Power Plant Capacity Planning

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A power company is doing long-range planning, and has projected the following needs for additional power plants:

year	1	2	3	4	5	6
#plants req'd	1	2	4	6	7	8
cost/plant ($million)	5.4	5.6	5.8	5.7	5.5	5.2

cumulative!

There is a fixed cost of $1.5 million if any plants are added in a year
At most 3 plants can be added in one year

When should each of the 8 needed plants be added?

($ might be saved by adding some plants in advance of the year in which they're needed, thereby avoiding the $1.5million fixed cost each year.)

DP Model

Stage: n = # of years remaining in the planning period

State: S_n = total # of plants which have been added

Decision: X_n = # of plants to be added in the current year

(Note that the stages are being numbered in reverse chronological order! That is, stage 6 is the first year in the planning period, etc.)

Immediate cost of stage n

$$g_n(S_n,X_n) = \begin{cases} +\infty & \text{if } S_n < R_{n+1} \\ 0 & \text{if } X_n = 0 \quad \& \ S_n \geq R_n \\ 1.5 + C_n X_n & \text{if } X_n = 1, 2, \text{ or } 3 \quad \& \ S_n \geq R_n \end{cases}$$

assures that sufficient plants have been added in previous year.

fixed cost *cost per plant*

Transition Function

If the decision X_n is made, when the current state is S_n, the resulting state will be

$$\boxed{T(S_n,X_n) = S_n + X_n}$$

cumulative number of plants at beginning of next year *current number of plants* *number of plants added*

Objective

☞ **Minimizing Total Cost**

☞ **Minimizing *Present Value* of Total Cost**

Definition of Optimal Value

$f_n(S_n)$ = minimum total cost of the current & remaining stages (n, n-1, ..., 0) if stage n is entered in state S_n

$$f_n(S_n) = \begin{cases} \displaystyle\min_{(R_n - S_n)^+ \leq X_n \leq 3} \left\{ g_n(S_n,X_n) + f_{n-1}(S_n + X_n) \right\} & \text{if } n \geq 1 \\ 0 & \text{if } n=0 \ \& \ S_0 \geq R_1 \\ +\infty & \text{if } n=0 \ \& \ S_0 < R_1 \end{cases}$$

$z^+ \equiv \max\{0, z\}$

APL code

Power Plant Capacity Planning

```
z←F N;t;S;G
⍝
⍝          Optimal Value Function
⍝       Power Plant Capacity Planning problem
⍝
→End IF N=0
⍝      Cost of adding plants in current year
G←((ρS)ρ0)  ∘.+  (x>0)×1.5+COST[N]×x

⍝    Cumulative # of plants added at beginning of next yr.
S← S∘.+x

⍝      Recursive definition of optimal value
z← MIN G + (F N-1)[TRANSITION S+BIG×(S<RQMT[N])]
→0
End:z←((ρS)ρ0),BIG
```


Stage 1

s \ x:	0	1
7	999999.99	6.70
8	0.00	999999.99

Cost per plant = C_1 = 5.2

Req'd # of plants = R_1 = 8

R_2 = 7

999999.99 represents infinity (infeasible decision).

Stage 1

s \ x:	0	1
7	999999.99	6.70
8	0.00	999999.99

State	Optimal Values $f_1(S_1)$	Optimal Decisions X_1^*	Resulting State S_0^*
7	6.70	1	8
8	0.00	0	8

Stage 2

$$f_2(S_2) = \underset{(R_2-S_2)^+ \le X_2 \le 3}{\text{minimum}} \; g_2(S_2,X_2) + f_1(S_2+X_2)$$

from computations just completed:

S_1	$f_1(S_1)$
7	6.70
8	0.00

Cost per plant = C_2 = 5.5

Req'd number of plants = R_2 = 7

s \ x:	0	1	2
6	999999.99	13.70	12.50
7	6.70	7.00	999999.99
8	0.00	999999.99	999999.99

$\boxed{1.5 + 5.5\times 1 + f_1(6+1)}$

Stage 2

s \ x:	0	1	2
6	999999.99	13.70	12.50
7	6.70	7.00	999999.99
8	0.00	999999.99	999999.99

State	Optimal Values $f_2(S_2)$	Optimal Decisions X_2^*	Resulting State S_1^*
6	12.50	2	8
7	6.70	0	7
8	0.00	0	8

Stage 3

S_2	$f_2(S_2)$
6	12.50
7	6.70
8	0.00

Cost per plant = C_3 = 5.7

Req'd number of plants = R_3 = 6

$$f_3(S_3) = \underset{(R_3-S_3)^+ \le X_3 \le 3}{\text{minimum}} \; g_3(S_3,X_3) + f_2(S_3+X_3)$$

S_3 \ X_3	0	1	2	3
4	999999.99	999999.99	25.40	25.30
5	999999.99	19.70	19.60	18.60
6	12.50	13.90	12.90	999999.99
7	6.70	7.20	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

1.5+5.7×2 + f_2(5+2)

Stage 3

s \ x:	0	1	2	3
4	999999.99	999999.99	25.40	25.30
5	999999.99	19.70	19.60	18.60
6	12.50	13.90	12.90	999999.99
7	6.70	7.20	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

State	Optimal Values $f_3(S_3)$	Optimal Decisions X_3^*	Resulting State S_2^*
4	25.30	3	7
5	18.60	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 4

S_3	$f_3(S_3)$
4	25.30
5	18.60
6	12.50
7	6.70
8	0.00

Cost per plant = C_4 = 5.8

Req'd number of plants = R_4 = 4

$$f_4(S_4) = \underset{(R_4-S_4)^+ \le X_4 \le 3}{\text{minimum}} \; g_4(S_4,X_4) + f_3(S_4+X_4)$$

s \ x:	0	1	2	3
2	999999.99	999999.99	38.40	37.50
3	999999.99	32.60	31.70	31.40
4	25.30	25.90	25.60	25.60
5	18.60	19.80	19.80	18.90
6	12.50	14.00	13.10	999999.99
7	6.70	7.30	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

Stage 4

s \ x:	0	1	2	3
2	999999.99	999999.99	38.40	37.50
3	999999.99	32.60	31.70	31.40
4	25.30	25.90	25.60	25.60
5	18.60	19.80	19.80	18.90
6	12.50	14.00	13.10	999999.99
7	6.70	7.30	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

State	Optimal Values $f_4(S_4)$	Optimal Decisions X_4^*	Resulting State S_3^*
2	37.50	3	5
3	31.40	3	6
4	25.30	0	4
5	18.60	0	5
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

S_4	$f_4(S_4)$
2	37.50
3	31.40
4	25.30
5	18.60
6	12.50
7	6.70
8	0.00

Stage 5

Cost per plant = C_5 = 5.6

Req'd number of plants = R_5 = 2

$$f_5(S_5) = \underset{(R_5-S_5)^+ \le X_5 \le 3}{\text{minimum}} \; g_5(S_5,X_5) + f_4(S_5+X_5)$$

s \ x:	0	1	2	3
1	999999.99	44.60	44.10	43.60
2	37.50	38.50	38.00	36.90
3	31.40	32.40	31.30	30.80
4	25.30	25.70	25.20	25.00
5	18.60	19.60	19.40	18.30
6	12.50	13.80	12.70	999999.99
7	6.70	7.10	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

Stage 5

s \ x:	0	1	2	3
1	999999.99	44.60	44.10	43.60
2	37.50	38.50	38.00	36.90
3	31.40	32.40	31.30	30.80
4	25.30	25.70	25.20	25.00
5	18.60	19.60	19.40	18.30
6	12.50	13.80	12.70	999999.99
7	6.70	7.10	999999.99	999999.99
8	0.00	999999.99		

State	Optimal Values $f_5(S_5)$	Optimal Decisions X_5^*	Resulting State S_4^*
1	43.60	3	4
2	36.90	3	5
3	30.80	3	6
4	25.00	3	7
5	18.30	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

S_5	$f_5(S_5)$
1	43.60
2	36.90
3	30.80
4	25.00
5	18.30
6	12.50
7	6.70
8	0.00

Stage 6

Cost per plant = C_6 = 5.4

Req'd number of plants = R_6 = 1

$$f_6(S_6) = \underset{(R_6-S_6)^+ \le X_6 \le 3}{\text{minimum}} \; g_6(S_6,X_6) + f_5(S_6+X_6)$$

s \ x:	0	1	2	3
0	999999.99	50.50	49.20	48.50
1	43.60	43.80	43.10	42.70
2	36.90	37.70	37.30	36.00
3	30.80	31.90	30.60	30.20
4	25.00	25.20	24.80	24.40
5	18.30	19.40	19.00	17.70
6	12.50	13.60	12.30	999999.99
7	6.70	6.90	999999.99	999999.99
8	0.00	999999.99	999999.99	999999.99

Stage 6

s \ x:	0	1	2	3
0	999999.99	50.50	49.20	48.50
1	43.60	43.80	43.10	42.70
2	36.90	37.70	37.30	36.00
3	30.80	31.90	30.60	30.20
4	25.00	25.20	24.80	24.40
5	18.30	19.40	19.00	17.70
6	12.50	13.60	12.30	999999.99
7	6.70	6.90		
8	0.00	999999.99		

State	Optimal Values $f_6(S_6)$	Optimal Decisions X_6^*	Resulting State S_5^*
0	48.50	3	3
1	42.70	3	4
2	36.00	3	5
3	30.20	3	6
4	24.40	3	7
5	17.70	3	8
6	12.30	2	8
7	6.70	0	7
8	0.00	0	8

Optimal Returns & Decisions

Stage 6:

State	Optimal Values	Optimal Decisions	Resulting State
0	48.50	3	3
1	42.70	3	4
2	36.00	3	5
3	30.20	3	6
4	24.40	3	7
5	17.70	3	8
6	12.30	2	8
7	6.70	0	7
8	0.00	0	8

Stage 5:

State	Optimal Values	Optimal Decisions	Resulting State
1	43.60	3	4
2	36.90	3	5
3	30.80	3	6
4	25.00	3	7
5	18.30	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
2	37.50	3	5
3	31.40	3	6
4	25.30	0	4
5	18.60	0	5
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
4	25.30	3	7
5	18.60	3	8
6	12.50	0	6
7	6.70	0	7
8	0.00	0	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
6	12.50	2	8
7	6.70	0	7
8	0.00	0	8

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
7	6.70	1	8
8	0.00	0	8

Optimal Solution

The initial state (entering stage 6) is 0

Power Plant Capacity Planning

*** Optimal value is 48.5 ***

STAGE	STATE	DECISION
6	0	3
5	3	3
4	6	0
3	6	0
2	6	2
1	8	0
0	8	

Discounting Future Costs

Let r = rate of interest

Then

$1 invested now has the same value as $\$1(1+r)^n$ after n periods,

and

$1 paid n periods hence is equivalent to

$$\$1 \frac{1}{(1+r)^n} = \$1\, \beta^n \quad \text{where } \beta = \frac{1}{1+r} \quad \boxed{\textit{discount factor}}$$

paid now.

Minimizing total discounted future costs, i.e., present value of future costs, in a DP model:

In the power plant capacity planning model:

$$f_n(S_n) = \begin{cases} \underset{(R_n - S_n)^+ \le X_n \le 3}{\text{minimum}} \left\{ g_n(S_n, X_n) + \beta f_{n-1}(S_n + X_n) \right\} & \text{if } n \ge 1 \\ 0 & \text{if } n=0 \ \& \ S_0 \ge R_1 \\ +\infty & \text{if } n=0 \ \& \ S_0 < R_1 \end{cases}$$

discount factor

= minimum present value of costs of last n stages *("present" ⟷ stage n)*

Power Plant Capacity Planning

State Vector

i	1	2	3	4	5	6	7	8	9
s[i]	0	1	2	3	4	5	6	7	8

Decision Vector

i	1	2	3	4
x[i]	0	1	2	3

Suppose that r = 20%, so that $\beta = \frac{1}{1.2} = 0.83333$

Power Plant Capacity Planning

```
z←F N;t;S;G
⍝
⍝            Optimal Value Function
⍝         Power Plant Capacity Planning problem
⍝         (minimizing present value of future costs, with
⍝          discount factor beta)
⍝
→End IF N=0

⍝       Cost of adding plants in current year
G←((ρs)ρ0)∘.+(x>0)×1.5+COST[N]×x

⍝       Cumulative # of plants added at beginning of next yr
S←s∘.+x
⍝       Recursive definition of optimal value
z← MIN G + beta×(F N-1)[TRANSITION S+BIG×(S<RQMT[N])]
→0
End:z←((ρs)ρ0),BIG

Note: recursion is backward
```

APL Code

Power Plant Capacity Planning

Recursion type: backward

——Stage 1——

s \ x:	0	1	2	3
5	9999.99	9999.99	9999.99	17.10
6	9999.99	9999.99	11.90	9999.99
7	9999.99	6.70	9999.99	9999.99
8	0.00	9999.99	9999.99	9999.99

State	Optimal Values	Optimal Decisions	Resulting State
5	17.10	3	8
6	11.90	2	8
7	6.70	1	8
8	0.00	0	8

$f_1(S_1)$

s \ x:	0	1	2	3
4	9999.99	9999.99	9999.99	23.58
5	9999.99	9999.99	18.08	18.00
6	9999.99	12.58	12.50	9999.99
7	5.58	7.00	9999.99	9999.99
8	0.00	9999.99	9999.99	9999.99

S_1	$f_1(S_1)$
5	17.10
6	11.90
7	6.70
8	0.00

---Stage 2---

State	Optimal Values	Optimal Decisions	Resulting State
4	23.58	3	7
5	18.00	3	8
6	12.50	2	8
7	5.58	0	7
8	0.00	0	8

$f_2(S_2)$

s \ x:	0	1	2	3
3	9999.99	9999.99	9999.99	29.02
4	9999.99	9999.99	23.32	23.25
5	9999.99	17.62	17.55	18.60
6	10.42	11.85	12.90	9999.99
7	4.65	7.20	9999.99	9999.99
8	0.00	9999.99	9999.99	9999.99

S_2	$f_2(S_2)$
4	23.58
5	18.00
6	12.50
7	5.58
8	0.00

---Stage 3---

State	Optimal Values	Optimal Decisions	Resulting State
3	29.02	3	6
4	23.25	3	7
5	17.55	2	7
6	10.42	0	6
7	4.65	0	7
8	0.00	0	8

$f_3(S_3)$

s \ x:	0	1	2	3
1	9999.99	9999.99	9999.99	38.28
2	9999.99	9999.99	32.48	33.53
3	9999.99	26.68	27.73	27.58
4	19.38	21.93	21.78	22.78
5	14.63	15.98	16.98	18.90
6	8.68	11.18	13.10	9999.99
7	3.88	7.30	9999.99	9999.99
8	0.00	9999.99	9999.99	9999.99

S_3	$f_3(S_3)$
3	29.02
4	23.25
5	17.55
6	10.42
7	4.65
8	0.00

---Stage 4---

State	Optimal Values	Optimal Decisions	Resulting State
1	38.28	3	4
2	32.48	2	4
3	26.68	1	4
4	19.38	0	4
5	14.63	0	5
6	8.68	0	6
7	3.88	0	7
8	0.00	0	8

$f_4(S_4)$

s \ x:	0	1	2	3
0	9999.99	9999.99	39.76	40.53
1	9999.99	34.16	34.93	34.45
2	27.06	29.33	28.85	30.49
3	22.23	23.25	24.89	25.53
4	16.15	19.29	19.93	21.53
5	12.19	14.33	15.93	18.30
6	7.23	10.33	12.70	9999.99
7	3.23	7.10	9999.99	9999.99
8	0.00	9999.99	99[illegible]	[illegible]

S_4	$f_4(S_4)$
1	38.28
2	32.48
3	26.68
4	19.38
5	14.63
6	8.68
7	3.88
8	0.00

---Stage 5---

State	Optimal Values	Optimal Decisions	Resulting State
0	39.76	2	2
1	34.16	1	2
2	27.06	0	2
3	22.23	0	3
4	16.15	0	4
5	12.19	0	5
6	7.23	0	6
7	3.23	0	7
8	0.00	0	8

$f_5(S_5)$

s \ x:	0	1	2	3
0	999999.99	35.37	34.85	36.23
1	28.47	29.45	30.83	31.16
2	22.55	25.43	25.76	27.86
3	18.53	20.36	22.46	23.73
4	13.46	17.06	18.33	20.39
5	10.16	12.93	14.99	17.70
6	6.03	9.59	12.30	9999.99
7	2.69	6.90	9999.99	9999.99
8	0.00	9999.99	[illegible]	[illegible]

S_5	$f_5(S_5)$
0	39.76
1	34.16
2	27.06
3	22.23
4	16.15
5	12.19
6	7.23
7	3.23
[illegible]	[illegible]

--Stage 6---

State	Optimal Values	Optimal Decisions	Resulting State
0	34.85	2	2
1	28.47	0	1
2	22.55	0	2
3	18.53	0	3
4	13.46	0	4
5	10.16	0	5
6	6.03	0	6
7	2.69	0	7
8	0.00	0	8

$f_6(S_6)$

Power Plant Capacity Planning

*** Optimal value is 34.85369084 ***

STAGE	STATE	DECISION
6	0	2
5	2	0
4	2	2
3	4	3
2	7	0
1	7	1
0	8	