

DYNAMIC PROGRAMMING

- an example from *Introduction to Dynamic Programming*.

by George Nemhauser
pages 71-76.

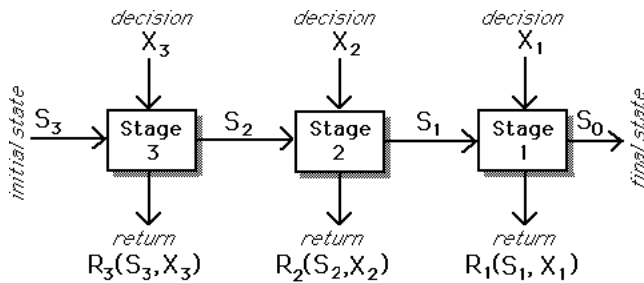
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A deterministic DP example with

- one state variable
- one decision variable
- three stages
- irregular (tabulated) returns & transitions



where

$$S_2 = T_3(S_3, X_3)$$

$$S_1 = T_2(S_2, X_2)$$

$$S_0 = T_1(S_1, X_1)$$

The stages are numbered so that stage number is the number of remaining stages!

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Both state & decision at each stage are discrete, with possible values:

State Vector

i	1	2	3	4	5
s(i)	1	2	3	4	5

Decision Vector

i	1	2	3	4
x(i)	1	2	3	4

Returns

Stage 3		Stage 2		Stage 1					
decision X_3		decision X_2		decision X_1					
1 2 3 4		1 2 3 4		1 2 3 4					
state S_3	1	3	4	1	4	2	1	3	0
	2	2	4	3	3	2	4	3	2
	3	3	4	5	4	3	3	5	4
	4	4	2	3	2	0	4	3	5
	5	0	0	0	0	0	0	0	4

The return, $R_n(S_n, X_n)$, at stage n is a function of both the state S_n and the decision X_n .

E.g., $R_3(2,1) = 2$ is the return if the system is in state 2 at stage 3, and the decision is $X_3 = 1$.

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Transitions

Stage 3		Stage 2		Stage 1						
decision X_3		decision X_2		decision X_1						
1 2 3 4		1 2 3 4		1 2 3 4						
state S_3	1	3	2	1	4	1	1	2	1	0
	2	4	3	3	4	2	2	4	3	2
	3	3	1	2	4	3	3	5	3	4
	4	2	4	2	1	0	4	0	4	3
	5	0	0	0	0	0	0	0	0	5

If the system is in state S_n at stage n, and the decision is X_n , a transition is made to the state $S_{n-1} = T_n(S_n, X_n)$, a function of both S_n & X_n .

Optimal Value Function

$f_n(S_n)$ = maximum value of current & remaining stages, if state is S_n

Recursive definition:

$$f_n(S_n) = \begin{cases} \text{maximum}_{X_n \in \{1,2,3,4\}} \{R_n(S_n, X_n) + f_{n-1}(T_n(S_n, X_n))\} & \text{for } n=3, 2, 1 \\ 0 & \text{for } n=0 \end{cases}$$

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APL code

Nemhauser DP example: tabulated returns & transitions

```

VALUE←F N;t
R
R      Optimal Value Function for Example DP Model
R
→LAST IF N=0
VALUE←MAX R[N;;]+(F N-1)[TRANSITION T[N;;]]
→0
LAST←VALUE←((ρS)ρ0),-BIG
    
```

Stage 1

s \ x	1	2	3	4
1	2.00	1.00	3.00	-999.99
2	4.00	3.00	2.00	-999.99
3	3.00	5.00	4.00	3.00
4	-999.99	4.00	3.00	5.00
5	-999.99	-999.99	4.00	3.00

Stage 1

s \ x	1	2	3	4
1	2.00	1.00	3.00	-999.99
2	4.00	3.00	2.00	-999.99
3	3.00	5.00	4.00	3.00
4	-999.99	4.00	3.00	5.00
5	-999.99	-999.99	4.00	3.00

decision X_1

state S_1	1	2	3	4
1	2	1	3	0
2	4	3	2	0
3	3	5	4	3
4	0	4	3	5
5	0	0	4	3

returns

decision X_1

state S_1	1	2	3	4
1	1	2	1	0
2	4	3	2	0
3	5	3	4	2
4	0	4	3	4
5	0	0	5	5

transitions

State	Optimal Values	Optimal Decisions	Resulting State
1	3.00	3	1
2	4.00	1	4
3	5.00	2	3
4	5.00	4	4
5	4.00	3	5

$f_1(S_1)$ X_1^* $T_1(S_1, X_1^*)$

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Stage 2

s \ x	1	2	3	4
1	-999.99	5.00	9.00	7.00
2	10.00	9.00	7.00	-999.99
3	7.00	7.00	8.00	-999.99
4	8.00	10.00	8.00	7.00

from previous computation

State	$f_1(S_1)$
1	3.00
2	4.00
3	5.00
4	5.00
5	4.00

decision X_2

state S_2	1	2	3	4
1	0	1	5	4
2	5	4	2	0
3	2	3	3	0
4	3	5	4	2
5	0	0	0	0

returns

decision X_2

state S_2	1	2	3	4
1	0	2	5	1
2	3	4	3	0
3	4	5	4	0
4	3	4	2	3
5	0	0	0	0

transitions

Stage 2

s \ x	1	2	3	4
1	-999.99	5.00	9.00	7.00
2	10.00	9.00	7.00	-999.99
3	7.00	7.00	8.00	-999.99
4	8.00	10.00	8.00	7.00

State	Optimal Values	Optimal Decisions	Resulting State
1	9.00	3	5
2	10.00	1	3
3	8.00	3	4
4	10.00	2	4

$f_2(S_2)$

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Stage 3

s \ x	1	2	3	4
1	11.00	14.00	10.00	14.00
2	12.00	12.00	11.00	13.00
3	11.00	13.00	15.00	14.00
4	14.00	12.00	13.00	11.00

from previous computation

State	$f_2(S_2)$
1	9.00
2	10.00
3	8.00
4	10.00

decision X_3

state S_3	1	2	3	4
1	3	4	1	4
2	2	4	3	3
3	3	4	5	4
4	4	2	3	2
5	0	0	0	0

returns

decision X_3

state S_3	1	2	3	4
1	3	2	1	4
2	4	3	3	4
3	3	1	2	4
4	2	4	2	1
5	0	0	0	0

transitions

Stage 3

s \ x	1	2	3	4
1	11.00	14.00	10.00	14.00
2	12.00	12.00	11.00	13.00
3	11.00	13.00	15.00	14.00
4	14.00	12.00	13.00	11.00

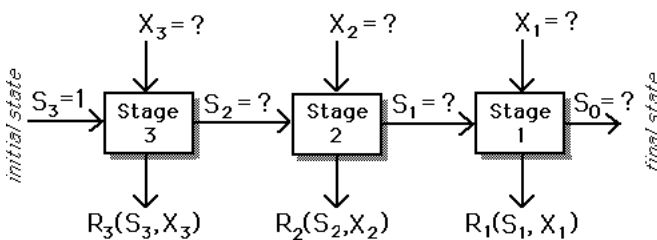
State	Optimal Values	Optimal Decisions	Resulting State
1	14.00	2	2
2	13.00	4	4
3	15.00	3	2
4	14.00	1	2

$f_3(S_3)$ ← alternate optimal

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Suppose that the system begins in the state $S_3 = 1$.
 What is the maximum return?
 What is the optimal sequence of states (trajectory) and decisions?



Optimal Solution No. 1

STAGE	STATE	DECISION
3	1	2
2	2	1
1	3	2
0		3

Optimal Solution No. 2

STAGE	STATE	DECISION
3	1	4
2	4	2
1	4	4
0		4



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