

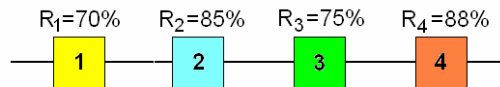
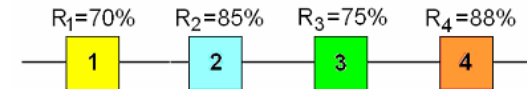
# Optimal Redundancy

## to Maximize System Reliability

- *Dynamic Programming Model*
- *Integer Programming Model*

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One of the systems of a communication satellite consists of four unreliable components (*e.g.*, antenna, transmitter, power source, amplifier, etc.) each of which are necessary for successful operation of the satellite—the probabilities that a component survives the planned lifetime of the satellite (i.e., the *reliabilities*) are shown below:

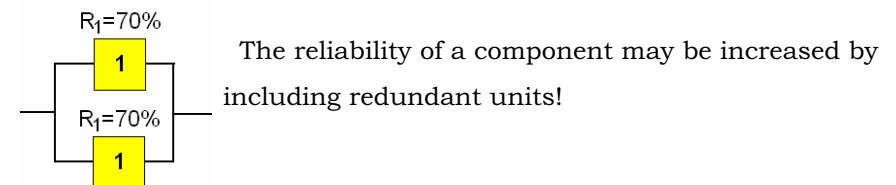


Assuming that component failures are independent,

### Reliability of system

$$\begin{aligned} &= P\{\text{components 1 through 4 survive}\} \\ &= P\{\#1 \text{ survives}\} \times P\{\#2 \text{ survives}\} \times P\{\#3 \text{ survives}\} \times P\{\#4 \text{ survives}\} \\ &= 0.70 \times 0.85 \times 0.75 \times 0.88 = \mathbf{39.27\%} \end{aligned}$$

This is an unacceptably low system reliability, and so redundant units of one or more components will be used in the design.

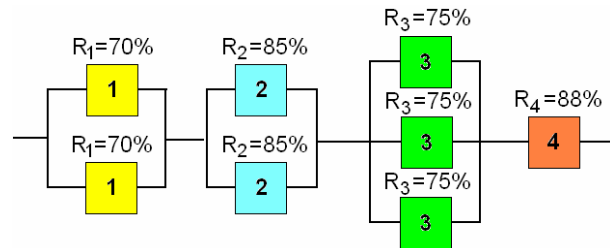


Reliability of component #1

$$\begin{aligned} &= P\{\text{at least one unit survives}\} \\ &= 1 - P\{\text{both units fail}\} \\ &= 1 - 0.30 \times 0.30 = 91\% \end{aligned}$$

*This assumes what is referred to as "hot standby", i.e., a standby unit may fail even before it is put into service!*

By using redundant units of each component, the system reliability can be dramatically increased—for example:



$$\left\{ \begin{array}{l} \text{System} \\ \text{Reliability} \end{array} \right\} = [1 - (0.30)^2] \times [1 - (0.15)^2] \times [1 - (0.25)^2] \times [0.88]$$

$$= 0.91 \times 0.9775 \times 0.984375 \times 0.88 = 77.0551\%$$

The problem faced by the designer is to maximize the system reliability, subject to a restriction on the total weight of the system.

Component	1	2	3	4
Weight (kg)	1	2	1	3

Total weight must not exceed 12 kg.

*(Total weight of one unit of each component is 7 kg, leaving 5 kg for redundant units.)*

### Reliability (%) vs. # redundant units

Component	1 unit	2 units	3 units
1	70	91	97.3
2	85	97.75	99.6625
3	75	93.75	98.4375
4	88	98.56	99.8272

*We will assume that no more than three units of any component will be included!*

### Dynamic Programming Model

**Stage:**  $n$  component type

**Decision:**  $x_n$  # of units of component  $n$  included in system

**State:**  $s_n$  slack weight, i.e., # kg available

We impose a sequential decision-making structure on the problem by supposing that we consider the components one at a time, deciding how many units to include based upon the available weight capacity.

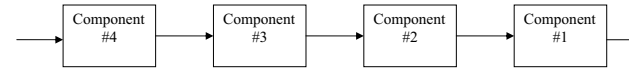


Arbitrarily we will use a “backward” order in what follows!

That is, imagine that we first consider how many units of component #4 are to be included when we begin with 12 kg of available capacity, while component #1 is the last to be considered.

### Optimal Value Function

$f_n(s_n)$  = maximum reliability of the subsystem consisting of devices n, n-1, ... 1, if  $s_n$  kg of available capacity remains to be allocated.



### Recursive definition of function

$$f_n(s_n) = \text{maximum}_{1 \leq x_n \leq \frac{s_n}{w_n}} \left\{ (1 - p_n^{x_n}) \times f_{n-1}(s_n - w_n x_n) \right\}$$

$$f_0(s_0) = \begin{cases} 1 & \text{if } s_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

### APL function definition

```

    ▽ z←F N;t
  [1]  A
  [2]  A Optimal redundancy to maximize reliability
  [3]  A
  [4]  :if N=0
  [5]    z←{(ps)ρ1),-BIG
  [6]  :else
  [7]  A Recursive definition of optimal value function
  [8]    z←Maximize {(ps)ρ1)°.-(1-R[N])×x}×(F N-1)[TRANSITION s°. -W[N]×x
  [9]  :endif
  ▽
  
```

Component #1: reliability = 70%, weight = 1 kg.

Stage 1				
s \ x:	1	2	3	Maximum
1	0.7000	99.9999	99.9999	0.7000
2	0.7000	0.9100	99.9999	0.9100
3	0.7000	0.9100	0.9730	0.9730

etc.

Component #2: reliability = 85%, weight = 2 kg.

Stage 2

s \ x:	1	2	3	Maximum
3	0.5600	-99.9999	-99.9999	0.5600
4	0.7280	-99.9999	-99.9999	0.7280
5	0.7784	0.6720	-99.9999	0.7784
6	0.7784	0.8736	-99.9999	0.8736
7	0.7784	0.9341	0.6944	0.9341
8	0.7784	0.9341	0.9027	0.9341

etc.

For example, suppose that we have 6 kg of capacity remaining, i.e.,  $s_2 = 6$ , and we choose to include 2 units of component #2. Then we obtain 97.75% reliability of subsystem #2 and arrive at stage 1 (component #1) with  $6 - 2 \times 2 = 2$  kg of capacity remaining, so that we can achieve 91% reliability ( $f_1(2) = 0.91$ ) in subsystem #1. Hence the subsystem of components 1&2 will have reliability  $0.9775 \times 0.91 = 0.8736$

Component #3: reliability = 75%, weight = 1 kg.

Stage 3

s \ x:	1	2	3	Maximum
4	0.4200	-99.9999	-99.9999	0.4200
5	0.5460	0.5250	-99.9999	0.5460
6	0.5838	0.6825	0.5513	0.6825
7	0.6552	0.7298	0.7166	0.7298
8	0.7006	0.8190	0.7662	0.8190
9	0.7006	0.8757	0.8600	0.8757

etc.

Component #4: reliability = 88%, weight = 3 kg.

Stage 4

s \ x:	1	2	3	Maximum
7	0.3696	-99.9999	-99.9999	0.3696
8	0.4805	-99.9999	-99.9999	0.4805
9	0.6006	-99.9999	-99.9999	0.6006
10	0.6422	0.4140	-99.9999	0.6422
11	0.7207	0.5381	-99.9999	0.7207
12	0.7706	0.6727	-99.9999	0.7706

Only the last row of this table need be computed to find the optimal reliability with 12 kg of capacity!

Summary of computations

Stage 4

Current State	Optimal Decision	Optimal Value	Next State
cap 7	1 units	0.3696	cap 4
cap 8	1 units	0.4805	cap 5
cap 9	1 units	0.6006	cap 6
cap 10	1 units	0.6422	cap 7
cap 11	1 units	0.7207	cap 8
cap 12	1 units	0.7706	cap 9

Stage 2

Current State	Optimal Decision	Optimal Value	Next State
cap 3	1 units	0.5600	cap 1
cap 4	1 units	0.7280	cap 2
cap 5	1 units	0.7784	cap 3
cap 6	2 units	0.8736	cap 2
cap 7	2 units	0.9341	cap 3
cap 8	2 units	0.9341	cap 4

Stage 3

Current State	Optimal Decision	Optimal Value	Next State
cap 4	1 units	0.4200	cap 3
cap 5	1 units	0.5460	cap 4
cap 6	2 units	0.6825	cap 4
cap 7	2 units	0.7298	cap 5
cap 8	2 units	0.8190	cap 6
cap 9	2 units	0.8757	cap 7

Stage 1

Current State	Optimal Decision	Optimal Value	Next State
cap 1	1 units	0.7000	cap 0
cap 2	2 units	0.9100	cap 0
cap 3	3 units	0.9730	cap 0
cap 4	3 units	0.9730	cap 1
cap 5	3 units	0.9730	cap 2
cap 6	3 units	0.9730	cap 3

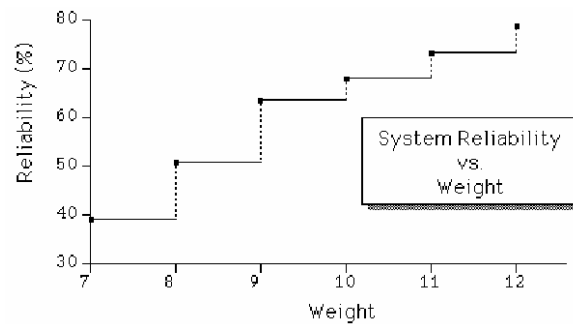
The maximum reliability, then, given a 12 kg weight restriction, is  $f_4(12) = 77.06\%$

By a “forward pass” through the tables, we can determine the optimal design:

stage	state	decision
4	cap 12	1 units
3	cap 9	2 units
2	cap 7	2 units
1	cap 3	3 units
0	cap 0	

That is, the optimal design includes 1 of component #4, 2 each of components #2 & #3, and 3 of component #1.

- What reduction in reliability would occur if the weight restriction were 11 kg rather than 12?
- What is the optimal design with a weight restriction of 11 kg?



### Integer Programming Model

Define *binary* decision variables:

$X_{in} = 1$  if  $n$  units of component  $i$  are included in the system

$X_{in} = 0$  otherwise

*Notation:*

Component $i$	$R_{i1}$	$R_{i2}$	$R_{i3}$
1	0.70	0.91	0.973
2	0.85	0.9775	0.996625
3	0.75	0.9375	0.984375
4	0.88	0.9856	0.998272

**Objective:**

In order to linearize the objective, we will instead maximize the **logarithm of the reliability**:

$$\text{Maximize } \sum_{i=1}^4 \sum_{n=1}^3 (\ln R_{in}) X_{in}$$

subject to

$$\sum_{i=1}^4 \sum_{n=1}^3 (W_{in}) X_{in} \leq W_{\max}$$

$$\sum_{n=1}^3 X_{in} = 1 \quad \forall i = 1, 2, 3, 4$$

$$X_{in} \in \{0, 1\} \quad \forall i \& n$$

Component i	ln R <sub>i1</sub>	ln R <sub>i2</sub>	ln R <sub>i3</sub>
1	-0.35667	-0.094311	-0.02737
2	-0.22314	-0.040822	-0.008032
3	-0.28768	-0.064539	-0.01575
4	-0.12783	-0.014505	-0.001729

### LINGO model:

```
SETS:
  COMPONENT / A B C D/:
    WEIGHT;
  UNITS / 1..3/;
  LOG(COMPONENT,UNITS): LNR, X;
ENDSETS

DATA:
  WEIGHT = 1 2 1 3;
  WMAX = 12;
  LNR = -0.35667 -0.094311 -0.027371
        -0.22314 -0.040822 -0.0080322
        -0.28768 -0.064539 -0.015748
        -0.12783 -0.014505 -0.0017295; ! LNR is log of reliability;
ENDDATA

MAX = @SUM( COMPONENT(I): @SUM( UNITS(N): LNR(I,N)*X(I,N) ) ) ;

@SUM( COMPONENT(I): @SUM( UNITS(N): WEIGHT(I)*N*X(I,N) ) ) <= WMAX;

@FOR (COMPONENT(I):
  @SUM (UNITS(N): X(I,N))=1;
);
@FOR (COMPONENT(I):
  @FOR (UNITS(N): @BIN (X(I,N)) ) );
```

### LINDO model:

```
MAX - .35667 X( A, 1) - .094311 X( A, 2) - .027371 X( A, 3)
      - .22314 X( B, 1) - .040822 X( B, 2) - .0080322 X( B, 3)
      - .28768 X( C, 1) - .064539 X( C, 2) - .015748 X( C, 3)
      - .12783 X( D, 1) - .014505 X( D, 2) - .0017295 X( D, 3)

SUBJECT TO
2] X( A, 1) + 2 X( A, 2) + 3 X( A, 3) + 2 X( B, 1) + 4 X( B, 2)
    + 6 X( B, 3) + X( C, 1) + 2 X( C, 2) + 3 X( C, 3) + 3 X( D, 1)
    + 6 X( D, 2) + 9 X( D, 3) <= 12
3] X( A, 1) + X( A, 2) + X( A, 3) = 1
4] X( B, 1) + X( B, 2) + X( B, 3) = 1
5] X( C, 1) + X( C, 2) + X( C, 3) = 1
6] X( D, 1) + X( D, 2) + X( D, 3) = 1

END
INTE 12
```

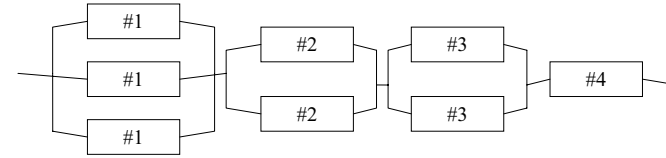
### Optimal Solution:

Objective value: - 0.2605620

Variable	Value	Reduced Cost
X( A, 3)	1.000000	0.2737100E-01
X( B, 2)	1.000000	0.4082200E-01
X( C, 2)	1.000000	0.6453900E-01
X( D, 1)	1.000000	0.1278300

*Note that*  $\exp\{-0.2605620\} = 0.77062$

which is in agreement with the dynamic programming solution.



Optimal Design