57:022 Principles of Design II

Homework #1 Solutions

Spring 1997

Be sure to state what probability distribution you assume in each problem! You may find the necessary probabilities in tables, compute them manually with a calculator, or use the APL workspace "ProbLib" which can be found on the ICAEN fileserver (in the folder for IE). (Using the APL workspace requires that you have a disk containing APL.68000 Level II, an interpreter for the APL language.)

1. Each throw of the pair of dice is considered a Bernouilli trial, with "success" = obtaining a 7 or 11. The value of "7" is achieved by six pairs (1,6), (2,5), (3,4), (4,3), (5,2), & (6,1), and since there are 36 pairs, P{7} = $^6/_{36} = {^1/_6}$. Likewise, the value "11" is achieved only by the two pairs (5,6) and (6,5), and therefore P{11} = $^2/_{36} = {^1/_{18}}$. Hence P{7 or 11} = $^1/_6 + {^1/_{18}} = {^4/_{18}} = {^2/_9}$.

   (a) Find the probability of throwing a "7 or 11" at least twice in five throws of a standard pair of dice.

   Solution: The number N_5 of successes in 5 trials has <u>Binomial</u> Distribution with (n,p) = (5, $^2/_9$). Therefore,

$$P\{N_5 = x\} = \frac{5!}{x!\,(5-x)!}\left(\frac{2}{9}\right)^x\left(\frac{7}{9}\right)^{5-x}$$

   Using this formula, the following table can be computed:

Binomial Distribution Function

n= 5, p= 0.22222222

x	P[x]	P{X≤x}	P{X>x}
0	0.28462802	0.28462802	0.71537198
1	0.40661146	0.69123948	0.30876052
2	0.23234940	0.92358889	0.07641111
3	0.06638554	0.98997443	0.01002557
4	0.00948365	0.99945808	0.00054192
5	0.00054192	1.00000000	0.00000000

   and so the probability of at least 2 successes is

$$P\{N_5 \geq 2\} = \sum_{x=2}^{5} P\{N_5 = x\} = 0.30876$$

   (b) What is the expected number of 7's & 11's obtained in 5 throws of a pair of dice?

   Solution: The expected value of N_5 having the Binomial distribution is np = $5(^2/_9) = {^{10}/_9}$ = 1.111111.

   (c) What is the expected number of throws of a pair of dice required in order to obtain a 7 or 11?

Solution : Let T_1 = the number of the throw of the dice in which the first success occurs. Then T_1 has the Geometric distribution, and $E[T_1] = 1/p = 9/2 = 4.5$.

2. *A certain production process has a fraction defective of 15%. Three good pieces are required. Pieces are produced until the 3 good pieces are obtained.*
Each production of a piece is a Bernouilli trial and "success" is the event that the piece produced is good rather than defective.

   a. *What are the expected value and standard deviation of the number of pieces needed to be produced?*

   Solution : Let T_3 = the number of the third piece which is determined to be good. Then T_3 has the Pascal distribution, with probability of success p = 85%. Since for the Pascal distribution,

$$E[T_k] = \frac{k}{p}\ ,\ Var[T_1] = \frac{k(1-p)}{p^2},$$

   the expected total number of pieces required to be produced in order to obtain three good pieces is 3.5294, while the variance is 0.62284 (so that the standard deviation is the square root of the variance, namely 0.78920.

   b. *Compute the probabilities of producing no more than 5 pieces in order to obtain 3 good pieces.*

   Solution : We need to compute the probability that T_3 is no more than 5, using the formula

$$P\{T_k = x\} = \binom{x-1}{k-1}(1-p)^{x-k}p^k,$$

   where k=3 and x = 3,4,5. (The minimum possible value of T_3 is 3.) We can then compute the table:

Pascal (Negative Binomial) Distribution

k= 3, p= 0.85

x	P(x)	P(X≤x)	P(X>x)
3	0.61412500	0.61412500	0.38587500
4	0.27635625	0.89048125	0.10951875
5	0.08290688	0.97338813	0.02661188
6	0.02072672	0.99411484	0.00588516
7	0.00466351	0.99877836	0.00122164
8	0.00097934	0.99975769	0.00024231
9	0.00019587	0.99995356	0.00004644
10	0.00003777	0.99999133	0.00000867

   and thus have the answer:

$$P10\{T_3 \le 5\} = \sum_{x=3}^{5} P\{T_3 = x\} = 0.97339$$

   c. *Suppose that instead of producing until 3 good pieces are obtained, a batch of size **n** is produced, and then the pieces are inspected. How large should **n** be in order to be 95% certain of obtaining at least 3 good pieces?*

Solution : We want to determine n such that

$$P\{T_k \le n\} \ge 95\%.$$

Compute the values of cumulative distribution function until a value of at least 0.95 is obtained. Consulting the table in Solution (b) above, this happens to be 5. If we wished to be 99.5% certain of obtaining at least 3 good pieces, we would need to produce 8, etc.

3. *A telephone exchange contains 10 lines. A line can be busy or available for calls and all lines act independently. If each line is busy 82% of the noon period (so that the probability that a specified line will be busy at any given time during the noon period is 82%),*
This is not a stochastic process, but we can consider testing each line to determine whether it is free ("success") or busy as a Bernouilli trial, so that N_{10}, the number of free lines of the possible ten, has a Binomial distribution with (n,p) = (10, 0.18).

a. ....what is the probability of there being at least four free lines at any given time during this period?

Solution : Using the formula

$$P\{N_{10} = x\} = \binom{10}{x} p^x (1-p)^{10-x},$$

we compute the table:

n= 10, p= 0.18

x	P[x]	P{X≤x}	P{X>x}
0	0.13744803	0.13744803	0.86255197
1	0.30171519	0.43916322	0.56083678
2	0.29803574	0.73719896	0.26280104
3	0.17445994	0.91165890	0.08834110
4	0.06701815	0.97867705	0.02132295
5	0.01765356	0.99633061	0.00366939
6	0.00322931	0.99955992	0.00044008
7	0.00040507	0.99996499	0.00003501
8	0.00003334	0.99999834	0.00000166
9	0.00000163	0.99999996	0.00000004
10	0.00000004	1.00000000	0.00000000

Hence, the probability of at least 4 free lines is

$$P10\{N_{10} \ge 4\} = 1 - \sum_{x=0}^{3} P\{N_{10} = 3\} = 0.08834$$

b. ...what is the expected number of free lines at any time during this period?

Solution : The expected value of the Binomial distribution is np and so the expected number of free lines is 10(0.18) = 1.8.

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Homework #2 Solutions

Spring 1997

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Be sure to state what probability distribution you assume in each problem. You may find the necessary probabilities in tables, compute them manually with a calculator, or use the APL workspace "PROBLIB" .

1. Along highway I-80 in Iowa, the probability that each passing car stops to pick up a hitchhiker is p=4%, i.e, an average of one in 25 drivers will stop; different drivers, of course, make their decisions whether to stop or not independently of each other.

a. Each car may be considered as a "trial" in a Bernouilli *process, with "success" defined as the car's stopping to pick up the hitchhiker.*

b. Given that a hitchhiker has counted 10 cars passing him without stopping, what is the probability that he will be picked up by the 25th car or before? 0.43533.

The process is memoryless... the probability that is requested is $P\{T_1 \leq 25 \mid T_1 > 10\}$ which is identical to $P\{T_1 \leq 15\} = 1 - (1-p)^{15} = 1 - 0.54209 = 0.45791$

Suppose that the arrivals of the cars form a Poisson *process, at the average rate of 10 per minute. Define "success" for the hitchhiker to occur at time t provided that both an arrival occurs at t and that car stops to pick him up. Let T_1 be the time (in seconds) of the first "success", i.e., the time that he finally gets a ride, when he begins his wait at time T_1=0.*

c. What is the arrival rate of "successes"? $\lambda = 0.04$ (10/minute) = 0.4/minute

d. What is the name of the probability distribution of T_1? Exponential

e. What is the value of $E(T_1)$? __$1/\lambda = 2.5$ _

What's the value of $Var(T_1)$? _ $1/\lambda^2 = \; 6.25 \text{ min}^2$___

f. What is the probability that he must wait less than 4 minutes for a ride ($P\{T_1 < 4\}$?

0.798193 $P\{T_1 < 4\} = 1 - e^{-4\lambda} = 1 - 0.201896 = 0.798193$

g. What is the probability that he must wait more than 4 minutes for a ride($P\{T_1 > 4\}$?

0.201896 $P\{T_1 > 4\} = e^{-4\lambda} = 0.201896$

h. What is the probability that he must wait exactly 4 minutes for a ride? ($P\{T_1 = 4\}$?

__zero__

Note that your answers in f, g, and h must have a sum equal to 1!

i. Suppose that after 4 minutes (during which 44 cars have passed by) he is still there waiting for a ride. Compute the conditional expected value of T_1 (expected total waiting time, given that he has already waited 4 minutes). 6.5 i.e., an additional 2.5 minutes

$$E[T_1 \mid T_1 > 4] = 4 + E[T_1] = 4 + 2.5 = 6.5$$

2. A bearing in a Grass Chopper mower's PTO mechanism fails randomly, with an expected lifetime of 250 hours. Assume that the lifetime of the bearing has an exponential distribution.

Define T_1 = lifetime (i.e., failure time) of the bearing. Then T_1 has an exponential distribution with expected value $1/\lambda = 250$ hours, or $\lambda = 0.004$ / hour

(a.) What is the probability that the bearing lasts longer than 250 hours? __0.367879_

$$E[T_1 > 250] = e^{-250\lambda} = e^{-1} = 0.367879$$

(b.) If the mower has already operated without failure, for 150 hours, what is the probability that the bearing will last at total of at least 250 hours? 0.67032

$$P\{T_1 \geq 250 \mid T_1 > 150\} = P\{T_1 \geq 250 - 150\}$$

$$= P\{T_1 \geq 100\} = e^{-100(0.004/\text{hour})} = 0.67032$$

(c.) If the bearing is replaced (with an identical bearing) when it experiences its first failure, what is the name of the probability distribution _2-Erlang_,

mean value __$2/\lambda = 500$ hours__ , *and*

variance __$2/\lambda^2$ = 125000 hours^2___ *of the total time (from first use of the mower) until the second failure?*

(d.) What is the probability that the bearing will fail (& be replaced) three or more times in 750 hours of mowing? (_0.57681_

$$P\{N_t \geq 3\} = 1 - P\{N_t \leq 2\} = 1 - \sum_{x=0}^{2} \frac{(\lambda t)^x}{x!} e^{-\lambda t}, \text{ and so}$$

$$P\{N_{750} \geq 3\} = 1 - P\{N_{750} \leq 2\} = 1 - \sum_{x=0}^{2} \frac{3^x}{x!} e^{-3}$$

$$= 1 - 0.049787 - 0.14936 - 0.22404 = 0.57681$$

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Homework #3 Solutions

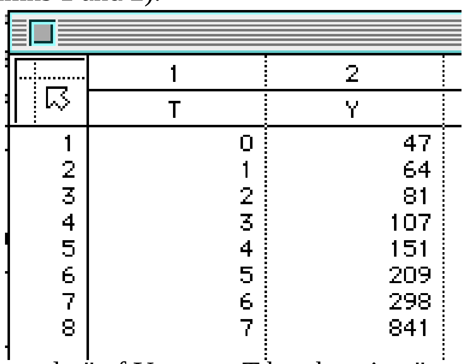
Due Wednesday, February 12, 1997

Regression Analysis. The number Y of bacteria per cubic centimeter found in a tillage after T hours is given in the following table:

T (hours)	0	1	2	3	4	5	6	7
Y (#/cm^3)	47	64	81	107	151	209	298	841

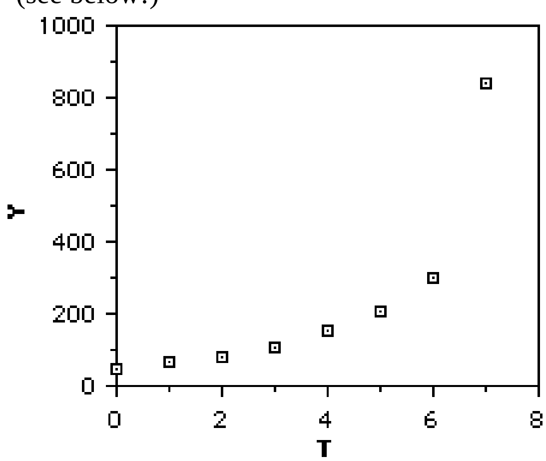
It is believed that the relation between the two variables is of the form $Y = \alpha\beta^T$.

a. Run the "Cricket Graph" software package (on the Macintosh), and enter the above observed values of T and Y (columns 1 and 2).



	1	2
	T	Y
1	0	47
2	1	64
3	2	81
4	3	107
5	4	151
6	5	209
7	6	298
8	7	841

b. Plot the "scatter plot" of Y versus T by choosing "scatter" on the Graph menu, and specifying T on the horizontal axis and Y on the vertical axis. Does the plot appear to be near-linear? **Solution:** __NO__ (see below!)



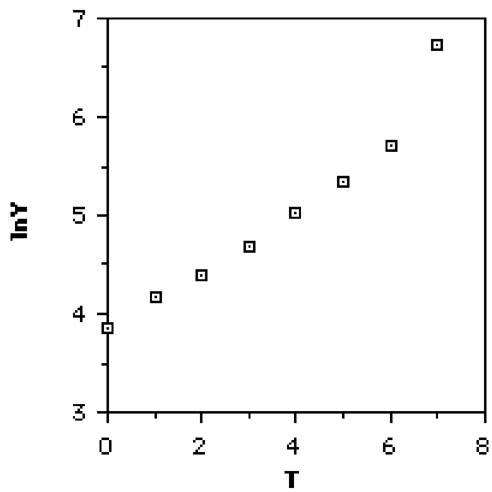
c. Write the linear form of the relationship $Y = \alpha\beta^T$.

Solution: Taking the logarithm of each side gives us the equation $\ln Y = (\ln \alpha) + (\ln \beta)T$, which, if we make a change of variable $Y' = \ln Y$, is a linear relationship between Y' and T.

d. Choose "transform" from the "data" menu to create a new variable lnY which is the *logarithm* of Y. (Put this new variable into Column 3.)

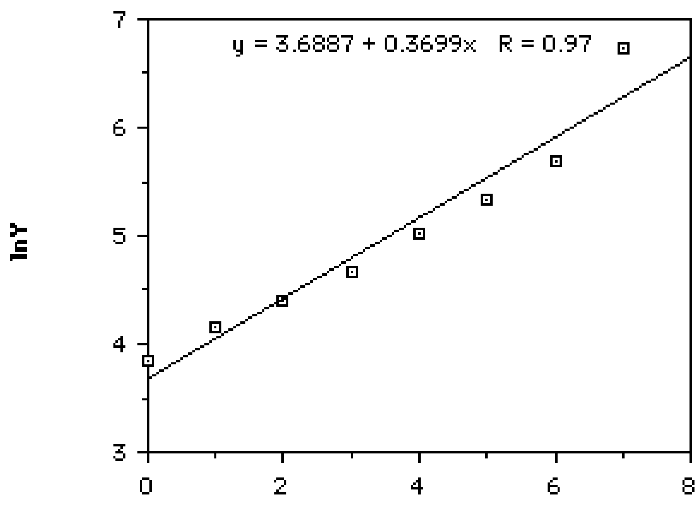
	1 T	2 Y	3 lnY
1	0	47	3.850
2	1	64	4.159
3	2	81	4.394
4	3	107	4.673
5	4	151	5.017
6	5	209	5.342
7	6	298	5.697
8	7	841	6.735

e. Plot the "scatter plot" of T (horizontal axis) versus lnY (vertical axis). Does this plot appear to be near-linear? **Solution:** YES, except perhaps for the point (7,841)



f. After plotting lnY versus T, select "Simple" from the "Curve Fit" menu, in order to fit a simple linear relationship between lnY and T, i.e., to determine a and b such that lnY $\approx a + bT$. What is the value of a? ___3.6887__ of b? __0.3699_

Solution: The linear relationship between (ln Y) and T is ln Y = 3.6887 + 0.3699T, as shown below:



g. What are the corresponding values of α & β in the relationship $Y = \alpha\beta^T$?
Solution: Exponentiating both sides of the equation ln Y = 3.6887 + 0.3699T in (f) above yields the equation $Y = 39.99 \times 1.4476^T$.

h. What is the prediction of the number of bacteria per cm^3 after 10 hours, based upon the above relationship?

Solution: $39.99 \times 1.4476^{10} = 1616$ bacteria per cm^3. Note that the relationship $Y = \alpha\beta^T$ may not be a good model outside of the interval [0,7], however. For example, congestion of the population might hinder further growth.

2. **Goodness-of-Fit test:** The numbers of arrivals during each of 100 one-minute intervals of what is believed to be a Poisson process were recorded.

Number of arrivals during each of the first 100 minutes

2	2	4	2	3	2	4	5	3	3	0	3	4	0	3	1	3	0	3	3	3	0	0	3	3
2	2	4	4	2	0	4	5	2	4	0	3	0	2	3	4	3	2	0	1	1	3	4	2	5
4	3	0	3	2	2	0	4	3	1	4	1	2	2	2	3	3	4	4	5	5	2	4	1	4
8	5	1	8	4	5	0	5	3	4	1	4	6	3	5	1	3	2	4	2	4	4	5	1	1

The observed numbers above ranged from zero to eight, with frequencies O_0 through O_8:

Number of arrivals i	0	1	2	3	4	5	6	7	8
observed frequency O_i	12	11	19	23	22	10	1	0	2

The average number of arrivals was 2.78/minute. We wish to test the "goodness-of-fit" of the assumption that the arrivals correspond to a Poisson process with arrival rate $\lambda = 2.78$/minute.
The first step is to compute the probability p_i of each observed value, i=0 through 8:

i	p_i
0	0.0620385
1	0.1724670
2	0.2397292
3	0.22215
4	0.1543936
5	0.0858428
6	0.0397738
7	0.0157959
8	0.0054890

a. What is the value missing above? (That is, the probability that the number of arrivals is exactly 3.)

Solution: N_1, the number of arrivals during a 1-minute interval, has the Poisson distribution according to the hypothesis that the arrivals form a Poisson process with arrival rate 2.78/minute. Therefore, since the Poisson distribution's probability function is

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

the probability that exactly 3 vehicles arrive during a one-minute interval should be

$$P\{N_1 = 3\} = \frac{(2.78)^3}{3!} e^{-2.78} = 0.22215 \backslash$$

b. Complete the table below:

i	p_i	E_i	O_i	$\|O_i - E_i\|$	$\frac{(O_i - E_i)^2}{E_i}$
0	0.062038507	6.2038507	12	5.7961493	5.4152409
1	0.17246705	17.246705	11	6.2467051	2.2625379
2	0.2397292	23.97292	19	4.97292	1.0315779
3	0.22215	22.215	23	0.785094	0.0277459
4	0.1543936	15.43936	22	6.5606404	2.7878101
5	0.085842839	8.5842839	10	1.4157161	0.2334792
6	0.039773849	3.9773849	1	2.9773849	2.2288064
7	0.0157959	1.57959	0	1.57959	1.57959
8	0.0054890752	0.54890752	2	1.4510925	3.8361095

What is the expected number of times in which we would observe three arrivals per minute? 22.215
Did we observe more or fewer than the expected number? Slightly more (23-22.215 = 0.785 more)

c. Because of the small number of observations of 6, 7, and 8 arrivals, we will group these observations together with 5 arrivals, so that we will have 6 cells (0, 1, ..., 4, and ≥5). Now, we can compute the expected number of observations in each of these cells, which we denote by E_0 through E_5.

i	p_i	E_i	O_i	$\|O_i - E_i\|$	$\frac{(O_i - E_i)^2}{E_i}$
0	0.062038507	6.2038507	12	5.7961493	5.4152409
1	0.17246705	17.246705	11	6.2467051	2.2625379
2	0.2397292	23.97292	19	4.97292	1.0315779
3	0.22215	22.215	23	0.785094	0.0277459
4	0.1543936	15.43936	22	6.5606404	2.7878101
5	0.14690166	14.690166	13	1.6901663	0.194458

d. What is the observed value of

$$D = \sum_i \frac{(E_i-O_i)^2}{E_i} \ ?$$

Solution: D = 11.719370.

e. Keeping in mind that the assumed arrival rate λ=2.78/minute was estimated from the data, what is the number of "degrees of freedom"? __4__

Solution: The number of degrees of freedom (6) must be reduced by 1 because λ=2.78/minute was estimated by using the average number of observed arrivals per minute), and also by 1 because the total number of observations was pre-specified to be 100. That is, if O_5 had not been reported, knowing that the sum of the other observations is 87 and that the total number of observations is 100, we can easily deduce that O_5=13. Likewise, if neither O_4 nor O_5 had been reported, but we know that the average # of arrivals/minute is 2.78 and the totaal number of observations is 100, then we would be able to compute both O_4 and O_5, by solving the pair of equations

$$12 + 11 + 19 + 23 + O_4 + O_5 = 100$$
$$(0[10] + 1[11] + 2[19] + 3[23] + 4O_4 + 5O_5)/100 = 2.78$$

f. Using a value of α = 5%, what is the value of $\chi^2_{5\%}$ such that D exceeds $\chi^2_{5\%}$ with probability 5% (if the assumption is correct that the arrivals form a Poisson process with arrival rate 2.78/minute)?

Solution: According to the chi-square probability distribution table appearing in your notes (see below), P{D ≥ 11.07} = 5%, i.e., $\chi^2_{5\%}$ = 9.488. That is, there is only a 5% probability that, if the arrival process really is Poisson with arrival rate 2.78/minute, the statistic D would exceed 9.488.

n	99%	95%	90%	•••	10%	5%	2%	1%
1	0.0002	0.004	0.0158		2.706	3.841	5.412	6.635
2	0.0201	0.103	0.211		4.605	5.991	7.824	9.210
3	0.115	0.352	0.584		6.251	7.815	9.837	11.341
4	0.297	0.711	1.064		7.779	9.488	11.668	13.277
5	0.554	1.145	1.610		9.236	11.070	13.388	15.086
6	0.872	1.635	2.204		10.645	12.592	15.033	16.812
7	1.239	2.167	2.833		12.017	14.067	16.622	18.475
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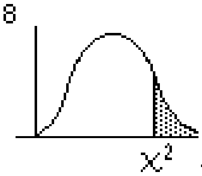

g. Is the observed value greater than or less than $\chi^2_{5\%}$? greater Should we accept or reject the assumption that the arrival process is Poisson with rate 2.78/minute?

Solution: We would reject this assumption, since there is less than 5% probability that the actual value of D, namely 11.719370, would ever be observed.

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Homework #4 Solution

Spring 1997

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1. **Generating Arrival Times in Poisson Process.** Suppose, in preparation for performing a manual simulation of the arrivals in a Poisson process (e.g., parts randomly arriving at a machine to be processed), you wish to generate some inter-arrival times, where the arrival rate is 4/hour, i.e., one every fifteen minutes. First, you need some uniformly-distributed random numbers. To obtain these, select a row from the following random number table, based upon the last digit of your ID#: if 1, use row #1; if 2, use row #2; ... if 0, use row #10.

Uniformly-Distributed "Random" Numbers

1	3735	5010	6806	0727	5863	7548	2955	0176	9132	8283
2	6613	9328	5159	5559	4755	3992	2960	7498	1480	4217
3	8768	7158	4764	2101	3525	9297	4171	1247	8098	3589
4	4326	1074	2295	7662	5421	8747	1089	7609	9694	3363
5	5081	0461	7977	8573	3930	8291	9051	9449	3346	1644
6	6595	5202	5096	2783	3414	8637	6043	6052	9561	7252
7	4055	1579	2020	6620	2424	8275	1389	0217	3510	6474
8	5134	1844	2788	5399	0629	4932	3028	4722	7346	2755
9	9652	2277	8133	1447	5645	8730	2435	4076	2632	0775
10	4655	6443	5269	0004	3687	3189	8049	9378	6781	3383

a. *What is the name of the probability distribution of the time T_1 of the first arrival?*
Solution: Exponential

b. *What will be the name of the probability distribution of the time τ_i between arrivals of parts i-1 and i (where i>1)?*
Solution: Exponential

c. *Use the inverse-transformation method to obtain random inter-arrival times $\tau_1, \tau_2, \dots \tau_{10}$ (where $T_1 = \tau_1$).*
Solution: The inverse transformation method requires a random variable X which has the uniform distribution over the interval [0,1] in order to generate times having an exponential distribution with mean $1/\lambda$. The inverse transformation is obtained by solving the equation $X = F(t) = 1 - e^{-\lambda t}$ for t in terms of X. This yields $t = F^{-1}(X) = -\dfrac{\ln(1-X)}{\lambda}$. Alternatively, because (1-X) has a uniform distribution in [0,1] if X does, we could use the transformation $t = -\dfrac{\ln \overline{X}}{\lambda}$ where $\overline{X}$ (=1-X), rather than X, is obtained from the table.

d. What are the arrival times ($T_1, T_2, \dots T_{10}$) of the first ten parts in your simulation?
Solution: The tables below give the ten simulated arrivals of the ten parts, one for each column of the random number table. Shown are the arrival times resulting from both transformations. For example, using column #1 for X results in the arrival time of the first part

$$\tau_1 = -\frac{\ln(1-0.3735)}{4} = -\frac{\ln 0.6265}{4} = 0.1169$$

while taking the value in the random number table to be X gives the arrival time

$\tau_1 = -\frac{\ln 0.3735}{4} = 0.2462$. The former is shown in the 3rd & 4th columns, below, while the latter is shown in the 5th & 6th columns. The columns labelled "T" are the cumulative sums of the columns labelled "tau", i.e., the "tau" column gives the inter-arrival times, and the "T" column gives the arrival times.

i	X	tau	T	tau	T
1	0.3735	0.1169	0.1169	0.2462	0.2462
2	0.6613	0.2707	0.3876	0.1034	0.3496
3	0.8768	0.5235	0.9110	0.0329	0.3825
4	0.4326	0.1417	1.0527	0.2095	0.5920
5	0.5081	0.1774	1.2301	0.1693	0.7612
6	0.6595	0.2693	1.4994	0.1041	0.8653
7	0.4055	0.1300	1.6294	0.2257	1.0909
8	0.5134	0.1801	1.8095	0.1667	1.2576
9	0.9652	0.8395	2.6490	0.0089	1.2665
10	0.4655	0.1566	2.8057	0.1912	1.4576

Column 1

i	X	tau	T	tau	T
1	0.5010	0.1738	0.1738	0.1728	0.1728
2	0.9328	0.6750	0.8488	0.0174	0.1902
3	0.7158	0.3145	1.1633	0.0836	0.2738
4	0.1074	0.0284	1.1917	0.5578	0.8316
5	0.0461	0.0118	1.2035	0.7692	1.6008
6	0.5202	0.1836	1.3871	0.1634	1.7642
7	0.1579	0.0430	1.4301	0.4614	2.2256
8	0.1844	0.0510	1.4810	0.4227	2.6483
9	0.2277	0.0646	1.5456	0.3699	3.0182
10	0.6443	0.2584	1.8041	0.1099	3.1281

Column 2

i	X	tau	T	tau	T
1	0.6806	0.2853	0.2853	0.0962	0.0962
2	0.5159	0.1814	0.4667	0.1655	0.2617
3	0.4764	0.1618	0.6285	0.1854	0.4470
4	0.2205	0.0623	0.6907	0.3780	0.8250
5	0.7977	0.3995	1.0902	0.0565	0.8815
6	0.5096	0.1781	1.2684	0.1685	1.0500
7	0.2020	0.0564	1.3248	0.3999	1.4499
8	0.2788	0.0817	1.4065	0.3193	1.7692
9	0.8133	0.4196	1.8260	0.0517	1.8209
10	0.5269	0.1871	2.0132	0.1602	1.9811

Column 3

i	X	tau	T	tau	T
1	0.0727	0.0189	0.0189	0.6554	0.6554
2	0.5559	0.2029	0.2218	0.1468	0.8021
3	0.2101	0.0590	0.2808	0.3900	1.1922
4	0.7662	0.3633	0.6441	0.0666	1.2588
5	0.8573	0.4868	1.1308	0.0385	1.2973
6	0.2783	0.0815	1.2124	0.3198	1.6170
7	0.6620	0.2712	1.4835	0.1031	1.7201
8	0.5399	0.1941	1.6776	0.1541	1.8742
9	0.1447	0.0391	1.7167	0.4833	2.3575
10	0.0004	0.0001	1.7168	1.9560	4.3135

Column 4

i	X	tau	T	tau	T
1	0.5863	0.2207	0.2207	0.1335	0.1335
2	0.4755	0.1613	0.3820	0.1858	0.3193
3	0.3525	0.1087	0.4906	0.2607	0.5800
4	0.5421	0.1953	0.6859	0.1531	0.7331
5	0.3930	0.1248	0.8107	0.2335	0.9666
6	0.3414	0.1044	0.9151	0.2687	1.2352
7	0.2424	0.0694	0.9845	0.3543	1.5895
8	0.0629	0.0162	1.0008	0.6916	2.2811
9	0.5645	0.2078	1.2086	0.1430	2.4240
10	0.3687	0.1150	1.3236	0.2494	2.6735

Column 5

i	X	tau	T	tau	T
1	0.7540	0.3514	0.3514	0.0703	0.0703
2	0.3992	0.1274	0.4788	0.2296	0.2999
3	0.9297	0.6637	1.1425	0.0182	0.3181
4	0.8747	0.5193	1.6618	0.0335	0.3516
5	0.8291	0.4417	2.1035	0.0469	0.3984
6	0.8637	0.4982	2.6017	0.0366	0.4351
7	0.8275	0.4393	3.0410	0.0473	0.4824
8	0.4932	0.1699	3.2109	0.1767	0.6591
9	0.8730	0.5159	3.7268	0.0340	0.6931
10	0.3189	0.0960	3.8228	0.2857	0.9788

Column 6

Column 7

i	X	tau	T	tau	T
1	0.2955	0.0876	0.0876	0.3048	0.3048
2	0.2960	0.0877	0.1753	0.3043	0.6091
3	0.4171	0.1349	0.3102	0.2186	0.8277
4	0.1089	0.0288	0.3391	0.5543	1.3821
5	0.9051	0.5887	0.9278	0.0249	1.4070
6	0.6043	0.2318	1.1596	0.1259	1.5329
7	0.1389	0.0374	1.1970	0.4935	2.0264
8	0.3028	0.0902	1.2871	0.2987	2.3251
9	0.2435	0.0698	1.3569	0.3532	2.6782
10	0.8049	0.4086	1.7655	0.0543	2.7325

Column 8

i	X	tau	T	tau	T
1	0.0176	0.0044	0.0044	1.0100	1.0100
2	0.7498	0.3464	0.3508	0.0720	1.0820
3	0.1247	0.0333	0.3841	0.5205	1.6024
4	0.7609	0.3577	0.7418	0.0683	1.6707
5	0.9449	0.7247	1.4665	0.0142	1.6849
6	0.6052	0.2323	1.6988	0.1255	1.8104
7	0.0217	0.0055	1.7043	0.9576	2.7681
8	0.4722	0.1598	1.8641	0.1876	2.9556
9	0.4076	0.1309	1.9950	0.2244	3.1800
10	0.9378	0.6944	2.6893	0.0161	3.1961

Column 9

i	X	tau	T	tau	T
1	0.9132	0.6110	0.6110	0.0227	0.0227
2	0.1480	0.0400	0.6511	0.4776	0.5003
3	0.8098	0.4149	1.0660	0.0527	0.5531
4	0.0694	0.0180	1.0840	0.6670	1.2200
5	0.3346	0.1018	1.1858	0.2737	1.4937
6	0.9561	0.7815	1.9673	0.0112	1.5050
7	0.3510	0.1081	2.0754	0.2617	1.7667
8	0.7346	0.3316	2.4070	0.0771	1.8438
9	0.2632	0.0764	2.4834	0.3337	2.1775
10	0.6781	0.2834	2.7667	0.0971	2.2746

Column 10

i	X	tau	T	tau	T
1	0.8283	0.4405	0.4405	0.0471	0.0471
2	0.4217	0.1369	0.5774	0.2159	0.2630
3	0.3589	0.1111	0.6886	0.2562	0.5191
4	0.3363	0.1025	0.7910	0.2724	0.7916
5	0.1644	0.0449	0.8359	0.4514	1.2429
6	0.7252	0.3229	1.1589	0.0803	1.3233
7	0.6474	0.2606	1.4195	0.1087	1.4320
8	0.2755	0.0806	1.5000	0.3223	1.7543
9	0.0775	0.0202	1.5202	0.6394	2.3936
10	0.3383	0.1032	1.6234	0.2710	2.6646

e. The expected number of arrivals in the first hour should, of course, be 4. In your simulation of the arrivals, however, what is the number of arrivals during the first hour? The number of arrivals in the first hour for each choice of random number sequence and each inverse transformation is shown below:

Random # column	Using 1-X from table	Using X from table
1	3	6
2	2	4
3	4	5
4	4	2
5	7	5
6	2	≥10
7	5	3
8	4	0
9	2	3
10	5	4

The number of arrivals varies considerably, depending upon which column of random numbers and which transformation you have used. Note, however, that the averages of these two columns are 3.8 and 4.2 (using 10 in the case above indicating "≥10").

2. Estimating Parameters of Weibull Distribution of Lifetime. Suppose that your company wishes to estimate the reliability of an electric motor. One hundred units are tested simultaneously for 30 days, and the number of failures each day are recorded:

t	#f	t	#f	t	#f
1	0	11	3	21	3
2	0	12	1	22	1
3	0	13	0	23	0
4	1	14	2	24	1
5	0	15	0	25	0
6	1	16	0	26	3
7	0	17	1	27	1
8	3	18	2	28	1
9	2	19	1	29	1
10	0	20	1	30	2

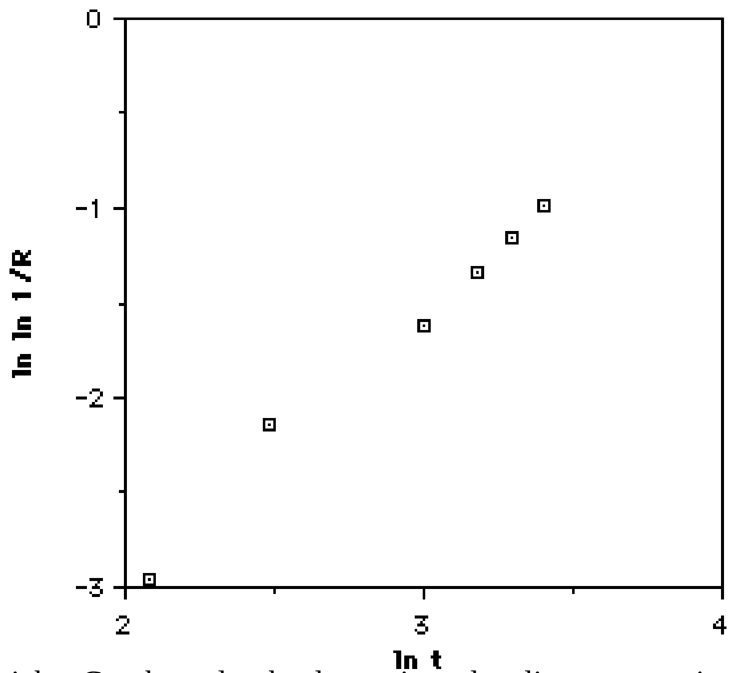
For the sake of convenience of doing this homework assignment, we will aggregate the data, rather than use the thirty data points:

t	f	NF	R
8	5	5	0.95
12	6	11	0.89
20	7	18	0.82
24	5	23	0.77
27	4	27	0.73
30	4	31	0.69

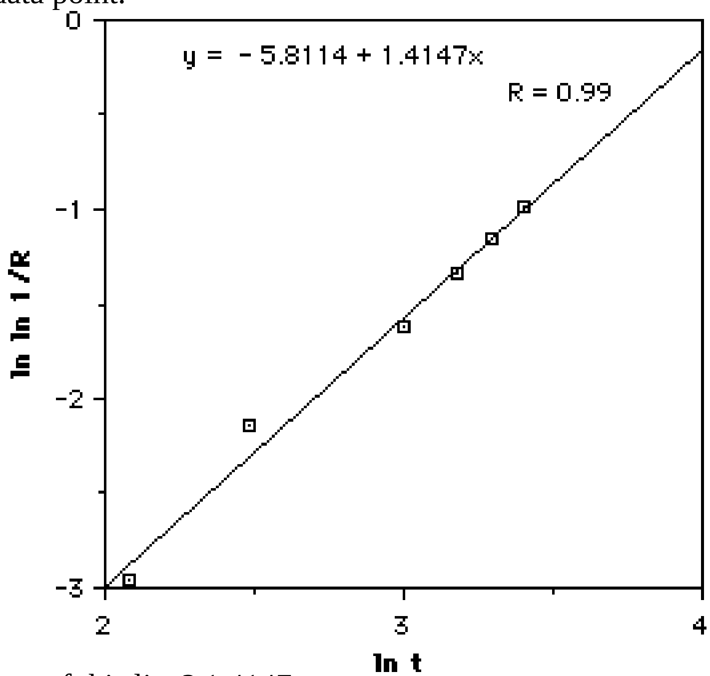
The column "f" displays the number of failures during the time intervals 0-8, 8-12, 12-20, etc. while the column "NF" displays the cumulative number of failures. Thus, there are five failures at the end of day #8, a total of 11 failures at the end of day 12, etc. The column "R" displays the fraction of the units which have survived at time t, e.g., at the end of day #24, 77% of the 100 units have survived.

t	f	NF	R	ln t	lnln 1/R
8	5	5	0.95	2.07944	¯2.9702
12	6	11	0.89	2.48491	¯2.14957
20	7	18	0.82	2.99573	¯1.61721
24	5	23	0.77	3.17805	¯1.34184
27	4	27	0.73	3.29584	¯1.1561
30	4	31	0.69	3.4012	¯0.991382

a. Either use *Cricket Graph* (or similar software) to plot the value of ($\ln \ln {}^{1}/_{R}$) on the vertical axis and ln T on the horizontal axis, or do the same manually with ordinary graph paper.



b. If you used Cricket Graph to plot the data points, do a linear regression to fit a straight line; if you have done it manually, draw a straight line (by "eyeballing it") which seems best to fit the data point.



c. What is the slope of this line? 1.4147

d. What is the y-intercept of this line? -5.8114

e. What is therefore your estimate of the parameters k and u of the Weibull distribution for the lifetimes of these motors? k= 1.4147 & u = 60.9169

Solution: slope=k and y-intercept = -k ln u, and so

$$\ln u = -\frac{\text{y-intercept}}{k} \qquad u = e^{-\text{y-intercept}/k}$$

$$u = e^{-(-5.8114)/1.4147} = 60.8169$$

f. What is the expected lifetime of the motors, according to your Weibull probability model? μ = 55.296 days.

(You may use the table below for the gamma function in the computation of μ. Values of $\Gamma(1+{}^{1}/_{k})$ are given for k=0.1, 0.2, ... 3.9)

$$\Gamma\left(1+\frac{1}{k}\right)$$

k	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	∞	3628800	120	9.2605	3.3234	2.0000	1.5046	1.2658	1.1330	1.0522
1	1.0000	0.9649	0.9407	0.9236	0.9114	0.9027	0.8966	0.8922	0.8893	0.8874
2	0.8862	0.8857	0.8856	0.8859	0.8865	0.8873	0.8882	0.8893	0.8905	0.8917
3	0.8930	0.8943	0.8957	0.8970	0.8984	0.8997	0.9011	0.9025	0.9038	0.9051

Solution: The relationship between the mean of the distribution and the parameters u & k is $\mu = u\,\Gamma\left(1+\frac{1}{k}\right)$. From the table below, when $\Gamma\left(1+\frac{1}{1.4}\right) = 0.9114$ and $\Gamma\left(1+\frac{1}{1.5}\right) = 0.9027$. Interpolating, we obtain

$$\Gamma\left(1+\frac{1}{1.4147}\right) \approx 0.9114 - 0.147(0.9114-0.9027) = 0.9101$$

Therefore the mean is approximately 60.9169 x 0.9101 = 55.44 days.

g. What fraction of the motors would you expect to have failed during the expected lifetime which you determined in (f)? 58.01%

Solution: The CDF of the Weibull distribution is

$$P\{T \le t\} = F(t) = 1 - e^{-(t/u)^k}$$

and so, substituting t=m and the parameters u & k above, we obtain

$$F(t) = 1 - e^{-(55.44/60.8169)^{1.4147}} = 0.5841$$

i.e., 58.41% of the motors are expected to have failed at 55.44 days.

h. Is the failure rate of the motors increasing or decreasing, according to the parameters which you have computed? increasing

Solution: Because k>1, the failure rate is increasing.

Perform a Chi-Square goodness of fit test to decide whether the Weibull probability distribution model which you have found is a "good" fit of the data.

i. Complete the table:

Solution:

Interval	O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
0 - 8	5	5.514	0.04795
8-12	6	4.062	0.92463
12-20	7	9.149	0.50478
20-24	5	4.812	0.00734
24-27	4	3.631	0.03750
27-30	4	3.618	0.04027
		Total: D =	1.5624

The values of the CDF (i.e., $P\{T \le t\} = F(t) = 1 - e^{-(t/u)^k}$)at t=8, 12, 20, 24, 27, and 30 are 0.05514, 0.09576, 0.18725, 0.23537, 0.27168, and 0.30787. By taking the differences, we get the probabilities of the failure time T falling into the respective intervals, namely 0.04062, 0.04062, 0.09151, 0.04810, 0.03631, and 0.3618.

j. *What is the number of "degrees of freedom" of the chi-square distribution?* __4
(Keep in mind that two parameters, u & k, were estimated based upon the data, but the total number of observed failures was not predetermined!)

Solution: There are six intervals in the table. Because the two parameters were estimated from the data, the degree of freedom is reduced from 6 to 4, however.

k. Using α = 5%, should the probability distribution be accepted or rejected as a model of the motor's lifetime? __accepted

Solution: According to the chi-square table (see below), P{D≥ 9.488} = 5%. Only if the computed value of D were to exceed 9.488 would we reject the probability distribution because the value of D is too unlikely (a value above 9.488 occurs with less than 5% probability).

n	99%	95%	90%	•••	10%	5%	2%	1%
1	0.0002	0.004	0.0158		2.706	3.841	5.412	6.635
2	0.0201	0.103	0.211		4.605	5.991	7.824	9.210
3	0.115	0.352	0.584		6.251	7.815	9.837	11.341
4	0.297	0.711	1.064		7.779	9.488	11.668	13.277
5	0.554	1.145	1.610		9.236	11.070	13.388	15.086
6	0.872	1.635	2.204		10.645	12.592	15.033	16.812
7	1.239	2.167	2.833		12.017	14.067	16.622	18.475
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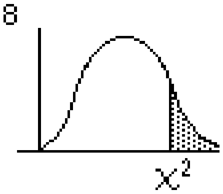

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57:022 Principles of Design II

Homework #5 Solutions

Spring 1997

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1. Generating Uniformly-Distributed Random Numbers: *Use the congruential method with $M= 2^8-1 = 255$ and $X_0=A=111$ to generate a sequence of 6 "pseudo-random" numbers with the uniform distribution. Then use the inverse transformation method to generate 6 lifetimes of the motors in HW #4, assuming a Weibull distribution with the parameters u & k found there (u=60.9169, k=1.4147).*

a. Complete the table:

i	X_i	AX_i	D_i ($\frac{AX_i}{M}$)	R_i	$U_i = R_i/M$	lifetime T_i
0	111	12321	48	81	0.317647	67.1106
1	81	8991	35	66	0.258824	76.3757
2	66	7326	28	186	0.729412	26.9535
3	186	20646	80	246	0.964706	5.8031
4	246	27306	107	21	0.082353	116.313
5	21	2331	9	36	0.141176	97.9421
6	36	3996	15	171	0.670588	(31.8523)

Note: Above, D_i and R_i denote the dividend and remainder, respectively, when AX_i is divided by M. The remainder R_i becomes the seed for the next random number. Since R_i is in the interval [0,254], we scale to the interval [0,1] by dividing by M=255.

Solution: The six required "pseudo-random" lifetimes are T_0, T_1, T_2, … T_5 above.

b. What is the average of the "randomly" generated lifetimes? 64.92 days.

(Note that the expected value, computed in HW#4, is 55.296 days.) The average of the seven lifetimes T_0 through T_6 is 60.19, nearer to the expected lifetime, however.

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2. Estimating Weibull Parameters Using Nonlinear Regression: Use the *Curve Fit 0.7e* software on the Macintosh to try fitting a Weibull distribution in its original (nonlinear) form to the data given in Homework #4 (and reproduced below). If x = upper limit of the intervals and y = fraction failed, then you should enter the function

f(x) = 1 - 2.718281828^(-(x/a)^b)

where a=u and b=k. (Note that *Curve Fit* allows only variables a, b, c, d, & e.)

From HW#4: *Suppose that your company wishes to estimate the reliability of an electric motor. One hundred units are tested simultaneously for 30 days, and the number of failures each day are recorded. For the sake of convenience of doing this homework assignment, we will aggregate the data, rather than use the thirty data points:*

t	f	NF	R
8	5	5	0.95
12	6	11	0.89
20	7	18	0.82
24	5	23	0.77
27	4	27	0.73
30	4	31	0.69

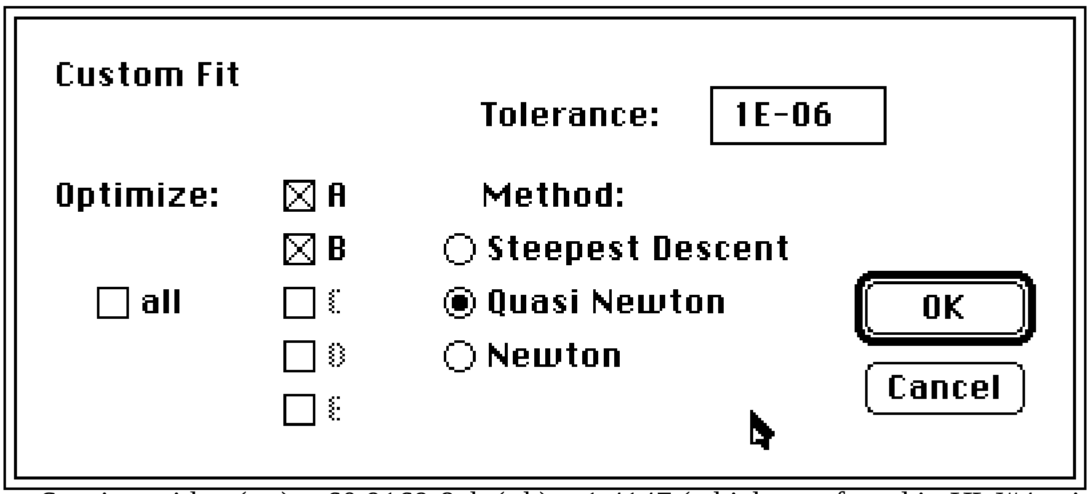
a. *Using as starting values for a & b the values which you found in HW#4, try fitting the curve to the data. What are the resulting values of a & b?*

a = _60.91699__ +/- ___3.23532____

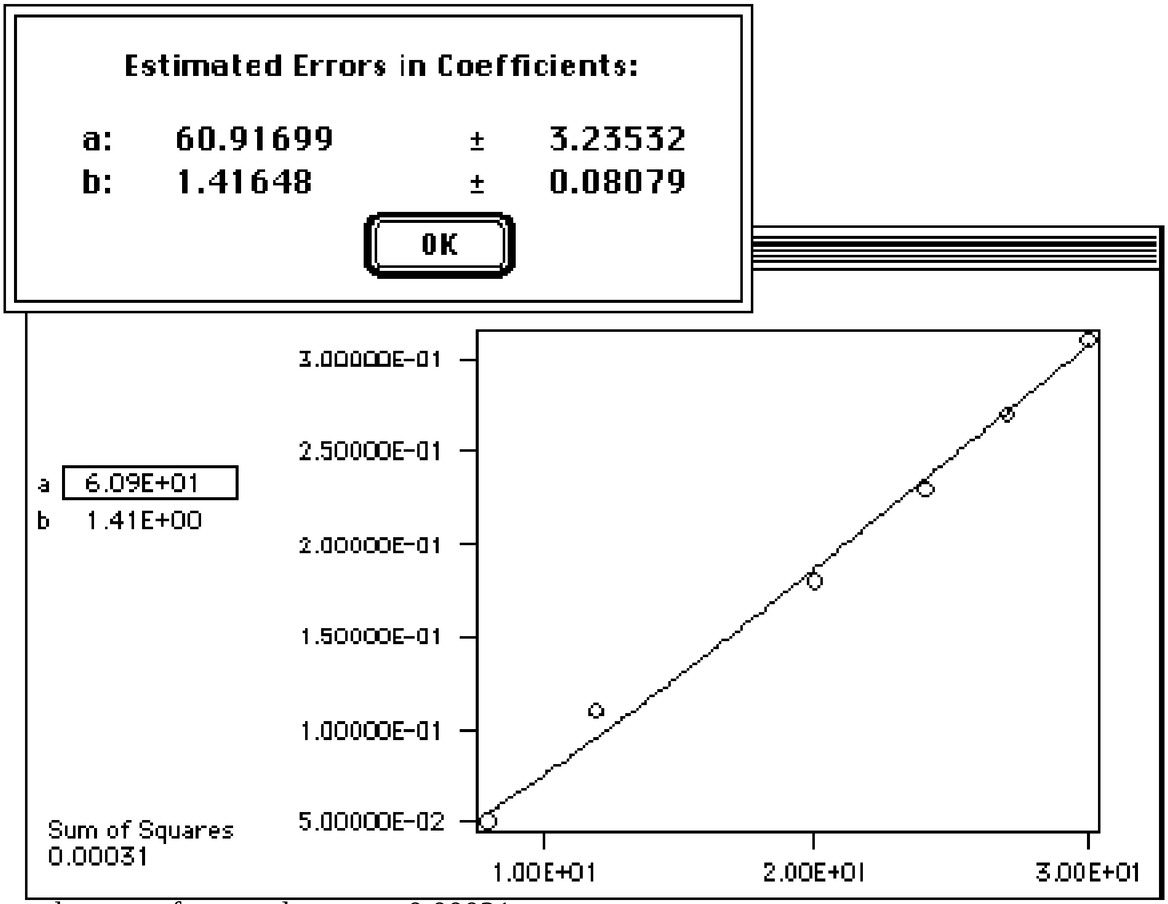
b = __1.41648__ +/- ___0.08079____

Solution:

We select the Quasi-Newton optimization algorithm to minimize the sum of the squares of the errors:

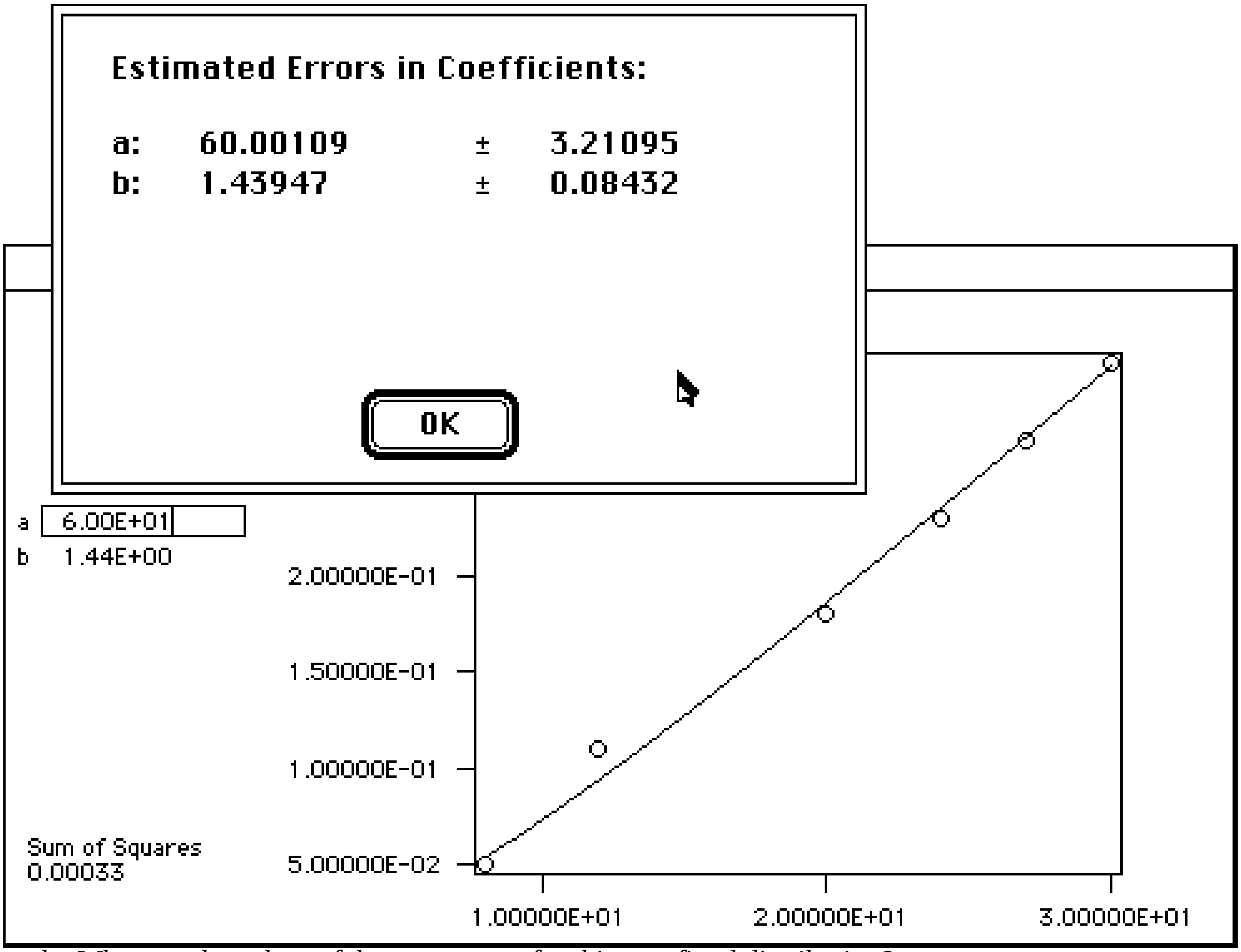


Starting with a (=u) = 60.9169 & b (=k) = 1.4147 (which was found in HW#4 using linear regression on the transformed data), the optimization algorithm terminates with the solution:



and a sum of squared errors = 0.00031.

Using a starting point further from the apparent optimum, namely a (=u) = 60 and b (=k) = 1.4, we get a slightly different result, with a slightly larger sum of squared errors (0.00033). These coefficients are well within the ranges of estimated errors shown earlier.



b. What are the values of the parameters for this new fitted distribution?
Solution: u = _60.91699__, k = __1.41648__

c. According to these values of u & k, what is
... the expected lifetime of the device? __55.4213__
... the probability that the device will have failed before its expected lifetime has passed?
__58.3%_

d. Using these new parameters u & k, compute the new expected number of failures (of the 100 units tested) in each of the intervals:
Solution:

Interval	O_i	Old E_i	New E_i
0 - 8	5	5.514	__5.482_
8-12	6	4.062	___4.046__
12-20	7	9.149	___9.126__
20-24	5	4.812	___4.804__
24-27	4	3.631	___3.626__
27-30	4	3.618	___3.614__

(Shown above are also the expected values using the Weibull distribution found in HW #4.)

e. According to these new estimates of u & k, at what time should

... 5% of the motors have failed? ___7.483__ days
... 50% of the motors? ___47.03__ days
... 90% of the motors? ___109.76__ days

Solution: To do this computation, it is necessary to find T such that F(T) = P = 0.05, 0.5, and 0.9, that is, to compute the inverse of the CDF of the Weibull distribution:

$$F(t) = P = 1 - e^{-\left(t/u\right)^k} \qquad t = F^{-1}(P) = u\left(-\ln\left[1-P\right]\right)^{1/k}$$

evaluated at the three probabilities. This gives the results above.

f. You needn't perform a "goodness-of-fit" test to confirm it, but state your opinion about which Weibull distribution seems to be a better fit of the original data.

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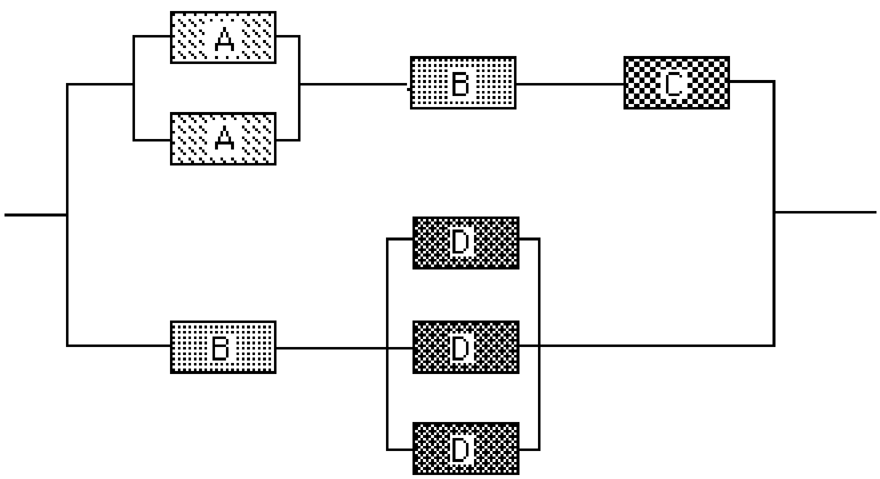
3. System Reliability. *A system contains 4 types of devices, with the system reliability represented schematically by*



It has been estimated that the lifetime probability distributions of the devices are as follows:

A: Weibull, with mean 800 days and standard deviation 1000 days

Note: This implies that the Weibull distribution has scale parameter **u**=709.9876 & shape parameter **k** = 0.80623 (which was determined by first solving the nonlinear equation for k by the Secant Method).

B: Exponential, with mean 4000 days

C: Normal, with mean 3000 days and standard deviation 1000 days

D: Exponential, with mean 1200 days

a.) Compute the reliabilities of a unit of each device for designed system lifetimes of 1000 and 2000 days:

Solution:

$R_A(t) = e^{-\left(t/u\right)^k} = e^{-\left(t/709.9876\right)^{0.80623}}$

$R_B(t) = 1 - F_B(t) = e^{-t/4000}$

$R_C(t) = 1 - F_C(t) = 1 - \Phi\left(\frac{t - 3000}{1000}\right)$ (which can be found in a standard table of the normal distribution).

$R_D(t) = 1 - F_D(t) = e^{-t/1200}$

Device	R(1000)	R(2000)
A	26.766%	9.978%
B	77.880%	60.653%

C	97.733%	84.088%
D	43.460%	18.888%

b.) Using the reliabilities in (b), compute the system reliability:

Solution:

$R_{AA}(t) = 1 - (1 - R_A(t))^2$

$R_{AABC}(t) = R_{AA}(t) \times R_B(t) \times R_C(t)$

$R_{DDD}(t) = 1 - (1 - R_D(t))^3$

$R_{BDDD}(t) = R_B(t) \times R_{DDD}(t)$

$R_{sys}(t) = 1 - (1 - R_{AABC}(t)) \times (1 - R_{BDDD}(t))$

Subsystem	R(1000)	R(2000)
AA	46.367 %	18.961 %
AA+B+C	35.293 %	9.670 %
DDD	81.925 %	46.634 %
B+DDD	63.813 %	28.285 %
Total system:	76.578 %	35.220 %

57:022 Principles of Design II

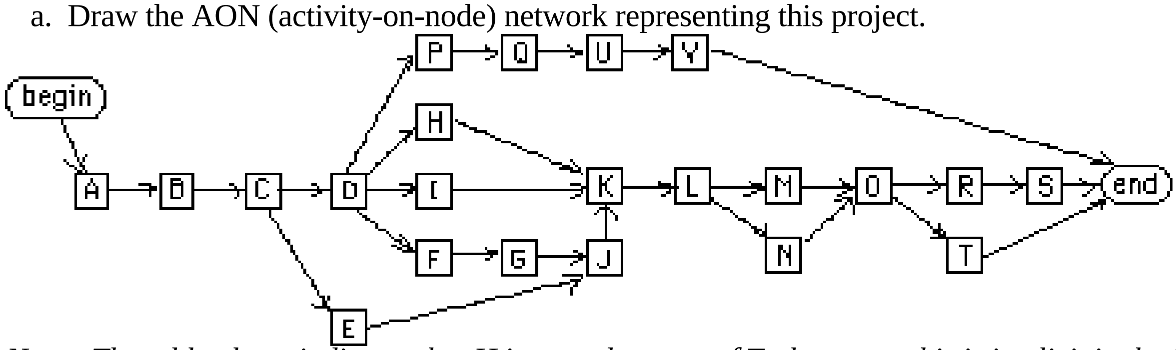
Homework #6 Solution

Spring 1997

1. **Project Scheduling:** A building contractor is preparing a project schedule for the construction of a house. The activity descriptions and estimated durations for the project are:

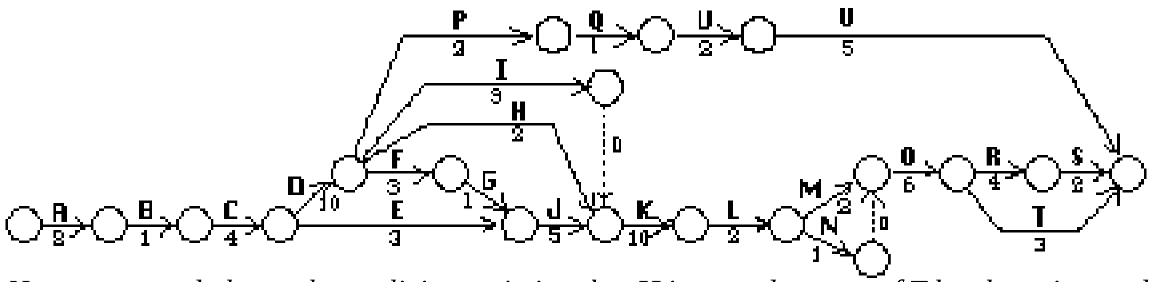
Activity	Description	Predecessor(s)	Duration: optimistic	Duration: most likely	Duration: pessimistic	Mean	Std Dev'n
A	Excavate foundations	none	1	2	4	$2\frac{1}{6}$	$\frac{1}{2}$
B	Pour footings	A	-	1	-	1	0
C	Pour foundations, including placing & removing forms	B	3	4	8	$4\frac{1}{2}$	$\frac{5}{6}$
D	Framing floors, walls, & roof	C	8	10	14	$10\frac{1}{3}$	1
E	Construct brick chimney	C	2	3	4	3	$\frac{1}{3}$
F	Install drains & rough plumbing	D	2	3	4	3	$\frac{1}{3}$
G	Pour basement floor	F	-	1	-	1	0
H	Install rough wiring	D	1	2	3	2	$\frac{1}{3}$
I	Install water lines	D	2	3	6	$3\frac{1}{3}$	$\frac{2}{3}$
J	Install heating ducts	D,E,G	4	5	7	$5\frac{1}{6}$	$\frac{1}{2}$
K	Lathe & plaster walls	H,I,J	7	10	12	$9\frac{5}{6}$	$\frac{5}{6}$
L	Finish flooring	K	-	2	-	2	0
M	Install kitchen equipment	L	1	2	3	2	$\frac{1}{3}$
N	Install bath plumbing	L	-	1	-	1	0
O	Cabinetwork	M,N	4	6	9	$6\frac{1}{6}$	$\frac{5}{6}$
P	Lay roofing	D	1	2	4	$2\frac{1}{6}$	$\frac{1}{2}$
Q	Install downspouts & gutters	P	-	1	-	1	0
R	Paint walls & trim	O	3	4	5	4	$\frac{1}{3}$
S	Sand & varnish floors	R	1	2	3	2	$\frac{1}{3}$
T	Install electric fixtures	H,O	2	3	5	$3\frac{1}{6}$	$\frac{1}{2}$
U	Grade lot	C,Q	1	2	4	$2\frac{1}{6}$	$\frac{1}{2}$
V	Landscape	U	4	5	7	$5\frac{1}{6}$	$\frac{1}{2}$
W	End	S,T,V	-	0	-	0	0

a. Draw the AON (activity-on-node) network representing this project.



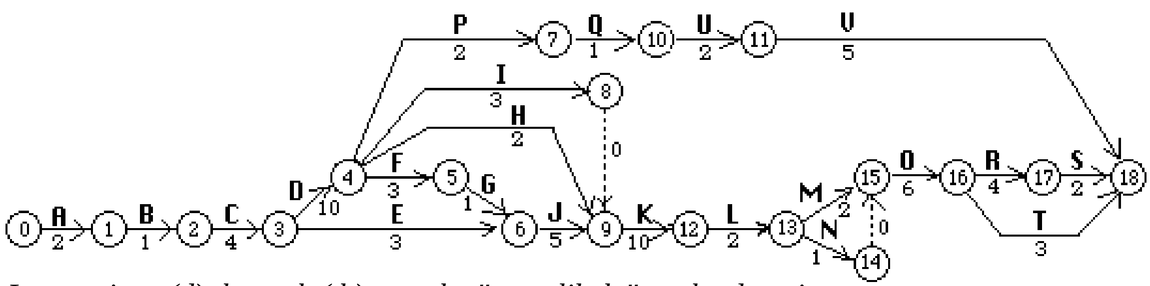
Note: The table above indicates that H is a predecessor of T; however, this is implicit in the fact that H precedes K, which precedes L, which precedes M, ... which precedes T. The arrow from H to T has therefore been omitted (for the sake of convenience... including it will have no effect on the results.)

b. Draw the AOA (activity-on-arrow) network representing this project. Explain the necessity for any "dummy" activities which you have included. **Solution:** see below



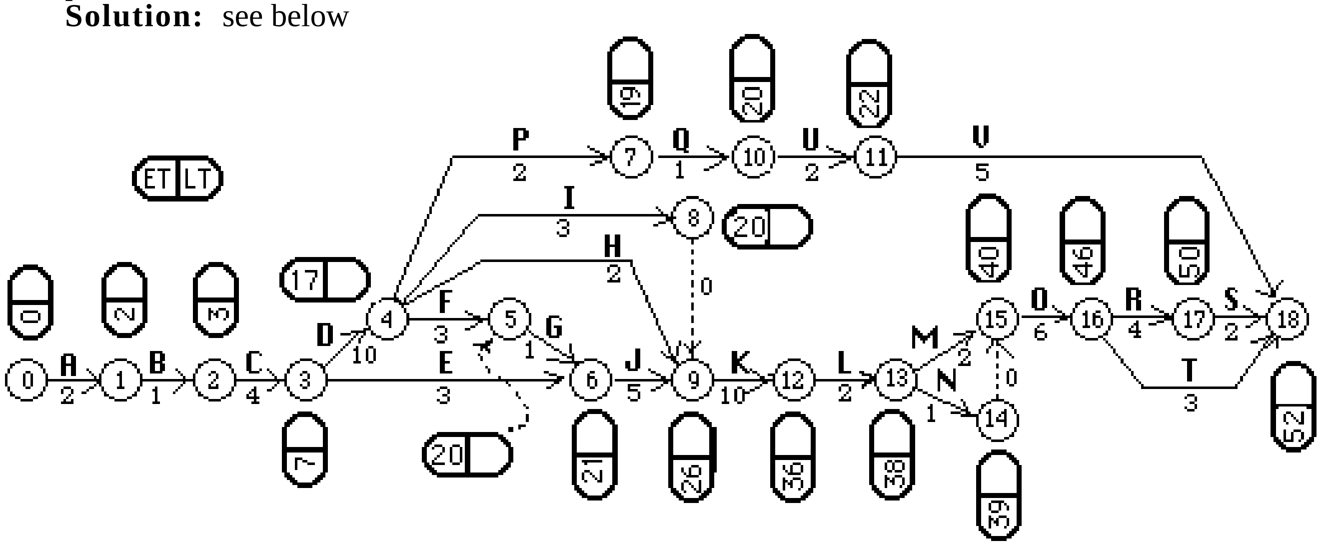
Note: as stated above, the explicit restriction that H is a predecessor of T has been ignored, since it is implicit in the other precedence restrictions.

c. Label the nodes of the AOA network, so that i < j if there is an activity with node i as its start and node j as its end node.
Solution: see below for one possible numbering scheme. (Several others are possible as well.)



In questions (d) through (h), use the "most likely" as the duration:

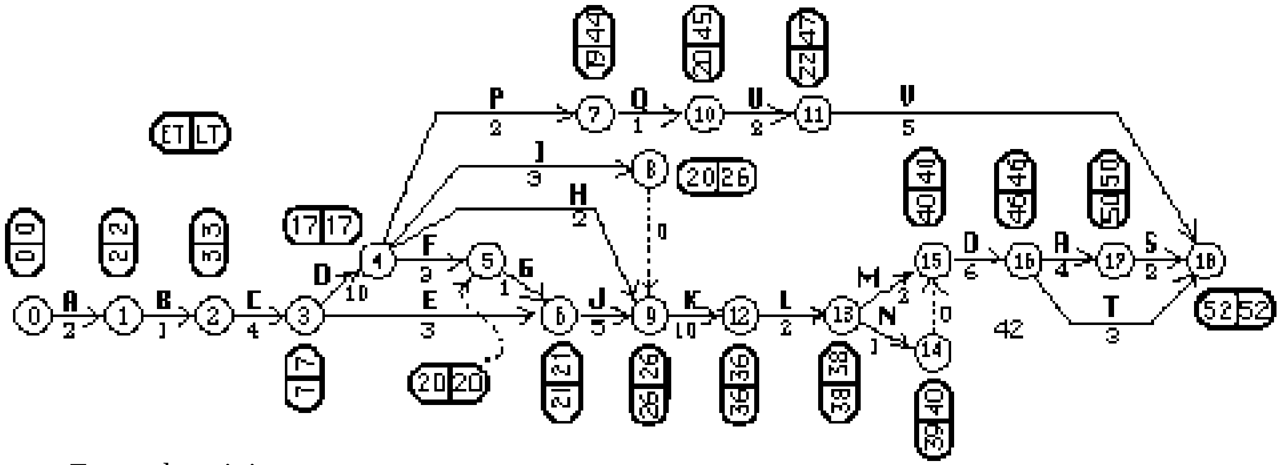
d. Perform the forward pass through the AOA network to obtain for each node i, ET(i) = earliest possible time for event i.
Solution: see below



e. What is the earliest completion time (# work days) for this project? **Solution:** __52__ days

f. Perform the backward pass through the AOA network to obtain, for each node i,
LT(i) = latest possible time for event i ,

assuming the project is to be completed in the time which you have specified in (e).
Solution: see below

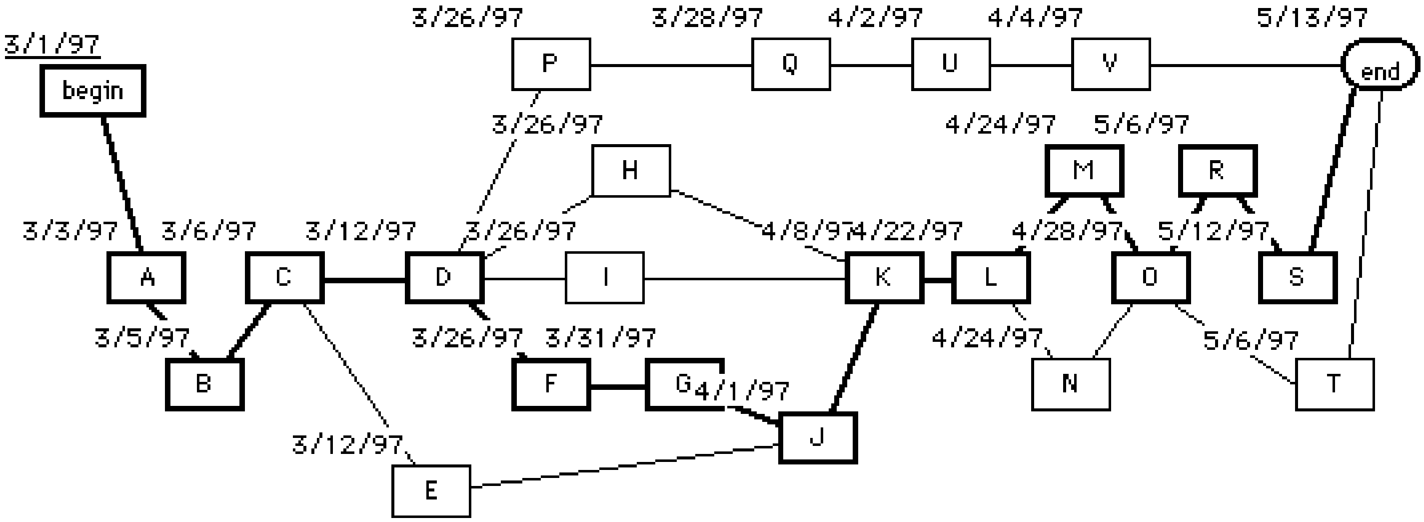


g. For each activity, compute:
ES = earliest start time EF = earliest finish time
LS = latest start time LF = latest finish time
TF = total float (slack)

Activity	Description	ES	EF	LS	LF	TF
A*	Excavate foundations	0	2	0	2	0
B*	Pour footings	2	3	2	3	0
C*	Pour foundations, including placing & removing forms	3	7	3	7	0
D*	Framing floors, walls, & roof	7	17	7	17	0
E	Construct brick chimney	7	10	17	20	10
F*	Install drains & rough plumbing	17	20	17	20	0
G*	Pour basement floor	20	21	20	21	0
H	Install rough wiring	17	19	24	26	7
I	Install water lines	17	20	23	26	6
J*	Install heating ducts	21	26	21	26	0
K*	Lathe & plaster walls	26	36	26	36	0
L*	Finish flooring	36	38	36	38	0
M*	Install kitchen equipment	38	40	38	40	0
N	Install bath plumbing	38	39	39	40	1
O*	Cabinetwork	40	46	40	46	0
P	Lay roofing	17	19	42	44	25
Q	Install downspouts & gutters	19	20	44	45	25
R*	Paint walls & trim	46	50	46	50	0
S*	Sand & varnish floors	50	52	50	52	0
T	Install electric fixtures	46	49	49	52	3
U	Grade lot	20	22	45	47	25
V	Landscape	22	27	45	52	23
W	End	52	____	____	52	____

h. Which activities are "critical", i.e., have zero float ("slack")? Place an * beside these in the table above. **Solution:** A, B, C, D, F, G, J, K, L, M, O, R, & S

Schedule this project by entering the AON network into MacProject Pro (found on the ICAEN fileserver for the Macintoshes). Specify that the start time for the project will be March 1, 1997.



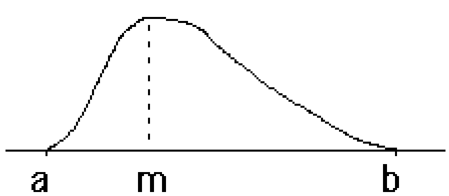
i. What is the earliest completion date for the project? 5/13/97
What are the critical activities? **Solution:** A, B, C, D, F, G, J, K, L, M, O, R, & S (same as in (h)).
(Note that 5-day work weeks and days off on holidays are assumed by default. It is possible to modify this calendar, if you choose to.)

Next, assume that the durations are random, with beta distributions.

j. What are the *expected value & standard deviation* of each critical activity's duration? (Specify in the first table above.) **Solution**: The expected values and standard deviations, computed by the forumulae below, are shown in the table on page 1 of these solutions.

The BETA distribution

is **unimodal** with **finite endpoints**



Mean: $\mu = \dfrac{a + 4m + b}{6}$

Standard deviation: $\sigma = \dfrac{b - a}{6}$

k. What is the expected value of the duration of the critical path found in (h)? **Solution:** 453$^{1}/_{6}$
(Assuming that the critical path found earlier using the "most likely" durations is always the critical path.)
What is the standard deviation of this duration? **Solution:** 2.0069 days
Note: The expected value (51) differs from the answer in (e) because you used the "most likely" (i.e. median) and not the "expected" (i.e. mean) value there.

The expected duration is found by summing the expected durations of the activities on this critical path, while the variance of the project duration is found by summing the variances of the activities. The standard deviation of the project duration is then found by computing the square root of its variance.

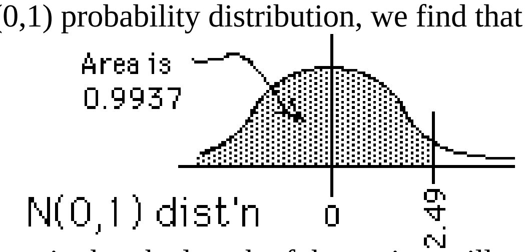
	Activity	mean	std. dev'n	Variance
A	Excavate foundations	$2\,{}^{1}/_{6}$	$^{1}/_{2}$	0.25
B	Pour footings	1	0	0
C	Pour foundations, including placing & removing forms	$4\,{}^{1}/_{2}$	$^{5}/_{6}$	0.6944
D	Framing floors, walls, & roof	$10\,{}^{1}/_{2}$	1	1
F	Install drains & rough plumbing	3	$^{1}/_{3}$	0.1111
G	Pour basement floor	1	0	0
J	Install heating ducts	$5\,{}^{1}/_{6}$	$^{1}/_{2}$	0.25
K	Lathe & plaster walls	$9\,{}^{5}/_{6}$	$^{5}/_{6}$	0.6944
L	Finish flooring	2	0	0
M	Install kitchen equipment	2	$^{1}/_{3}$	0.1111
O	Cabinetwork	$6\,{}^{1}/_{6}$	$^{5}/_{6}$	0.6944
R	Paint walls & trim	4	$^{1}/_{3}$	0.1111
S	Sand & varnish floors	2	$^{1}/_{3}$	0.1111
	MEAN:	$53^{1}/_{6}$	VARIANCE:	4.0278

m. Using the PERT technique, what is the probability that the project duration will not exceed the expected duration by more than 5 days?

Solution: According to one of the assumptions of PERT, the length of the critical path has a normal distribution, i.e., the distribution N(51, 2.0069). The requested probability is therefore $P\{T \le 51+5\}$, which can be calculated by first "standardizing" the N(51, 2.0069) to express the required probability in terms of the N(0,1) distribution. (This is done by subtracting the mean and dividing by the standard deviation.) Thus,

$$P\left\{T \le 53\tfrac{1}{6}+5\right\} = P\left\{\frac{T-53\frac{1}{6}}{2.0069} \le \frac{58\frac{1}{6}-53\frac{1}{6}}{2.0069}\right\} = P\left\{\frac{T-53\frac{1}{6}}{2.0069} \le 2.4914\right\}$$

Consulting a table of the N(0,1) probability distribution, we find that the CDF at 2.4914 is 0.9937.

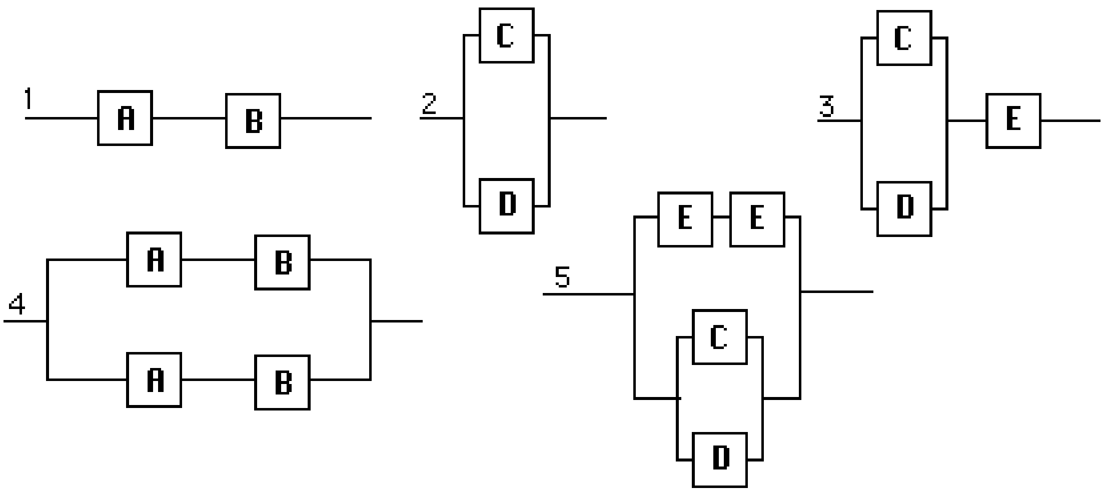


That is, we can be 99.37% certain that the length of the project will not extend more than 5 days past the expected completion date!

2. **System Reliability**: Devices A through E are basic components of five different systems (#1 through #5). They are subject to failure, with the probability that each fails during its first year of use:

Device	Failure Probability
A	10%
B	5%
C	20%
D	25%
E	15%

Compute, for each system below, the reliability, i.e., the probability that it will survive for at least one year.



System #	Reliability	
1	85.5 %	$= R_A R_B = (0.9)(0.95) \equiv R_{AB}$
2	95 %	$= 1 - (1 - R_C)(1 - R_D) = 1 - (0.2)(0.25) \equiv R_{C\|\|D}$
3	80.75 %	$= R_{C\|\|D} R_E = (0.95)(0.85)$
4	97.89 %	$= 1 - (1 - R_{AB})(1 - R_{AB}) = 1 - (1 - 0.855)^2$
5	98.61 %	$= 1 - (1 - R_E R_E)(1 - R_{C\|\|D})$

Note that the reliability is found by subtracting the above failure probabilities from 1.0. Thus, R_A=90%, R_B=95%, R_C= 80%, R_D= 75%, and R_E= 85%.

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57:022 Principles of Design II

Homework #7 Solutions

Wednesday, March 12, 1997

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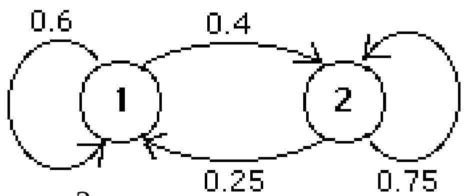
1. Consider the Markov chain having 2 states, with transition probability matrix P=

$$\begin{bmatrix} .60 & .40 \\ .25 & .75 \end{bmatrix}$$

a. Draw a diagram, with the states represented by nodes and the transitions by arrows.

Solution:



b. Compute the matrix P^2.

Solution:

$$P^2 = \begin{bmatrix} 0.6\times0.6 + 0.4\times0.25 & 0.6\times0.4 + 0.4\times0.75 \\ 0.25\times0.6 + 0.75\times0.25 & 0.25\times0.4 + 0.75\times0.75 \end{bmatrix}$$

$$= \begin{bmatrix} 0.46 & 0.54 \\ 0.3375 & 0.6625 \end{bmatrix}$$

c. If the system is initially in state #1, what is the probability that it is in state #2 after 1 time period? **Solution:** $p_{12} = 0.4$ ... after 2 time periods? **Solution:** $p_{12}^{(2)} = 0.54$

d. Write the linear equations which can be solved to obtain the steady state probabilities Π_1 and Π_2.

Solution: $$[\pi_1\ \pi_2] = [\pi_1\ \pi_2]\begin{bmatrix} 0.6 & 0.4 \\ 0.25 & 0.75 \end{bmatrix}$$

or, $$\begin{cases} \pi_1 = 0.6\pi_1 + 0.25\pi_2 \\ \pi_2 = 0.4\pi_1 + 0.75\pi_2 \end{cases}$$

i.e., $$\begin{cases} -0.4\pi_1 + 0.25\pi_2 = 0 \\ 0.4\pi_1 - 0.25\pi_2 = 0 \end{cases}$$

These two equations are linearly dependent; solving either one, together with the "normalizing" equation $\pi_1 + \pi_2 = 1$, will yield the steady state distribution π.

e. Solve the equations in (d).

Solution: $\pi_1 = 0.3846$; $\pi_2 = 0.6154$

f. Suppose that there is a cost of $5 per time period when the system is in state #1 and $8 per time period when the system is in state #2. What is the average cost per time period when the system is in steady state?

Solution: \$ $5\pi_1 + 8\pi_2$ = \$6.846 /period.

2. Consider an (s,S) inventory system in which the number of items on the shelf is checked at the end of each day. The maximum number on the shelf is 8. If 3 or fewer units are on the shelf, the shelf is refilled overnight. The demand distribution is as follows:

x	0	1	2	3	4	5	6
P{D=x}	.1	.15	.25	.25	.15	.05	.05

(Assume that the demand never exceeds 6.)

The system is modeled as a Markov chain, with the state defined as the number of units on the shelf at the end of each day. The probability transition matrix is:

Transition Probabilities

from \ to	1	2	3	4	5	6	7	8	9
1	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
2	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
3	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
4	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
5	0.25	0.25	0.25	0.15	0.1	0	0	0	0
6	0.1	0.15	0.25	0.25	0.15	0.1	0	0	0
7	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0	0
8	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0
9	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1

a. Explain the derivation of the values P_{19}, P_{35}, P_{51}, P_{83} above. (Note that state 1=inventory level 0, etc.)

The steady-state distribution of the above Markov chain is:

Steady State Distribution

i	π_i
1	0.06471513457
2	0.07698357218
3	0.1304613771
4	0.1355295351
5	0.16322964
6	0.1698706746
7	0.1384131423
8	0.0754980776
9	0.04529884656

b. Write two of the equations which define this steady-state distribution.

Solution: Besides the "normalizing" equation,

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 + \pi_9 = 1$$

there are the equations represented in matrix form by $\pi = \pi P$, namely

$\pi_1 = 0.25\pi_5 + 0.1\pi_6 + 0.05\pi_7$

$\pi_2 = 0.25\pi_5 + 0.15\pi_6 + 0.05\pi_7 + 0.05\pi_8$

$\pi_3 = 0.05\pi_1 + 0.05\pi_2 + 0.05\pi_3 + 0.05\pi_4 + 0.25\pi_5 + 0.25\pi_6 + 0.15\pi_7 + 0.05\pi_8 + 0.05\pi_9$

etc.

How many equations must be solved to yield the solution above? Solution: nine

c. What is the expected number on the shelf at the end of each day?
Solution: $0\pi_1 + 1\pi_2 + 2\pi_3 + 3\pi_4 + 4\pi_5 + 5\pi_6 + 6\pi_7 + 7\pi_8 + 8\pi_9 = 3.968$

The mean first passage matrix is:

Mean First Passage Times

from \ to	1	2	3	4	5	6	7	8	9
1	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
2	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
3	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
4	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
5	12.2711	10.4926	6.64702	7.25983	6.12634	6.35574	7.50281	13.7757	23.1867
6	14.3163	11.5197	6.47734	6.42023	5.85772	5.88683	7.68799	13.9609	23.3719
7	14.632	12.4406	7.01794	6.24734	5.2518	5.79028	7.22475	14.3004	23.7114
8	15.2275	12.1857	7.62978	6.77218	5.19576	5.29848	7.10625	13.2454	24.1281
9	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756

d. If the shelf is full Monday morning, what is the expected number of days until the shelf is first emptied ("stockout")? **Solution:** m_{91} = 15.4523 days

e. What is the expected time between stockouts? **Solution:** m_{91} = 15.4523 days

f. How frequently will the shelf be restocked? (or equivalently, what is the average number of days between restocking?)
Solution: The steady-state probability of a restocking is $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 0.40767$, i.e., the frequency will be once every $^1/_{0.40767}$ = 2.453 days

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57:022 Principles of Design II

Homework #8 Solutions

Spring 1997

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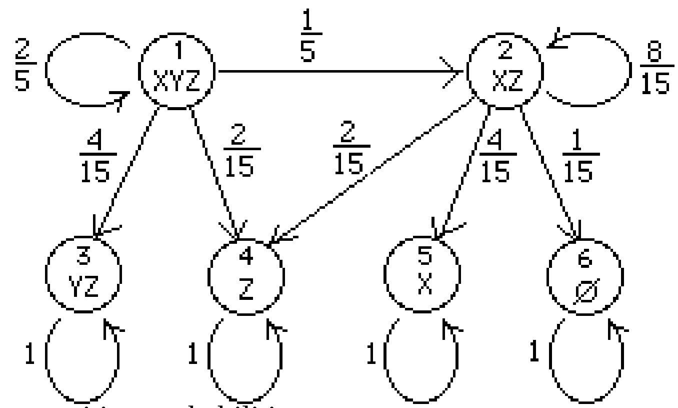
1. Three tanks (X,Y, & Z) are fighting a battle, with Y and Z united against X. In each "round" of this battle, the tanks fire simultaneously, and X always fires at Y unless Y has already been destroyed, in which case he fires at Z. Tank X has probability $^1/_3$ of destroying the tank it fires at, while for Y and Z the probabilities are $^1/_4$ and $^1/_5$, respectively. They continue to fire until either X is destroyed or both Y and Z are destroyed. Using as states the set of currently surviving tanks, set up a Markov chain model.

a. How many states are in this chain?

Solution: There are a total of six states, as shown below. For example, "XYZ" denotes the state in which X, Y, and Z are all "alive", "XZ" denotes that X and Z have survived (but Y has been destroyed), ... " " denotes that none of the three tanks has survived.



Computation of the transition probabilities:

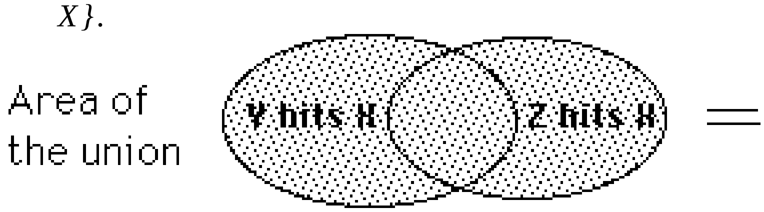
p_{11} = P{X misses Y **and** Y misses X **and** Z misses X}$=\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{2}{5}$

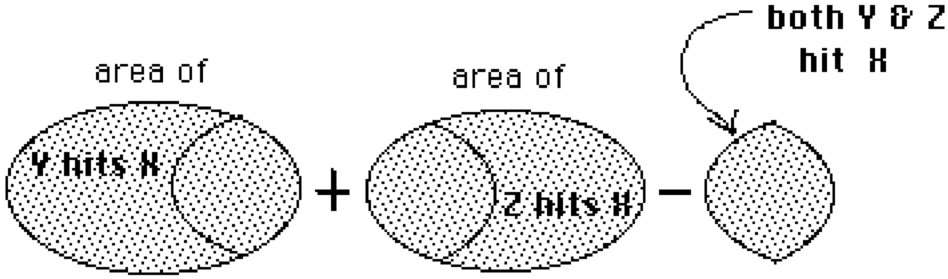
p_{12} = P{X hits Y **and** Y misses X **and** Z misses X} $=\frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{5}$

p_{13} = P{X misses Y **and** (Y hits X **or** Z hits Z)} $=\frac{2}{3}\times\left[\frac{1}{4}+\frac{1}{5}-\left(\frac{1}{4}\times\frac{1}{5}\right)\right]=\frac{4}{15}$

p_{14} = P{X hits Y **and** (Y hits X **or** Z hits Z)} $=\frac{1}{3}\times\left[\frac{1}{4}+\frac{1}{5}-\left(\frac{1}{4}\times\frac{1}{5}\right)\right]=\frac{2}{15}$

*Note that P{Y hits X **or** Z hits X} = P{Y hits X} + P{Z hits X} - P{Y hits X **and** Z hits X}.*





p_{22} = P{X misses Z **and** Z misses X} = $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

p_{24} = P{X misses Z **and** Z hits X} = $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

p_{25} = P{Z hits Z **and** Z misses Z } = $\frac{1}{3} \times \frac{4}{5} = \frac{4}{15}$

p_{26} = P{X hits Z **and** Z hits X} = $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

Transition Probability Matrix

from \ to	1	2	3	4	5	6
1	0.4	0.2	0.2667	0.1333	0	0
2	0	0.5333	0	0.1333	0.2667	0.0667
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1

b. How many are absorbing? **Solution:** 4 states are absorbing: YZ, Z, X, and .

c. Find the expected length of the battle, i.e., the number of rounds fired.

Solution: Each time the system is in a transient state, a round of shots will be fired. From the matrix $E = (I - Q)^{-1}$, we see that if the system begins in state #1 (XYZ), we expect that it will be in state #1 for 1.66667 stages (i.e., the initial visit **plus** 0.66667 additional visits) and in state #2 for 0.714285 stages. Therefore, we expect the system to spend 1.66667 + 0.714285 = **2.380955** stages in transient states before being absorbed.

A = Absorption Probabilities

from \ to	3	4	5	6
1	0.444445	0.317459	0.190476	0.0476193
2	0	0.285713	0.571429	0.142858

E = Expected No. Visits to Transient States

from \ to	1	2
1	1.66667	0.714285
2	0	2.14286

d. Find the probability that X ultimately survives.

Solution: X will survive only if the system is absorbed into state "X" (#5). According to the matrix A above, the probability that the system is absorbed into state 5, starting in state 1, is 0.190476, i.e., X has approximately 19% probability of surviving this battle.

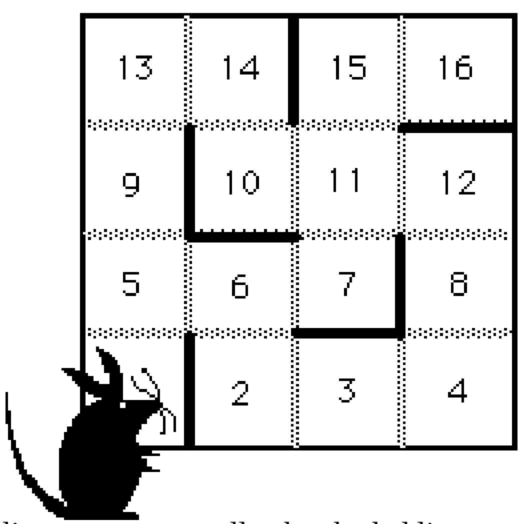
e. If, during the first round, Y is destroyed, what is your revised estimate of the length of the battle and the probability that X will ultimately survive?

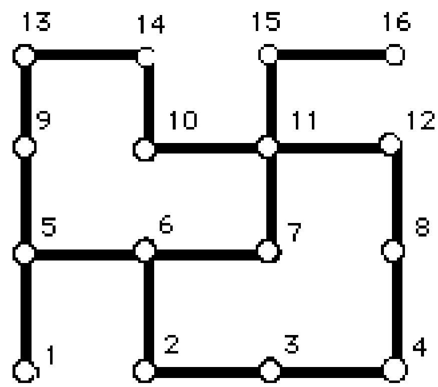
Solution: Again, according to A, starting in state "XZ" (#2), the system will eventually be absorbed into state 5 with probability 0.571429, i.e., if X is able to destroy Y without being destroyed, its chances of ultimately surviving this battle has tripled, i.e., its chances of ultimately winning the battle will be over 57%.
A simulation was performed of ten battles with the results shown below. Note that in these simulated battles, X was the victor 3 times!

Simulation results:

Run # \ Stage	0	1	2	3	4	5	6	7	8	9	10
1	1	4	4	4	4	4	4	4	4	4	4
2	1	3	3	3	3	3	3	3	3	3	3
3	1	1	2	2	5	5	5	5	5	5	5
4	1	4	4	4	4	4	4	4	4	4	4
5	1	2	5	5	5	5	5	5	5	5	5
6	1	1	3	3	3	3	3	3	3	3	3
7	1	3	3	3	3	3	3	3	3	3	3
8	1	1	2	2	2	5	5	5	5	5	5
9	1	3	3	3	3	3	3	3	3	3	3
10	1	1	3	3	3	3	3	3	3	3	3

2. We wish to model the passage of a rat through a maze. Sketch a maze in the form of a 4x4 array of boxes, such as the one below on the left:

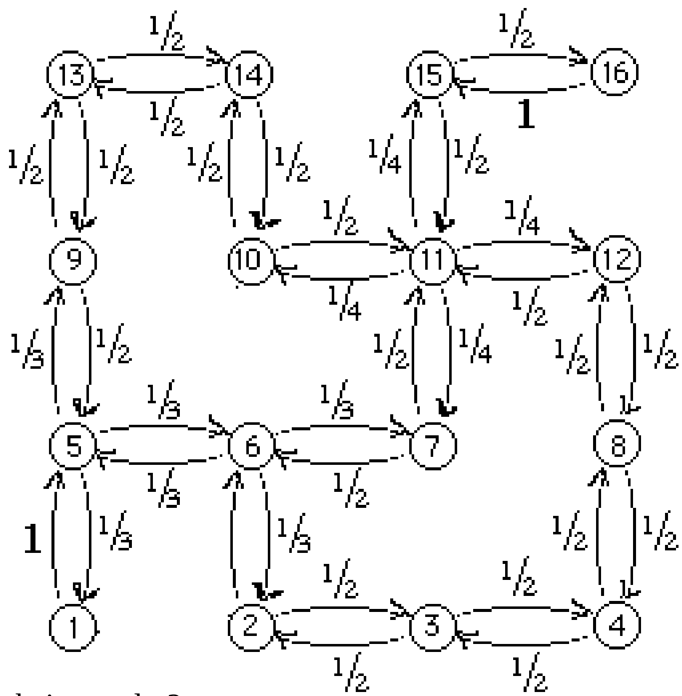




The solid lines represent walls, the shaded lines represent doors. We will assume that a rat is placed into box #1. While in any box, the rat is assumed to be equally likely to choose each of the doors leaving the box (including the one by which he entered the box). For example, when in box #2 above, the probability of going next to boxes 3 and 6 are each $^1/_2$, regardless of the door by which he entered the box. *This assumption implies that no learning takes place if the rat tries the maze several times.*
Once you have sketched a 4x4 maze (**not** identical to the one above!), load the **APL** workspace named **RAT** and enter the maze which you have designed. (Be sure that your maze is connected, i.e. each box is reachable from any other box.) The functions in this workspace construct a Markov chain model of the movement of the rat through the maze.
Note: in the solutions given below, I have used the maze which appears above!

a. On the diagram representing your Markov chain, write the transition probabilities on each transition in each direction.

Solution:



b. Is your Markov chain regular?

Solution: A Markov chain is <u>regular</u> if each pair of states "communicate" (i.e. each state in the pair can be reached from the other) <u>and</u> if the states are not periodic. Although every pair of states above communicate, this particular Markov chain isn't regular, because, for example, if the system begins in state #**i** then it will <u>never</u> be in state #**i** after an odd number of stages, so that its period is 2. This is apparent because all the "loops" are of even length (such as the loop from box 5 through 9, 13, 14, 10, 11, 7, 6, and back to 5). The same will be true however you design the maze, if the rat cannot move diagonally between two boxes.

c. Compute the steady-state distribution of the rat's location. Which box will be visited most frequently by the rat?

Solution: The steady-state probabilities, as computed by the RAT workspace, are:

Steady State Distribution

i	P{i}
1	0.02941
2	0.05882
3	0.05882
4	0.05882
5	0.08824
6	0.08824
7	0.05882
8	0.05882
9	0.05882
10	0.05882
11	0.1176
12	0.05882
13	0.05882
14	0.05882
15	0.05882
16	0.02941

From this we see that the probability of each box is proportional to the number of doors which enter that room, so that the most frequently visited state should be #11, where π_{11} = 11.76%.

d. Select some box far from box #1 in which a reward (e.g. food) might be placed for the rat. What is the expected number of moves of the rat required to reach this reward?
Solution: Let's choose #16 as the box with the reward. Then the "mean first passage time" $m_{1,16}$ is the expected number of moves required to first reach #16 from #1. The Mean First Passage Time matrix computed by the RAT workspace is

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
34	31	43.6	49.1	1	11.3	23.1	47.6	19.6	33	23.3	39	31.1	35.6	54.3	87.3
59.6	17	18.2	29.4	26.6	8.6	20.4	33.6	39.6	36	20.6	30.6	45.4	44.2	51.6	84.6
65.2	11.2	17	15.7	32.2	15.3	23.7	24.4	44	37	20.5	26	48.7	46.4	51.5	84.5
68.7	20.4	13.7	17	35.7	20	25	13.2	46.4	36	18.4	19.3	50	46.5	49.4	82.4
33	30	42.6	48.1	11.3	10.3	22.1	46.6	18.6	32	22.3	38	30.1	34.6	53.3	86.3
52.1	20.8	34.5	41.2	19.1	11.3	15.2	40.8	33.2	33	18.8	33.3	40.2	40.1	49.8	82.8
60.7	29.4	39.7	43	27.7	12	17	39.2	38.4	28	10.4	28.3	42	38.5	41.4	74.4
70.2	27.6	25.4	16.2	37.2	22.6	24.2	17	46.8	33	14.2	10.6	49.2	44.6	45.2	78.2
43.8	35.2	46.6	51.0	10.8	16.6	25.0	48.4	17	27	21.8	38.6	16.0	25.0	52.8	85.8
64.4	38.8	46.8	47.8	31.4	23.6	21.8	41.8	34.2	17	8.4	28.6	29.8	18.4	39.4	72.4
67.3	36	42.9	42.8	34.3	22	16.8	35.6	41.6	21	8.5	21.3	41.8	34.9	31	64
69.8	32.8	35.2	30.5	36.8	23.3	21.5	18.8	45.2	28	8.1	17	46.5	40.8	39.1	72.1
52.7	38.4	48.7	52	19.7	21	26	48.2	13.4	20	19.4	37.3	17	13.5	50.4	83.4
59.6	39.6	48.8	50.9	26.6	23.3	24.9	46	24.8	11	14.9	34	15.9	17	45.9	78.9
70.3	39	45.9	45.8	37.3	25	19.8	38.6	44.6	24	3	24.3	44.8	37.9	17	33
71.3	40	46.9	46.8	38.3	26	20.8	39.6	45.6	25	4	25.3	45.8	38.9	1	34

From this we see that $m_{1,16}$ = 87.3, that is, the rat requires an average of 87.3 moves to first find the reward.

e. Count the minimum number of moves (M) required to reach the reward. What is the probability that the rat reaches the reward in **exactly** this number of moves?
Solution: By examining the diagram above, we see that it is possible for the rat to reach #16, starting in #1, in only six moves. By displaying the first passage probability $f^{(6)}_{1,16}$, we see that the probability that state #16 is reached in exactly six moves is only 0.006944.

n	$f^{(n)}_{1,16}$
1	0
	⋮
5	0
6	0.006944
7	0
8	0.01264
9	0
10	0.01649
11	0
12	0.0189
13	0
14	0.0203
15	0
16	0.02102

f. What is the probability that the rat reaches the reward with no more than 4 unnecessary moves?
Solution: The probability that the rat reaches #16 in no more than 6+4=10 moves is 0.006944 + 0.01264 + 0.01649 = 0.036074.

g. Simulate twice the first 2M moves made by the rat. Did he reach the reward in either simulation?
Solution: Below are the results of **ten** simulations of the first 12 stages of this Markov chain. Surprisingly, in the tenth simulation the rat reached #16 in only 8 moves, even though this result is very unlikely to occur!

Simulation results:

Run # \ Stage	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	5	9	13	9	5	1	5	9	13	9	13	9
2	1	5	9	13	9	13	14	13	9	13	14	10	11
3	1	5	9	5	6	2	6	2	6	5	6	7	6
4	1	5	9	5	6	2	3	4	3	2	3	2	3
5	1	5	9	5	6	7	11	10	11	12	8	12	11
6	1	5	9	5	6	5	9	5	6	2	6	5	6
7	1	5	6	7	11	10	11	7	11	7	11	10	14
8	1	5	1	5	1	5	1	5	1	5	9	5	6
9	1	5	1	5	9	13	9	13	14	13	14	13	9
10	1	5	9	13	14	10	11	15	16	15	16	15	16

h. Briefly discuss the utility of this model in testing a hypothesis that a *real* rat is able to learn, thereby finding the reward in relatively few moves after he has made several trial runs through the maze.
Solution: The "null" hypothesis could be that the probability distribution of $N_{1,16}$ (the number of moves required to reach 16, starting in 1) is given by the first-visit probability distribution $\{f^{(n)}_{1,16}$, n=6,7,8,9,10,11, ... $\}$ given by the Markov chain ("memoryless") model. One might then run the rat through the maze many times, recording the number of moves required to reach box #16 from #1. By performing a chi-square "goodness of fit" test with a pre-selected a, one might then be able to reject this null hypothesis and to therefore conclude that the rat is not memoryless, i.e., that he has accomplished learning. (This would perhaps require that you first verify that the Markov chain model above is valid for a single run, by testing a large number of rats, each a single time.)

i. Briefly discuss how you might modify your Markov chain model if the rat will never exit a box by the same door through which he entered, unless he has reached a "dead end".
Solution: With these modified transition probabilities, the original stochastic process is no longer "memoryless", i.e., the probability of passing through a door depends no only on the current location of the rat, but also the door by which he entered his current location! To restore the memorylessness property, we must define the state to include information not only about the rat's location, but also about the door by which he entered that location. For example, state #5 above would be replaced by three states, such as 5_S, 5_E, and 5_N indicating that the rat entered box 5 from the south, the east, and the north, respectively. The number of states perhaps increases by a factor between 2 and 3. The transition probabilities for this new Markov chain are easily defined, e.g.,
$P\left\{5_S \rightarrow 6_W\right\} = \frac{1}{2}$, $P\left\{5_S \rightarrow 9_S\right\} = \frac{1}{2}$, but $P\left\{5_S \rightarrow 1_N\right\} = 0$.

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57:022 Principles of Design II

Homework #10 Solutions

Spring 1997

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1. Barges arrive at the La Crosse lock on the Mississippi River at an average rate of one every two hours. It requires an average of 30 minutes to move a barge through the lock.
Assuming that the arrival process is Poisson and that the time to move the barge through the lock has exponential distribution, find:

a. The average number of barges in the system, i.e., either using or waiting to use the lock.

$$L = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\rho}{1-\rho} = \frac{1/4}{1-1/4} = 1/3$$

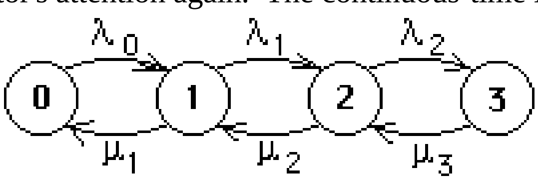
b. The average time spent by a barge at the lock (including waiting time).

$$W = \frac{1}{\mu-\lambda} = \frac{1}{\frac{2}{\text{hr.}} - \frac{1/2}{\text{hr.}}} = 3/2 \text{ hr.}$$ minutes.

c. The fraction of the time that the lock is busy. $1-\pi_0 = \lambda/\mu = 1/4 = 25\%$

Note: This system can be modelled as an M/M/1 queue, with λ = 0.5/hour & μ = 2/hr. Note also that the above formulae for L and W are valid for M/M/1 queues only!

2. A machine operator has sole responsibility for keeping three semi-automatic machines busy. The time required to ready a machine (unloading & reloading jobs) has exponential distribution with mean 15 minutes. The machine will then process its job without human attendance for an average of 1 hour (but with actual time having exponential distribution) before it requires the operator's attention again. The continuous-time Markov chain model for this system is:



a. Is this a birth-death process? Yes

b. Indicate the values of λ_i and μ_i on the diagram above.

$\lambda_2 = 1$/hr.
$\lambda_1 = 2 \times \lambda_2 = 2$/hr.
$\lambda_0 = 3 \times \lambda_2 = 3$/hr.
& $\mu_1 = \mu_2 = \mu_3 = 4$/hr.

c. Compute the steady-state probability π_0 .

$$\frac{1}{\pi_0} = 1 + \frac{3}{4} + \frac{3}{4} \times \frac{2}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4}$$
$$= 2.2187$$
$$\pi_0 = 0.4507$$

d. The operator will be busy what fraction of the time? $1 - \pi_0 = 54.93\ \%$

e. What fraction of the time will the operator be busy but with no machine waiting to be serviced? $\pi_1 = \frac{3}{4}\pi_0 = 33.8\%$

f. What is the value of L (average number of machines in the system)?

$$L = \sum_{i=0}^{3} i\,\pi_i = 0\times0.4507 + 1\times0.33803 + 2\times0.169 + 1\times0.04225$$
$$= 0.8028$$

g. What is the value of $\underline{\lambda} = \sum_{i=0}^{3} \lambda_i \pi_i$ (the average arrival rate)?

$$\underline{\lambda} = \sum_{i=0}^{3} \lambda_i\,\pi_i = 3/\text{hr.}\times0.4507 + 2/\text{hr.}\times0.33803 + 1/\text{hr.}\times0.169 + 0/\text{hr.}\times0.04225$$
$$= 2.1972/\text{hr.}$$

h. According to Little's Law, what is the average time from a machine's completion of one job until it begins processing its next job?

$$W = \frac{L}{\underline{\lambda}} = \frac{0.8028}{2.1972/\text{hr.}} = 0.36538\ \text{hr.} = 21.923\ \text{min.}$$

i. What will be the utilization of this group of 3 machines?

$$U = \frac{\sum_{i=0}^{3} (3-i)\,\pi_i}{3} = \frac{3\times0.4507 + 2\times0.33803 + 1\times0.169 + 0\times0.04225}{3}$$
$$= \frac{2.1972}{3} = 73.24\%$$

j. What is the expected number of jobs per 40-hour week which this machine center will process?

$$73.24\% \times 3\ \text{machines} \times 1\ \frac{\text{job}}{\text{hr}} \times \frac{40\ \text{hrs}}{\text{week}} = 87.88\ \frac{\text{job}}{\text{week}}$$

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Use ARENA to simulate this same system. Specify a simulation run of length 50 hours, with a warm-up period of 10 hours. (That is, the statistics will be collected between the 10th hour, when the system should approximate steady-state, and the 50th hour.) According to the ARENA output,

k. The operator will be busy what fraction of the time? 52.2 %
(c.f. (d) above, which predicts 54.93%.)

l.. The operator will be idle what fraction of the time? 47.8 %
(c.f. (c) above, which predicts 45.07%.)

m.. What is the value of L (average number of machines in the system)? 0.81 machines
(average # in queue + utilization of server [i.e.,average number being served]).
(c.f. (f) above, which predicts 0.8029 machines.)

n. What is the average time from a machine's completion of one job until it begins processing its next job? 23.88 minutes (average time in queue + average service time).
(c.f. (h) above, which predicts 21.923 minutes.)

o. What will be the utilization of this group of 3 machines? 73 % (c.f. (i) above, which predicts 73.24%.)

p. What is the number of jobs processed by this machine center during the 40-hour period for which you kept statistics? 77 jobs/week
(c.f. (j) above, which predicts 87.888 jobs/week.)

Suppose that the processing time for the jobs no longer has exponential distribution, but instead has normal distribution with the same mean value (60 minutes) but standard deviation 10 minutes (compared to 60 minutes for the exponential distribution). Modify your ARENA model to reflect this change and re-run the simulation.

q. What is the number of jobs processed by this machine center during the 40-hour period for which you kept statistics? 88 jobs

r. Did reducing the variation in processing times increase or decrease the throughput of this machine center? increased