OPTIMAL LOT SIZE

by Dynamic Programming

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- A company requires ***n*** units of a customized electronic component, which is ordered from a supplier.
- When a lot is received, it is immediately inspected, and the company pays an amount ***c*** for each unit passing inspection.
- The rejection rate is $q = 1 - p$.
- Any units in surplus of the number required yields a salvage value ***v*** per unit.
- If insufficient acceptable units are received, another lot must be ordered. There is a fixed cost ***K*** for reordering.

The smallest lotsize for which the expected yield of acceptable units is equal to at least ***n*** is, of course, $\left\lceil n/p \right\rceil$, but the optimal lot size will, in general, be larger in order to avoid the reordering cost ***K***.

Example data

- ***n*** = **20** units
- ***q*** = rejection rate = **15%**
- ***c*** = cost per acceptable unit = **$20**
- ***v*** = salvage value for surplus units = **$5**
- ***K*** = reordering cost = **$500**

We will assume that the outcome of each inspection is independent and identically distributed, so that the acceptable yield of a lot of size N would have binomial distribution with parameters (**N**, **p**). Hence we would expect that a lot size of $\left\lceil 20/0.85 \right\rceil = \left\lceil 23.5294 \right\rceil = 24$ would yield the required **20** units. However, there would be approximately

$$\sum_{j=0}^{19} p_x(j) = 28.66\%$$

probability that a deficit would remain so that reordering would be required, where

$$p_x(j) = \binom{x}{j} p^j (1-p)^{x-j}$$

is the probability that j units of a lot of size x will pass inspection.

Binomial Distribution Table

P {j units accepted | x units ordered}

x\j	0	1	2	3	4	5	6	7	8
1	15000	85000							
2	02250	25500	72250						
3	00338	05738	32513	61412					
4	00051	01148	09754	36848	52201				
5	00008	00215	02438	13818	39150	44371			
6	00001	00039	00549	04145	17618	39933	37715		
7	00000	00007	00115	01088	06166	20965	39601	32058	
8	00000	00001	00023	00261	01850	08386	23760	38469	27249

For example, if 6 units are ordered, the probability that exactly 4 units are accepted is 0.17618.

For the original n required units and each possible deficit, what are the lot sizes which will minimize the total expected cost (minus salvage value received for surplus units)?

Dynamic Programming Model

Define an optimal value function

f(n) = minimum expected cost of acquiring *n* acceptable units.

x*(n) **=** optimal lot size when n acceptable units are required.

We wish to determine the values of f(20) and x(20).*

Recursive Definition of the Optimal Value Function

$$f(n) = \min_{x \geq n} \left\{ c\sum_{j=0}^{x} j p_x(j) - v\sum_{j=n+1}^{x}(j-n)p_x(j) + \sum_{j=0}^{n-1}\left[K + f(n-j)\right]p_x(j) \right\}$$

where

$c\sum_{j=0}^{x} j p_x(j)$ is the expected cost of acceptable units in a lot of size x

$v\sum_{j=n+1}^{x}(j-n)p_x(j)$ is the expected salvage value of surplus units

$\sum_{j=0}^{n-1}\left[K + f(n-j)\right]p_x(j)$ is the expected cost of reordering

Note that *f(n)* appears on both left and right of the "="!

Denote the optimal x by $\hat{x}$.

$$f(n) - p_{\hat{x}}(0) f(n) =$$
$$c\sum_{j=0}^{\hat{x}} j p_{\hat{x}}(j) - v\sum_{j=n+1}^{\hat{x}}(j-n)p_{\hat{x}}(j) + \sum_{j=0}^{n-1}\left[K + f(n-j)\right]p_{\hat{x}}(j) + K p_{\hat{x}}(0)$$

Solving for $f(n)$ yields the recursion

$$f(n) =$$
$$\min_{x \geq n} \left\{ \frac{c\sum_{j=0}^{x} j p_x(j) - v\sum_{j=n+1}^{x}(j-n)p_x(j) + \sum_{j=0}^{n-1}\left[K + f(n-j)\right]p_x(j) + K p_x(0)}{1 - p_x(0)} \right\}$$

Computation of f(1):

x	purchase	salvage	reorder	Total
1	17	0.00000	75.00000	108.2353
2	34	¯3.61250	11.25000	42.5959
3	51	¯7.76688	1.68750	45.0727
4	68	¯12.00253	0.25312	56.2791
5	85	¯16.25038	0.03796	68.7928
6	102	¯20.50006	0.00569	81.5066
7	119	¯24.75001	0.00085	94.2510
8	136	¯29.00000	0.00012	107.0002

f(1) = 42.5959 with lotsize = 2

Example Calculation: Suppose the lotsize is x=3, so that the probability distribution of the number of acceptable pieces is

x	j= 0	1	2	3
3	00338	05738	32513	61412

$$p_x(j) = \binom{x}{j} p^j (1-p)^{x-j}$$

Then the expected purchase price is $c\sum_{j=1}^{x} j p_x(j) =$

$20[1(0.05738) +2(0.32513) +3(0.61412)]
= $20[0.057375+0.65025 +1.84237] = $20[2.55] = **$51**

The expected **salvage value** is $v\sum_{j=n+1}^{x} (j-n)\, p_x(j) =$

$5 [1×0.32513 + 2×0.61412] = $5 [0.325125 + 1.22825]
= $5[1.55338] = **$7.77**

The expected **reorder cost** is $\sum_{j=0}^{n-1}\left[K + f(n-j)\right]p_x(j) + Kp_x(0) =$

$500×0.00338 = **$1.6875**

Summing and dividing by $1-p_x(0) = 0.996625$ yields

$$\frac{51 - 7.77 + 1.6875}{0.996625} = \mathbf{\$45.07}$$

Computation of f(2):

x	purchase	salvage	reorder	Total
2	34	0.00000	1.39708E2	177.7068
3	51	¯3.07063	3.05188E1	78.7138
4	68	¯7.06244	6.01219E0	66.9837
5	85	¯11.26152	1.11698E0	74.8612
6	102	¯15.50205	1.99821E¯1	86.6988
7	119	¯19.75036	3.48142E¯2	99.2846
8	136	¯24.00006	5.94828E¯3	112.0059

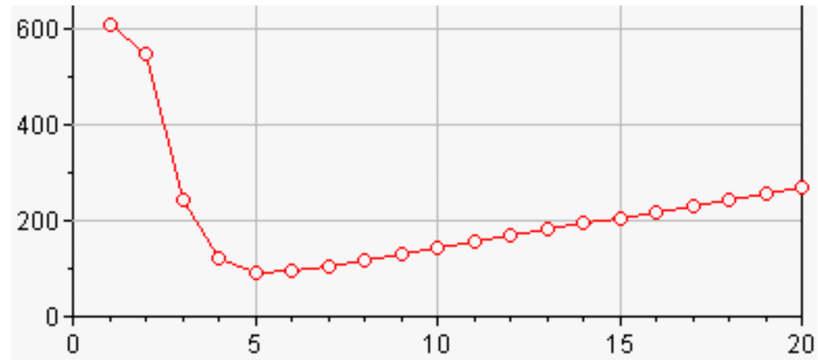
f(2) = 66.9837 with lotsize = 4

Computation of f(3):

x	purchase	salvage	reorder	Total
4	68	¯2.61003	5.52821E1	120.7332
5	85	¯6.39458	1.34027E1	92.0151
6	102	¯10.53148	2.95984E0	94.4294
7	119	¯14.75646	6.13824E¯1	104.8575
8	136	¯19.00127	1.21666E¯1	117.1204
9	153	¯23.25024	2.33059E¯2	129.7731
10	170	¯27.50005	4.34687E¯3	142.5043
11	187	¯31.75001	7.93567E¯4	155.2508
12	204	¯36.00000	1.42349E¯4	168.0001

f(3) = 92.0151 with lotsize = 5

Note that the minimand is unimodal, although not convex:



Computation of f(4):

x	purchase	salvage	reorder	Total
5	85	¯2.21853	8.35848E1	166.379
6	102	¯5.76817	2.39300E1	120.163
7	119	¯9.81698	6.10536E0	115.289
8	136	¯14.01554	1.43755E0	123.422
9	153	¯18.25341	3.19049E¯1	135.066
10	170	¯22.50072	6.76673E¯2	147.567
11	187	¯26.75015	1.38449E¯2	160.264
12	204	¯31.00003	2.75127E¯3	173.003
13	221	¯35.25001	5.33687E¯4	185.751
14	238	¯39.50000	1.01441E¯4	198.500
15	255	¯43.75000	1.89498E¯5	211.250
16	272	¯48.00000	3.48733E¯6	224.000

f(4) = 115.289 with lotsize = 7

etc.

# Required	Lotsize	Expected yield	Expected cost
0	0	0.00	0.0000
1	2	1.70	42.5959
2	4	3.40	66.9837
3	5	4.25	92.0151
4	7	5.95	115.2886
5	8	6.80	137.6817
6	9	7.65	161.7710
7	11	9.35	183.2215
8	12	10.20	205.0304
9	13	11.05	227.9728
10	15	12.75	249.7202
11	16	13.60	270.8886
12	17	14.45	292.8776
13	19	16.15	315.5066
14	20	17.00	336.1120
15	21	17.85	357.3386
16	22	18.70	379.2370
17	24	20.40	401.0926
18	25	21.25	421.7098
19	26	22.10	442.8532
20	27	22.95	464.5606

Summary

If 20 usable parts are required, a lot of size 27 should be ordered.

The expected yield is 22.95 (nearly 23, i.e., 3 more than required),

and the expected cost is \$464.56.

If, for example, the yield is 23, the cost would be $\$20 \times 23 = \$460$, and the extra 3 parts could be salvaged for $\$5\times3 = \$15$, a net cost of \$445 (about \$19.56 less than the expected cost).

If the yield were only 18, however, the cost of this lot would be $\$20\times18 = \$360$, and two additional parts are needed, so that another lot of size 4 should be ordered. (This would cost an additional \$500 for re-ordering, plus the cost of the acceptable parts, etc.

