Legislative Redistricting

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A state's legislature has R representatives. The state is sectioned into S districts ($S \le R$), where district J has population P_j.

Under strictly proportional representation ("one man-one vote"), district j would receive

$$SHARE(j) = \left(\frac{P_j}{\sum_{i=1}^{S} P_i} \right) R$$

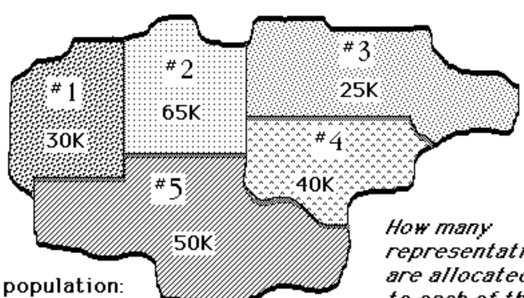
This allocation is generally not feasible, however, since SHARE(j) is not in general integer-valued.

The problem is to allocate the representatives to the districts so that
the maximum deviation from the target allocation, i.e., SHARE, is minimized.

Example

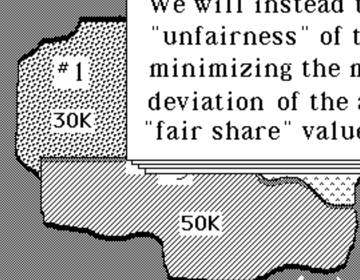
A state has R=10 representatives, and S=5 districts, with populations (in thousands):

District j	Population P_j	"Fair" Share	
1	30	1.4286	(14%)
2	65	3.0952	(31%)
3	25	1.1905	(12%)
4	40	1.9048	(19%)
5	50	2.3809	(24%)
	210	10	(100%)



Total population: 210 K

How many representatives are allocated to each of the 5 districts?



We cannot allocate each district its "fair share", which is non-integer... We will instead try to minimize the "unfairness" of the allocation, by minimizing the maximum of the deviation of the allocation from the "fair share" value.

DP Model

We impose a sequential decision-making structure, in which we consider the districts one at a time, beginning with district 5 and continuing through district 1, deciding upon the number of representatives to be assigned.

Stage: n = district whose allocation of representatives is considered

State: S_n = # of representatives not yet assigned to a district

Decision X_n = # of representatives assigned to district #n

Optimal Value Function

$f_n(S_n)$ = minimum deviation from the target shares of districts n, n−1, ..., 1, if S_n representatives have not yet been assigned.

Recursive Definition

$$f_n(S_n) = \underset{0 \le X_n \le S_n}{\text{Minimum}} \left[\max \left\{ \left| X_n - SHARE(j) \right|, f_{n-1}(S_n - X_n) \right\} \right] \quad n=5,4,3,2,1$$

$$f_0(S_0) = 0$$

APL Code

```
VALUE←F N;t;diff
⍝
⍝          Optimal Value Function of DP model
⍝        for 'Legislative Redistricting' problem
⍝
→LAST IF N=0
  diff ← ((⍴s)⍴0) ∘.+ x-SHARE[N]
  t ← TRANSITION s ∘.- x
  VALUE←MIN (|diff) ⌈ (F N-1)[t]
  →0

LAST:VALUE←0,(⍴s)⍴BIG
```


Stage 1

Computation of f_1 is trivial, since all remaining representatives must be assigned:

s \ x	1	2	3	4	5	6	$f_1(S_1)$	Optimal Decisions
1	0.43	99.99	99.99	99.99	99.99	99.99	0.43	1
2	99.99	0.57	99.99	99.99	99.99	99.99	0.57	2
3	99.99	99.99	1.57	99.99	99.99	99.99	1.57	3
4	99.99	99.99	99.99	2.57	99.99	99.99	2.57	4
5	99.99	99.99	99.99	99.99	3.57	99.99	3.57	5
6	99.99	99.99	99.99	99.99	99.99	4.57	4.57	6

Stage 2

"fair share" = 3.0952

s \ x	1	2	3	4	5	6
0	99.99	99.99	99.99	99.99	99.99	99.99
1	99.99	99.99	99.99	99.99	99.99	99.99
2	2.10	99.99	99.99	99.99	99.99	99.99
3	2.10	1.10	99.99	99.99	99.99	99.99
4	2.10	1.10	0.43	99.99	99.99	99.99
5	2.57	1.57	0.57	0.90	99.99	99.99
6	3.57	2.57	1.57	0.90	1.90	99.99
7	4.57	3.57	2.57	1.57	1.90	2.90
8	99.99	4.57	3.57	2.57	1.90	2.90
9	99.99	99.99	4.57	3.57	2.57	2.90
10	99.99	99.99	99.99	4.57	3.57	2.90

S_1	$f_1(S_1)$
1	0.43
2	0.57
3	1.57
4	2.57
5	3.57
6	4.57

Stage 2

s \ x	1	2	3	4	5	6
0	99.99	99.99	99.99	99.99	99.	
1	99.99	99.99	99.99	99.99	99.	
2	2.10	99.99	99.99	99.99	99.99	99.99
3	2.10	1.10	99.99	99.99	99.99	99.99
4	2.10	1.10	0.43	99.99	99.99	99.99
5	2.57	1.57	0.57	0.90	99.99	99.99
6	3.57	2.57	1.57	0.90	1.90	99.99
7	4.57	3.57	2.57	1.57	1.90	2.90
8	99.99	4.57	3.57	2.57	1.90	2.90
9	99.99	99.99	4.57	3.57	2.57	2.90
10	99.99	99.99	99.99	4.57	3.57	2.90

S_2	$f_2(S_2)$	X^*_2	Resulting State
2	2.10	1	1
3	1.10	2	1
4	0.43	3	1
5	0.57	3	2
6	0.90	4	2
7	1.57	4	3
8	1.90	5	3
9	2.57	5	4
10	2.90	6	4

"fair share" = 1.1905

Stage 3

s \ x	1	2	3	4	5	6
0	99.99	99.99	99.99	99.99	99.99	99.99
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	2.10	99.99	99.99	99.99	99.99	99.99
4	1.10	2.10	99.99	99.99	99.99	99.99
5	0.43	1.10	2.10	99.99	99.99	99.99
6	0.57	0.81	1.81	2.81	99.99	99.99
7	0.90	0.81	1.81	2.81	3.81	99.99
8	1.57	0.90	1.81	2.81	3.81	4.81
9	1.90	1.57	1.81	2.81	3.81	4.81
10	2.57	1.90	1.81	2.81	3.81	4.81

S_2	$f_2(S_2)$
2	2.10
3	1.10
4	0.43
5	0.57
6	0.90
7	1.57
8	1.90
9	2.57
10	2.90

Stage 3

s \ x	1	2	3	4	5	
0	99.99	99.99	99.99	99.99	99.99	9
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	2.10	99.99	99.99	99.99	99.99	99.99
4	1.10	2.10	99.99	99.99	99.99	99.99
5	0.43	1.10	2.10	99.99	99.99	99.99
6	0.57	0.81	1.81	2.81	99.99	99.99
7	0.90	0.81	1.81	2.81	3.81	99.99
8	1.57	0.90	1.81	2.81	3.81	4.81
9	1.90	1.57	1.81	2.81	3.81	4.81
10	2.57	1.90	1.81	2.81	3.81	4.81

S_3	$f_3(S_3)$	X^*_3	Resulting State
3	2.10	1	2
4	1.10	1	3
5	0.43	1	4
6	0.57	1	5
7	0.81	2	5
8	0.90	2	6
9	1.57	2	7
10	1.81	3	7

Stage 4

"fair share" = 1.9048

s \ x	1	2	3	4	5	6
0	99.99	99.99	99.99	99.99	99.99	99.99
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	99.99	99.99	99.99	99.99	99.99	99.99
4	2.10	99.99	99.99	99.99	99.99	99.99
5	1.10	2.10	99.99	99.99	99.99	99.99
6	0.90	1.10	2.10	99.99	99.99	99.99
7	0.90	0.43	1.10	2.10	99.99	99.99
8	0.90	0.57	1.10	2.10	3.10	99.99
9	0.90	0.81	1.10	2.10	3.10	4.10
10	1.57	0.90	1.10	2.10	3.10	4.10

S_3	$f_3(S_3)$
3	2.10
4	1.10
5	0.43
6	0.57
7	0.81
8	0.90
9	1.57
10	1.81

Stage 4

s \ x	1	2	3	4	5	
0	99.99	99.99	99.99	99.99	99.99	
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	99.99	99.99	99.99	99.99	99.99	99.99
4	2.10	99.99	99.99	99.99	99.99	99.99
5	1.10	2.10	99.99	99.99	99.99	99.99
6	0.90	1.10	2.10	99.99	99.99	99.99
7	0.90	0.43	1.10	2.10	99.99	99.99
8	0.90	0.57	1.10	2.10	3.10	99.99
9	0.90	0.81	1.10	2.10	3.10	4.10
10	1.57	0.90	1.10	2.10	3.10	4.10

S_4	$f_4(S_4)$	X^*_4	Resulting State
4	2.10	1	3
5	1.10	1	4
6	0.90	1	5
7	0.43	2	5
8	0.57	2	6
9	0.81	2	7
10	0.90	2	8

"fair share" = 2.3809

Stage 5

s \ x	1	2	3	4	5	6
0	99.99	99.99	99.99	99.99	99.99	99.99
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	99.99	99.99	99.99	99.99	99.99	99.99
4	99.99	99.99	99.99	99.99	99.99	99.99
5	2.10	99.99	99.99	99.99	99.99	99.99
6	1.38	2.10	99.99	99.99	99.99	99.99
7	1.38	1.10	2.10	99.99	99.99	99.99
8	1.38	0.90	1.10	2.10	99.99	99.99
9	1.38	0.43	0.90	1.62	2.62	99.99
10	1.38	0.57	0.62	1.62	2.62	3.62

S_4	$f_4(S_4)$
4	2.10
5	1.10
6	0.90
7	0.43
8	0.57
9	0.81
10	0.90

Stage 5

s \ x	1	2	3	4	5	
0	99.99	99.99	99.99	99.99	99.99	
1	99.99	99.99	99.99	99.99	99.99	99.99
2	99.99	99.99	99.99	99.99	99.99	99.99
3	99.99	99.99	99.99	99.99	99.99	99.99
4	99.99	99.99	99.99	99.99	99.99	99.99
5	2.10	99.99	99.99	99.99	99.99	99.99
6	1.38	2.10	99.99	99.99	99.99	99.99
7	1.38	1.10	2.10	99.99	99.99	99.99
8	1.38	0.90	1.10	2.10	99.99	99.99
9	1.38	0.43	0.90	1.62	2.62	99.99
10	1.38	0.57	0.62	1.62	2.62	3.62

S_5	$f_5(S_5)$	X_5^*	Resulting State
5	2.10	1	4
6	1.38	1	5
7	1.10	2	5
8	0.90	2	6
9	0.43	2	7
10	0.57	2	8

Optimal Returns & Decisions

Stage 5

State	Optimal Values	Optimal Decisions	Resulting State
5	2.10	1	4
6	1.38	1	5
7	1.10	2	5
8	0.90	2	6
9	0.43	2	7
10	0.57	2	8

Stage 4

State	Optimal Values	Optimal Decisions	Resulting State
4	2.10	1	3
5	1.10	1	4
6	0.90	1	5
7	0.43	2	5
8	0.57	2	6
9	0.81	2	7
10	0.90	2	8

Stage 3

State	Optimal Values	Optimal Decisions	Resulting State
3	2.10	1	2
4	1.10	1	3
5	0.43	1	4
6	0.57	1	5
7	0.81	2	5
8	0.90	2	6
9	1.57	2	7
10	1.81	3	7

Stage 2

State	Optimal Values	Optimal Decisions	Resulting State
2	2.10	1	1
3	1.10	2	1
4	0.43	3	1
5	0.57	3	2
6	0.90	4	2
7	1.57	4	3
8	1.90	5	3
9	2.57	5	4
10	2.90	6	4

Stage 1

State	Optimal Values	Optimal Decisions	Resulting State
1	0.43	1	0
2	0.57	2	0
3	1.57	3	0
4	2.57	4	0
5	3.57	5	0
6	4.57	6	0

Legislative Redistricting Problem

*** Optimal value is 0.5714285714 ***

STAGE	STATE	DECISION
5	10	2
4	8	2
3	6	1
2	5	3
1	2	2
0	0	

Optimal Allocation

District j	"Fair" Share	# of Representatives
1	1.4286	2
2	3.0952	3
3	1.1905	1
4	1.9048	2
5	2.3809	2