«»«»«»«»«» 57:022 Principles of Design II «»«»«»«»«»

Homework Solutions, Spring 1993

Prof. Dennis L Bricker, Dept. of Industrial Engineering
University of Iowa

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #2 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

(1.) The probability that a given driver stops to pick up a hitchhiker is p=0.05; different drivers, of course, make their decisions to stop or not independently of each other.

(a) Given that a hitchhiker has counted 30 cars passing him without stopping, what is the probability that he will be picked up by the 37th car or before?

The arrival process is "memoryless", so the fact that 30 cars have passed without stopping is not relevant in computing whether one of the next seven cars will stop. Consider the arrival of the cars to be a Bernouilli process, with "success" indicating that the car stops. Then the number N_1 of the first car to stop will have geometric distribution, with p=0.05, i.e.,

$$P\{N_1 = 30+n\} = (0.95)^{n-1}(0.05)$$

We need $P\{N_1 \le 37\}$, which can be computed in several ways:

$$P\{N_1 \le 37\} = \sum_{n=1}^{7} (0.95)^{n-1}(0.05)$$

Each term can be computed and summed. As an alternative, we could use the fact that the sum is a finite geometric series (minus its first term, 1) and use the appropriate formula. The easiest method, however, is to compute $P\{N_1 \le 37\} = 1 - P\{N_1 > 37\} = 1 - (0.95)^7 = 1 - 0.6983373 = 0.3016627$. (That is, the event $\{N_1 > 37\}$ occurs *if & only if* the next 7 cars do not stop.)

Suppose that the cars arrive according to a Poisson process, at the average rate of 5 per minute. Then "success" for the hitchhiker occurs at time t provided that both an arrival occurs at t ***and*** *that car stops to pick him up. Let T be the time (in seconds) that he finally gets a ride, when he begins his wait at time zero.*

(b) Find the distribution of T; compute E(T) and Var(T).

If the arrival of all cars is a Poisson process with l = 5/minute, then the arrival of cars which stop is a Poisson process with arrival rate pl = (0.05)(5/minute) = 0.25/minute. Therefore, T must have exponential distribution with mean E(T) = 1/pl = 60 sec./0.25 = 240 sec. = 4 minutes, and variance V(T) = $[E(T)]^2 = 1/(pl)^2 = 57600 \text{ sec}^2 = 16 \text{ min}^2$.

(c) Given that after 4 minutes (during which 20 cars pass by) he is still there waiting for a ride, compute the expected value of T.

We use the fact that the exponential distribution is "memoryless", so that the expected value of T, *given that T>4 minutes*, is 4 minutes + E(T) = 8 minutes (= 480 sec.)

(2.) (a.) Using the last four digits of your ID# as the "seed" for the "Midsquare" technique, generate a sequence of 10 pseudo-random numbers uniformly distributed in the interval [0,1].

Each student should compute different numbers, assuming that the last 4 digits of their IDs are different. For example, suppose that Joe's ID# ends with the digits "1234". Then he will compute the sequence R_i, i=1, 2, ...10:

i	R_i	$(R_i)^2$
0	1234	01522756
1	5227	27321529
2	3215	10336225
3	3362	11303044
4	3030	09180900
5	1809	03272481

6	2724	07420176
7	4201	17648401
8	6484	42042256
9	0422	00178084
10	1780	

(Each number R_i above should then be multiplied by 0.0001 so as to get a sequence of numbers in the interval [0,1].)

(b.) Using the "Inverse Transformation" technique and the 10 numbers generated in (a.), generate the interarrival times for 8 vehicles which form a Poisson process with arrival rate l=5/minute.

As explained on page D2 of your notes (Hypercard stack "Random Numbers"), we generate interarrival times t_i having exponential distribution by the transformation

$$\tau_i = -\frac{\ln(1 - R_i)}{\lambda} = -\frac{\ln \overline{R}_i}{\lambda}$$

If we use the numbers R_i generated above, we obtain t_1 = - (0.2 min.) ln (1-0.5227) = - (0.2 min.)ln(0.4773) = - (0.2 min.)(-0.73961) = 0.147922 minute, etc. To save a bit of computational effort, we could instead use the sequence of numbers above as $1-R_i$ instead of R_i , since if R_i is uniformly distributed in [0,1], then $(1-R_i)$ is also. In this case, we would compute t_1 = - 0.2 ln (0.5227) = (-0.2)(-0.6487475) = 0.1297495 minute, etc. The arrival times are found by computing partial sums of the interarrival times:

$$T_i = \sum_{k=1}^{i} \tau_k$$

The results of performing these transformations on the 10 uniformly generated random numbers are:

i	R_i	$1-R_i$	$\tau_i = \frac{-\ln R_i}{\lambda}$		$\tau_i = \frac{-\ln(1-R_i)}{\lambda}$	
			τ_i	T_i	τ_i	T_i
1	0.5227	0.4773	0.12975	0.12975	0.147922	0.147922
2	0.3215	0.6785	0.226952	0.356701	0.0775742	0.225496
3	0.3362	0.6638	0.21801	0.574711	0.0819549	0.307451
4	0.303	0.697	0.238804	0.813515	0.072194	0.379645
5	0.1809	0.8191	0.341962	1.15548	0.0399098	0.419555
6	0.2724	0.7276	0.260097	1.41557	0.0636008	0.483156
7	0.4201	0.5799	0.173453	1.58903	0.10898	0.592136
8	0.6484	0.3516	0.0866495	1.67568	0.209052	0.801188
9	0.0422	0.9578	0.633067	2.30874	0.00862326	0.809811
10	0.178	0.822	0.345194	2.65394	0.039203	0.849014

(c.) What is the expected number of arrivals during the first minute? What is the actual # of arrivals in your simulation? What is the probability that you would observe exactly this number of arrivals in this Poisson process?

The expected number of arrivals during the first minute is, of course, lt = (5/minute)(1 minute) = 5. The first sequence of numbers (generated using R_1=0.5227) 4 arrivals in the first minute. (The fifth arrival occurs later than 1 minute, i.e., at 1.15548 minutes.) In this case, the probability of observing exactly this value (4) is:

$$P\{N_t = x\} = \frac{(5)^4}{4!} e^{-5} = \frac{625}{24}\, 0.0067379 = 0.1754673$$

The second sequence (generated using $1-R_1$ = 0.5227) shows 10 arrivals in less than one minute (0.849014 minutes). The probability that there are exactly 10 arrivals in 0.849014 minute is rather unlikely:

$$P\{N_t = x\} = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

where x=10, t=0.849014, and l = 5/minute. We compute

$$P\{N_t = x\} = \frac{(4.24507)^{10}}{10!} e^{-4.24507} = \frac{1900415.5}{3628800} 0.0143347 = 0.007507$$

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #11 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. *Barges arrive at the La Crosse lock on the Mississippi River at an average rate of one every two hours. It requires an average of 30 minutes to move a barge through the lock.*
Assuming that the time to move the barge through the lock has exponential distribution, find:

- *a. The average number of barges in the system, i.e., either using or waiting to use the lock.*
- *b. The average time spent by a barge at the lock.*
- *c. The fraction of the time that the lock is busy.*
- *d. The standard deviation of the time to move a barge through the lock.*

If the time to move a barge through the lock has, not exponential, but normal distribution N(30,15), what are the revised values in (a), (b), and (c)?

The lock system forms an M/M/1 queueing system (with infinite capacity), with arrival rate l = 0.5/hr. and service rate m = 2/hr. The utilization r = l/m = 0.25.

a. L = average number of "customers" in the system = r/(1-r) = 0.25/0.75 = 0.33333

b. W = average time in the system per customer (including going through the lock) = L/l = 0.33333/(0.5/hr) = 2/3 hr. = 40 minutes. Of these 40 minutes, 30 minutes is, on average, spent going through the lock, leaving an average of 10 minutes (0.083333 hr.) waiting in the queue.

c. The fraction of the time which the lock is busy is $1\text{-}p_0$ = r = 25%.

d. Under the assumption that the time going through the lock has exponential distribution, the standard deviation of this time equals the mean value, i.e., 30 minutes or 0.5 hr., and s^2=0.25 hr^2.

Suppose that the time required to go through the lock is N(30,15^2), i.e., mean value is, as before, 30 minutes, and the standard deviation is 15 minutes, i.e., 0.25 hr. and s^2= 1/16 hr^2. We cannot use the formulas for M/M/1 system now, but rather M/G/1 (i.e., service time distribution is "General".)

We have

$$\pi_0 = 1 - \rho$$

where

$$\rho = \lambda / \mu$$

and

$$L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

Since r = l/m is unchanged, the answer to (c) is the same as before, i.e., the lock will be busy 25% of the time. The average length of the queue (not including a barge in the process of moving through the lock) is

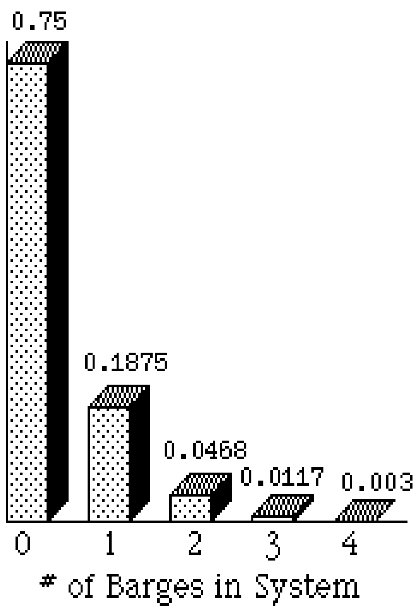
$$L_q = \frac{(0.5)^2 (0.25)^2 + (0.25)^2}{2(1-0.25)} = 0.0520833$$

From this we can use Little's formula L_q=lW_q to find W_q: W_q = 0.0520833/(0.5/hr.) = 0.104166 hr. This is the average time in the queue, so we add the average time in the lock, 0.5 hr., to get W=0.604166 hr., the revised answer to (b). Then we use Little's formula again to get L=lW = 0.3020833. This would be the revised answer to (a), which you will note is less than when the time had exponential distribution. This illustrates the fact that, in general, if the variance of the service time is reduced, both the average queue length and the waiting time in the queue will be reduced. The output of the M/M/C APL workspace, for the original M/M/1 queue, is as follows:

i	Pi	CDF
0	0.750000	0.750000
1	0.187500	0.937500
2	0.046875	0.984375
3	0.011719	0.996094
4	0.002930	0.999023
5	0.000732	0.999756
6	0.000183	0.999939
7	0.000046	0.999985
8	0.000011	0.999996
9	0.000003	0.999999
10	0.000001	1.000000

Steady-State Distribution

Mean Queue Length (L) = 0.083333
Mean number of servers busy = 0.25
Probability that at least one server is idle = 0.75



2. *In a particular manufacturing cell, one repairman has to maintain four machines. For the machines, the time between breakdowns is exponentially distributed with an average of 4 hours. On the average, it takes half an hour to fix a machine (exponentially distributed).*

a. What is the steady-state probability distribution of the number of machines which are broken down?
b. What fraction of the time will the repairman be busy?
c. What is the average number of machines in need of repair (including those in the process of being repaired)?
d. What is the average time between a machine breakdown and that machine being restored to operating condition?

a. We use the formulas for the M/M/1/4/4 queueing system. That is, there is a finite source population (the 4 machines). Machines in need of repair then queue in front of the single server, i.e., the repairman. The steady state probability that the system is empty, i.e., no machines need repair, is

$$\pi_0 = \frac{1}{\sum_{j=0}^{N} \frac{N!}{(N-j)!}\rho^j} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

In our case here, N=4, l = arrival rate <u>per machine</u> = 1/(4hr.) = 0.25/hr., and m = repair rate = 1/(0.5hr.) = 2/hr. We therefore obtain r = (0.25/hr.)/(2/hr.) = 1/8 and

$$\pi_0 = \frac{1}{\frac{4!}{4!}\left(\frac{1}{8}\right)^0 + \frac{4!}{3!}\left(\frac{1}{8}\right)^1 + \frac{4!}{2!}\left(\frac{1}{8}\right)^2 + \frac{4!}{1!}\left(\frac{1}{8}\right)^3 + \frac{4!}{0!}\left(\frac{1}{8}\right)^4}$$

$$= \frac{1}{1 + 0.5 + 0.1875 + 0.046875 + 0.005859} = \frac{1}{1.7402344} = 0.5746$$

We then use p_0 and the terms in the denominator above to get the remaining probabilities:
$p_1 = 0.5p_0$, $p_2 = 0.1875p_0$, $p_3 = 0.046875p_0$, and $p_4 = 0.005859p_0$, as confirmed by the computer output:

```
Steady-State Distribution

 i     Pi        CDF
 -  --------  --------
 0  0.574635  0.574635
 1  0.287318  0.861953
 2  0.107744  0.969697
 3  0.026936  0.996633
 4  0.003367  1.000000

The mean number of customers in the system
  (including the one being served) is:
                                0.5970819304

The average arrival rate of customers
                           is 0.8507295174

Using Little's formula, the average time
   spent in the system, per customer, is
                          W = 0.7018469657
```

b. p_0 is the fraction of the time that the repairman is idle, so $1-p_0 = 42.5\%$ is the fraction of the time he is busy, i.e., the server utilization.

c. The probabilities p_i give us the probability distribution of the number of machines in need of repair, so the average number of machines needing repair is the mean value of this distribution:

$$L = \sum_{j=0}^{4} j\,\pi_j = 0\pi_0 + 1\pi_1 + \cdots + 4\pi_4$$

which is approximately 0.5971, as given in the output above.

d. The quantity requested is denoted by W, the average time in the system each time a machine fails. This can be obtained from Little's formula $L=\underline{l}W$ if we first calculate $\underline{l}$, the average arrival rate:
$\underline{l} = 4(0.25/hr)p_0 + 3(0.25/hr)p_1 + 2(0.25/hr)p_2 + 1(0.25/hr)p_3 + 0(0.25/hr.)p_4$
= approximately 0.85/hour, as given in the computer output above. (That is, when there are 0 machines "in the system", i.e., all 4 machines are "up & running", failures will occur at the rate of 4 times the failure rate of each machine, but if 1 machine is in the system, only 3 machines are running and the failure rate is only 3 times the failure rate of each machine, etc.)
Now Little's formula gives us $W=L/\underline{l} = 0.5971/(0.85/hr)= 0.7018$ hour

3. *Customers arrive at a service center with two servers at the rate 10/hour. Average time for serving a customer is 10 minutes (exponentially distributed). Compare the average customer waiting times and the server utilization of two alternative systems:*

a. Each server has a queue, and customers are equally likely to enter either queue (so that in effect, the arrival rate for each queue is 5/hour).

b. All customers enter the same queue, and a server selects the next customer to be served from the head of this queue.

a. Analysis of the M/M/1 queueing system for **each** of the two servers:
Using the formula given in the notes,

$$L = \frac{\rho}{1-\rho}, \text{ where } \rho = \frac{\lambda}{\mu}$$

we obtain L = average number of customer in the system (including the one being served) = (5/6)/(1/6) = 5. We can then use Little's formula, L=lW, to get W=L/l= 5/(5/hr.) = 1 hour. Ten minutes of this time is the service time, leaving W_q = 50 minutes = average waiting time.

Results obtained from the computer output:

i	ρ	Pi	CDF
0	0.833333	0.166667	0.166667
1	0.833333	0.138889	0.305556
2	0.833333	0.115741	0.421296
3	0.833333	0.096451	0.517747
4	0.833333	0.080376	0.598122
5	0.833333	0.066980	0.665102
6	0.833333	0.055816	0.720918
7	0.833333	0.046514	0.767432
8	0.833333	0.038761	0.806193
9	0.833333	0.032301	0.838494
10	0.833333	0.026918	0.865412

M/M/1 system, with

λ = 5/hour
μ = 6/hour

Mean Queue Length (L) = 4.1667
Mean number of servers busy = 0.83333
Probability that at least one server is idle = 0.16667

(To obtain average waiting time given the average number in queue obtained from the computer output above, we use Little's formula L_q=lW_q, to get W_q = L_q/l = 4.1667/(5/hr.) = 0.83333 hr. = 50 minutes.)

b. Analysis of the single M/M/2 queueing system:

First compute p_0, the probability that system is empty:

$$\pi_0 = \frac{1}{\sum_{n=0}^{1}\frac{(c\rho)^n}{n!} + \frac{(c\rho)^2}{2!}\frac{1}{1-\rho}} = \frac{1}{\frac{\left(\frac{10}{6}\right)^0}{0!} + \frac{\left(\frac{10}{6}\right)^1}{1!} + \frac{\left(\frac{10}{6}\right)^2}{2!} \times \frac{1}{1-\frac{10}{12}}}$$

$$= \frac{1}{1+1.66667+8.3333} = \frac{1}{11} = 0.0909$$

The formula for the average number of customers in the queue is

$$L_q = \frac{\rho\,(c\rho)^c}{c!}\,\pi_0\left(\frac{1}{1-\rho}\right)^2$$

$$= \frac{\frac{10}{12}\left(\frac{10}{6}\right)^2}{2!} \times \frac{1}{11} \times \left(\frac{1}{1-\frac{10}{12}}\right)^2$$

$$= 0.1157407 \times 0.090909 \times 36 = 0.37878$$

This agrees with the output of the computer program:

i	ρ	Pi	CDF
0	1.666667	0.090909	0.090909
1	1.666667	0.151515	0.242424
2	1.666667	0.126263	0.368687
3	1.666667	0.105219	0.473906
4	1.666667	0.087682	0.561588
5	1.666667	0.073069	0.634657
6	1.666667	0.060891	0.695547
7	1.666667	0.050742	0.746289
8	1.666667	0.042285	0.788575
9	1.666667	0.035238	0.823812
10	1.666667	0.029365	0.853177

M/M/2 system, with

λ = 10/hr.
μ = 6/hr.

Mean Queue Length (L) = 3.7879
Mean number of servers busy = 1.6667
Probability that at least one server is idle = 0.24242

To obtain average waiting time, given the average queue length obtained from the computer output, we use Little's formula L=lW, to get W = L/l = 3.7879/(10/hr.) = 0.37879 hr. = 22.73 minutes.

Use of system (b), the M/M/2 queue, would result in approximately 55% reduction in waiting time for the customers!

«»