTE    Y6
INTE    Y7
INTE    Y8
```

```
: GO
  LP OPTIMUM FOUND  AT STEP    14

     OBJECTIVE FUNCTION VALUE

 1)       295.000000
```


VARIABLE	VALUE	REDUCED COST
X1	1.000000	-40.000000
X2	1.000000	-30.000000
X3	1.000000	-35.000000
X4	1.000000	-20.000000
X5	1.000000	-15.000000
X6	1.000000	-50.000000
X7	1.000000	-45.000000
X8	1.000000	-60.000000

VARIABLE	VALUE	REDUCED COST
Y1	1.000000	0.000000
Y2	0.000000	0.000000
Y3	0.000000	0.000000
Y4	0.000000	0.000000
Y5	0.000000	0.000000
Y6	1.000000	0.000000
Y7	0.000000	0.000000
Y8	0.000000	0.000000

```
NO. ITERATIONS=      14
 BRANCHES=    0 DETERM.=  1.000E  0
  FIX ALL VARS.(    8)  WITH RC >   15.0000

LP OPTIMUM IS IP OPTIMUM
```