Bernoulli & Related Processes

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Bernoulli Random Variable

A random variable **X** has the Bernoulli distribution with parameter **p** if

$$P\{X = 1\} = p$$
$$P\{X = 0\} = 1-p = q$$

e.g., outcome of an experiment can be classified as a "*success*" (X=1) or a "*failure*" (X=0)

Bernoulli Random Variable

Mean Value: $E(X) = 1 \cdot p + 0(1-p) = p$

Variance:
$$\begin{aligned} Var(X) &= E[X-E(X)]^2 \\ &= (1-p)^2 p + (0-p)^2(1-p) \\ &= (1-p)\,p \end{aligned}$$

Bernoulli Process

The stochastic process $\{X_n; n=1, 2, 3, \dots\}$ is a Bernoulli process if

- $X_1, X_2, \dots$ are independent
- X_n has the Bernoulli distribution with parameter **p** for each **n**.

Examples

- At a certain intersection, about 30% of the vehicles turn left. We define $X_n = 1$ if the n^{th} vehicle turns left, and 0 otherwise. Then $\{X_n; n=1, 2, \dots\}$ is a Bernoulli process with parameter p=0.30.

- Diameters of bearings coming off a production line are measured, and those that do not meet specifications are rejected. Let Y_n be the diameter of the n^{th} bearing, a normally-distributed random variable with mean 3 and standard deviation 0.002. Let a=2.994 and b=3.006 be the lower & upper tolerances, so that the bearing is not rejected if $2.994 \le Y_n \le 3.006$. Let $X_n = 1$ if the n^{th} bearing meets specifications. Then $\{X_n; n=1, 2, \dots\}$ is a Bernoulli process with parameter 0.9974.

Number of successes

Let N_n be the number of "successes" of the first n trials of the Bernoulli process $\{X_n\}$, i.e.,

$$N_n = \sum_{i=1}^{n} X_i$$

$\{N_n; n=1, 2, \dots\}$ is a ***counting process,*** and N_n has the ***binomial distribution:***

$$P\{N_n = k\} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots n$$

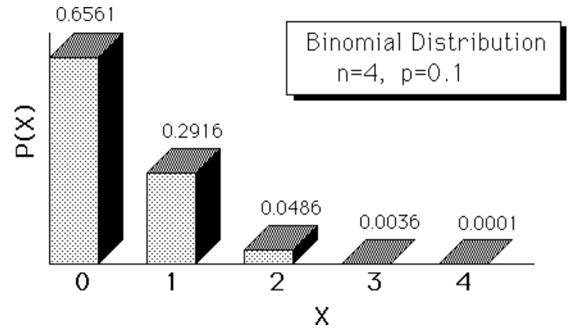
Binomial Distribution

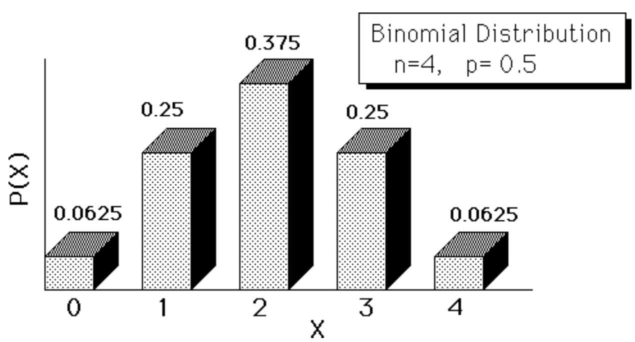
$$P\{N_n = k\} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k = 1, 2, \ldots n$$

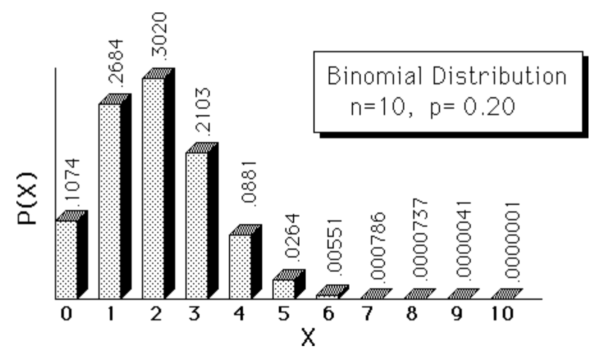
Mean Value $E(N_n) = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i) = np$

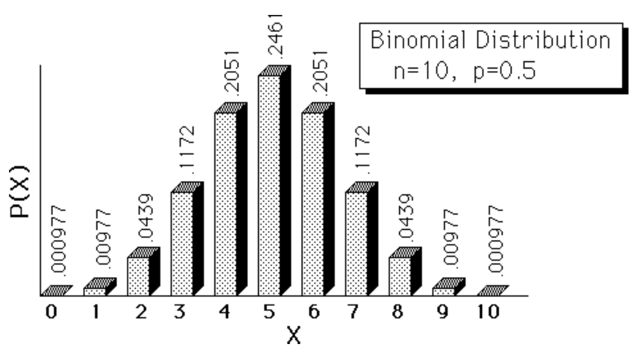
Variance $Var(N_n) = Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var(X_i)$

$= n(1-p)p$

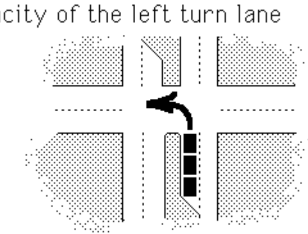






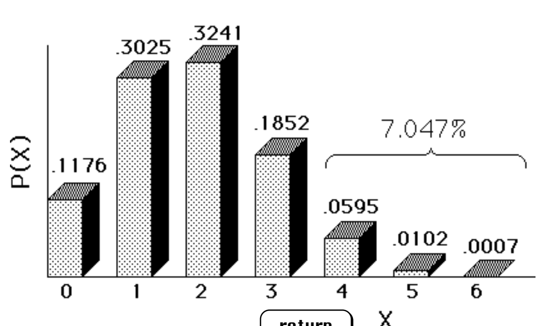


Example A left-turn lane at an intersection has a capacity of 3 autos. 30% of autos arriving at the intersection wish to turn left.
If 6 autos arrive during a red light, what is the probability that the capacity of the left turn lane will be insufficient?



Solution

P{X of 6 autos turn left}



return

Times of successes

Define a stochastic process $\{T_k\,;\, k=1,2,\ldots\}$ where T_k is the number of the trial in which the k^{th} success occurs.

In order for $T_1 = n$, it is necessary that

- the first n-1 trials must have been failures
- the n^{th} trial must be a success

Therefore,

$$P\{T_1 = n\} = (1-p)^{n-1} p$$

geometric distribution

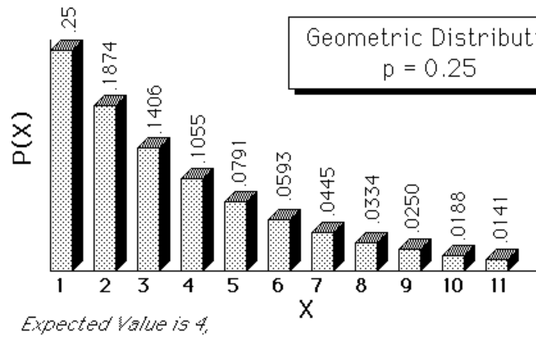
geometric distribution $P\{T_1 = n\} = (1-p)^{n-1}\, p$

CDF $F_{T_1}(n) = P\{T_1 \le n\} = 1 - P\{T_1 > n\}$
$= 1 - P\{n \text{ failures in first } n \text{ trials}\}$
$= 1 - (1-p)^n$

Mean Value $E(T_1) = \dfrac{1}{p}$ *"return period"*

Variance $Var(T_1) = \dfrac{1-p}{p^2}$

Geometric Distribution
p = 0.25



Expected Value is 4, but smaller values are much more likely!

Time of k^{th} success

Let τ_i = additional trials performed after $(i-1)^{th}$ success, in order to achieve i^{th} success

$$T_k = \sum_{i=1}^{k} \tau_i$$

Negative Binomial (Pascal) distribution

Mean Value $E(T_k) = E\left(\sum_{i=1}^{k}\tau_i\right) = \sum_{i=1}^{k} E(\tau_i) = kE(T_1) = \dfrac{k}{p}$

Variance $Var(T_k) = Var\left(\sum_{i=1}^{k}\tau_i\right) = \sum_{i=1}^{k} Var(T_1) = \dfrac{k(1-p)}{p^2}$

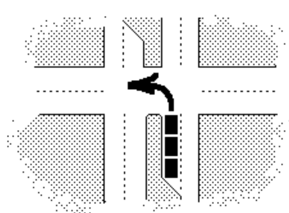
The k^{th} success occurs on trial n if & only if

- there are exactly k−1 successes in the first n−1 trials
  probability: $\binom{n-1}{k-1}(1-p)^{n-k}p^{k-1}$
- there is a success on the n^{th} trial
  probability: p

$$\therefore P\{T_k = n\} = \binom{n-1}{k-1}(1-p)^{n-k}p^k$$

Example A left-turn lane at an intersection has a capacity of 3 autos. 30% of autos arriving at the intersection wish to turn left.

What is the probability distribution of the number of arrivals which cause the capacity of the left-turn lane to be exceeded?



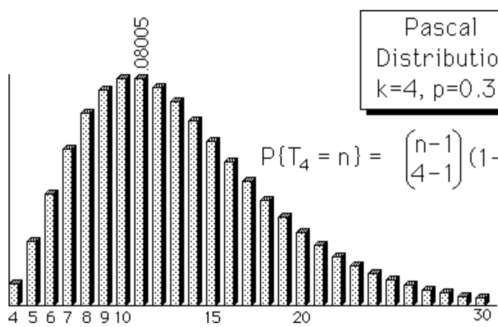
Pascal Distribution
k=4, p=0.3

Probability that arrival # n causes the overflow of the left-turn lane

$$P\{T_4 = n\} = \binom{n-1}{4-1}(1-p)^{n-4}p^4$$

n	P{n}
4	0.00810000
5	0.02268000
6	0.03969000
7	0.05556600
8	0.06806835
9	0.07623655
10	0.08004838
11	0.08004838
12	0.07704657
13	0.07191013
14	0.06543822
15	0.05829950
16	0.05101206
17	0.04394886
18	0.03735653
19	0.03137948
20	0.02608419
21	0.02148110
22	0.01754290
23	0.01421898
24	0.01144628
25	0.00915702
26	0.00728400
27	0.00576386
28	0.00453904
29	0.00355861
30	0.00277845

$P\{n \le 30\} = 99.068\%$



Pascal Distribution
k=4, p=0.3

$$P\{T_4 = n\} = \binom{n-1}{4-1}(1-p)^{n-4}p^4$$