«»«»«»«»«» 57:022 Principles of Design II «»«»«»«»«»

Homework Assignments, Spring 1995

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«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #1 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. The foreman of a casting section in a certain factory finds that on the average, 1 in every 9 castings made is defective.

   a. If the section makes 15 castings a day, what is the probability that 2 of these will be defective?
   b. What is the probability that 3 or more defective castings are made in one day?

2. A city's population is 55% in favor of a school bond issue. Suppose that the local newspaper conducts a "random poll" of the citizenry.
   a. What is the probability that, if ten citizens are polled, the majority of those polled will *oppose* the issue?
   b. What is this probability if twenty citizens are polled?

3. The probability that each car stops to pick up a hitchhiker is p=3%; different drivers, of course, make their decisions to stop or not independently of each other.

   a. Given that a hitchhiker has counted 20 cars passing him without stopping, what is the probability that he will be picked up by the 25^{th} car or before?

   Suppose that the cars arrive according to a Poisson process, at the average rate of 10 per minute. Then "success" for the hitchhiker occurs at time t provided that both an arrival occurs at t **and** that car stops to pick him up. Let T be the time (in seconds) that he finally gets a ride, when he begins his wait at time zero.

   b. What is the distribution of T? What are E(T) and Var(T)?
   c. Given that after 4 minutes (during which 42 cars have passed by) he is still there waiting for a ride, compute the expected value of T (his total waiting time, including the 4 minutes he has already waited).

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #2 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. **Generating Arrival Times in Poisson Process.** Suppose, in preparation for performing a manual simulation of the arrivals in a Poisson process (e.g., parts randomly arriving at a machine to be processed), you wish to generate some inter-arrival times, where the arrival rate is 4/hour. First, you need some uniformly-distributed random numbers. To obtain these, select a row from the table which appeared in the Hypercard stack:

3821	4876	3071	5268	8684	0169	1746	6658	8605	9638
0218	3519	0707	3695	6478	3977	2017	3644	7993	5547
5105	8147	7365	2901	7228	2307	7241	4225	6078	9344
4549	1468	4395	3808	9446	5954	6851	2930	9217	5668
6758	7233	0503	0981	5955	4881	5916	3197	8532	9810
8431	5742	0744	3115	4411	5132	2175	8044	5668	3463
5072	1129	0723	1390	0722	6669	8144	0434	3014	9675
1797	8050	3603	9301	2162	8267	6733	5878	9918	3984
5280	5063	6663	6449	6400	0863	2414	4309	0851	3393
7223	4603	1542	9279	7217	2279	4575	5332	0000	6645

Select a row based upon the last digit of your ID#: if 1, use row #1; if 2, use row #2; ... if 0, use row #10.

a. What is the probability distribution of the time T_1 of the first arrival?

b. What will be the probability distribution of the time τ_i between arrivals of parts i-1 and i (i>1)?

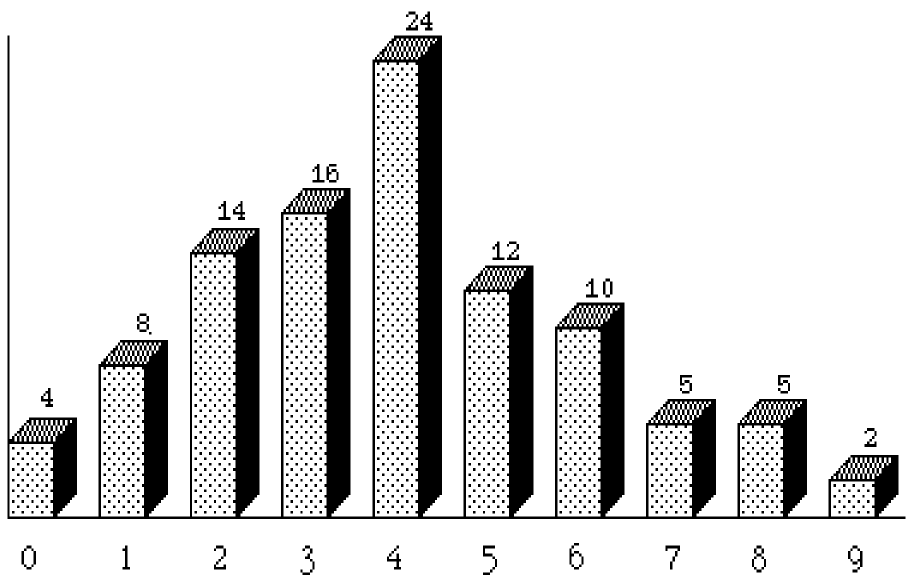
c. Use the inverse-transformation method to obtain random inter-arrival times $\tau_1, \tau_2, \ldots \tau_8$ (where $T_1 = \tau_1$).

d. What are the arrival times ($T_1, T_2, \ldots T_8$) of the first eight parts in your simulation?

e. The expected number of arrivals in the first hour should, of course, be 4. In your simulation of the arrivals, however, what is the number of arrivals during the first hour?

2. Manual Simulation of Drive-In Teller Window In the Hypercard stack, the arrival of the first three autos at the drive-in teller window was manually simulated. Continue the simulation manually until the departure of the 10^{th} auto, and give the event log and the event schedule at that time. What was the maximum length of the waiting line during the simulation?

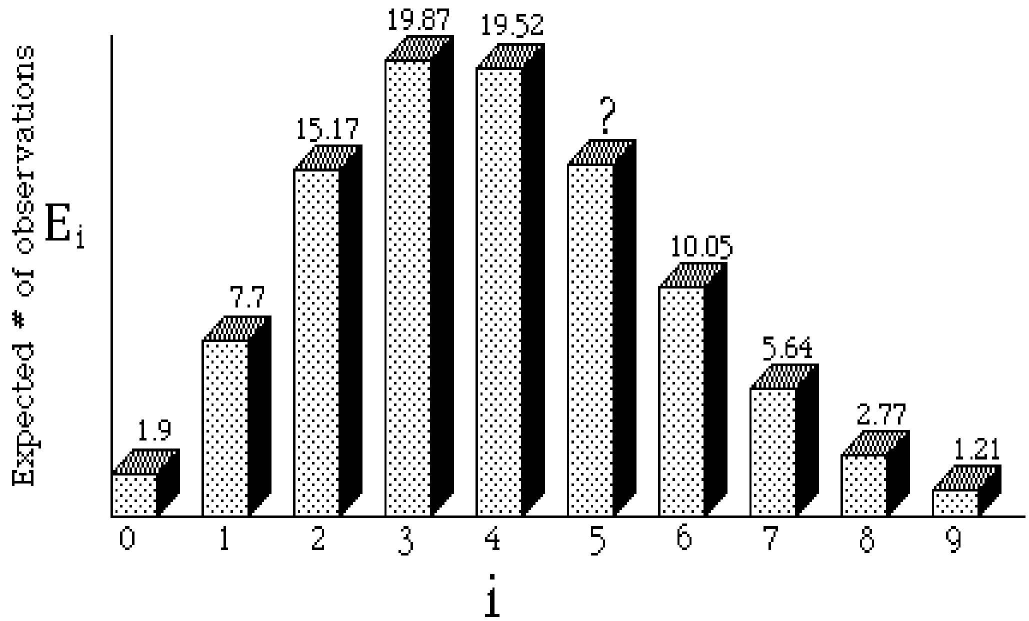
3. The numbers of arrivals during 100 hours of what is believed to be a Poisson process were recorded. The observed numbers ranged from zero to nine, with frequencies O_0 through O_9:



The average number of arrivals was 3.93/hour. We wish to test the "goodness-of-fit" of the assumption that the arrivals correspond to a Poisson process with arrival rate 3.93/hour.
The first step is to compute the probability of each observed value, 0 through 9:

x	P[x]
0	0.01964367
1	0.07719963
2	0.15169728
3	0.19872344
4	0.19524578
5	
6	0.10051838
7	0.05643389
8	0.02772315
9	0.01210578

a. What is the value missing above? (That is, the probability that the number of arrivals is exactly 5.)



b. Now, we can compute the expected number of observations of each of the values 0 through 9, which we denote by E_0 through E_9. What is the expected number of times in which we would observe five arrivals per hour? Did we observe more or fewer than the expected number?

c. Complete the table below:

i	P_i	E_i	O_i	$(E_i-O_i)^2$
0	0.019644	1.9644	4	4.1438
1	0.077200	7.7200	8	0.0784
2	0.151697	15.1697	14	1.3683
3	0.198723	19.8723	16	14.9950
4	0.195246	19.5246	24	20.0294
5			12	
6	0.100518	10.0518	10	0.0027
7	0.056434	5.6434	5	0.4139
8	0.027723	2.7723	5	4.9626
9	0.012106	1.2106	2	0.6232

d. Ignoring the suggestion that cells should be aggregated so that they contain at least five observations, what is the observed value of

$$D = \sum_i \frac{(E_i-O_i)^2}{E_i} \ ?$$

e. Keeping in mind that the assumed arrival rate λ=3.93/hour was estimated from the data, what is the number of "degrees of freedom"?

f. Using a value of α = 5%, what is the value of $\chi^2_{5\%}$ such that D exceeds $\chi^2_{5\%}$ with probability 5% (if the assumption is correct that the arrivals form a Poisson process with arrival rate 3.93/hour)?

g. Is the observed value greater than or less than $\chi^2_{5\%}$? Should we accept or reject the assumption that the arrival process is Poisson with rate 3.93/hour?

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #3 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. **Monte-Carlo Simulation to Estimate Reliability.** Suppose that a certain component fails if the "stress" s on the component exceeds the "strength" S. It is assumed that the strength S has a Weibull distribution with mean 1000 psi and standard deviation 150 psi, while the stress s has a Gumbel distribution with mean 900 psi with mean 200psi .

a. What are the parameters of the Gumbel distribution? α = ____________ , u = _____________

b. What is the "coefficient of variation" of the Weibull distribution? σ/μ = _____________

c. Using the table in the Hypercard stack, estimate the parameters of the Weibull distribution: first, the parameter k = ____________ , and then the parameter u = _____________

You are to randomly generate 10 pairs (s,S) and test whether a failure occurs, i.e., s>S. To generate the random values of s, select a row from the table which appeared in the Hypercard stack and is reprinted below: if the last digit of your ID# is 1, use row #1; if 2, use row #2; ... if 0, use row #10, etc.

3821	4876	3071	5268	8684	0169	1746	6658	8605	9638
0218	3519	0707	3695	6478	3977	2017	3644	7993	5547
5105	8147	7365	2901	7228	2307	7241	4225	6078	9344
4549	1468	4395	3808	9446	5954	6851	2930	9217	5668
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1797	8050	3603	9301	2162	8267	6733	5878	9918	3984
5280	5063	6663	6449	6400	0863	2414	4309	0851	3393
7223	4603	1542	9279	7217	2279	4575	5332	0000	6645

To randomly generate S, select a column, based upon the first digit of your ID#. (If 1, use column #1; if 2, use column #2; ... if 0, use column #10, etc.)

ID#: _____-___-______

Simulation #i	stress s_i	Strength S_i	Failure? (circle)
1	________	________	Yes / No
2	________	________	Yes / No
3	________	________	Yes / No
4	________	________	Yes / No
5	________	________	Yes / No
6	________	________	Yes / No
7	________	________	Yes / No
8	________	________	Yes / No
9	________	________	Yes / No

10	________	________	Yes / No

Total # of failures: _______
Estimated Reliability: ___/10 = ______ %

2. **Regression Analysis.** Tests on the fuel consumption of a vehicle traveling at different speeds yeilded the following results:

Speed s (mph)	20	30	40	50	60	70	80	90
Consumption C (mile/gal.)	11.2	18.1	22.1	25.3	26.4	27.9	28.7	29.4

It is believed that the relation between the two variables is of the form $C = a + b/s$.

a. Run the "Cricket Graph" software package (on the Macintosh), and enter the above observed values of s and C (columns 1 and 2).

Untitled Data

	1	2	3
	s	C	Column 3
1	20	11.2	
2	30	18.1	
3	40	22.1	
4	50	25.3	
5	60	26.4	
6	70	27.9	
7	80	28.7	
8	90	29.4	

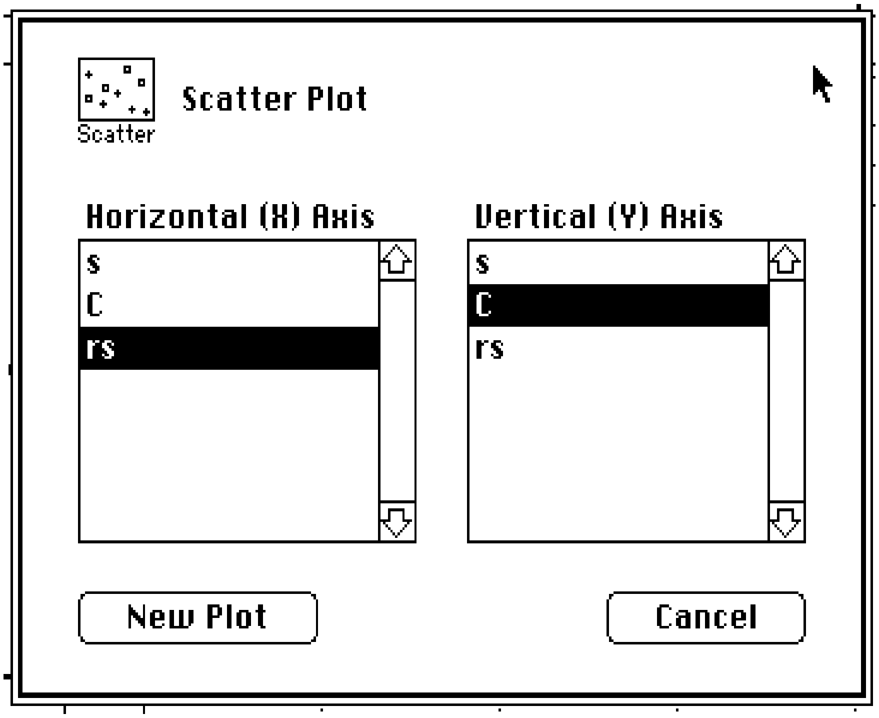
b. Plot the "scatter plot" of C versus s by choosing "scatter" on the Graph menu, and specifying s on the horizontal axis and C on the vertical axis. Does the plot appear to be linear? _______

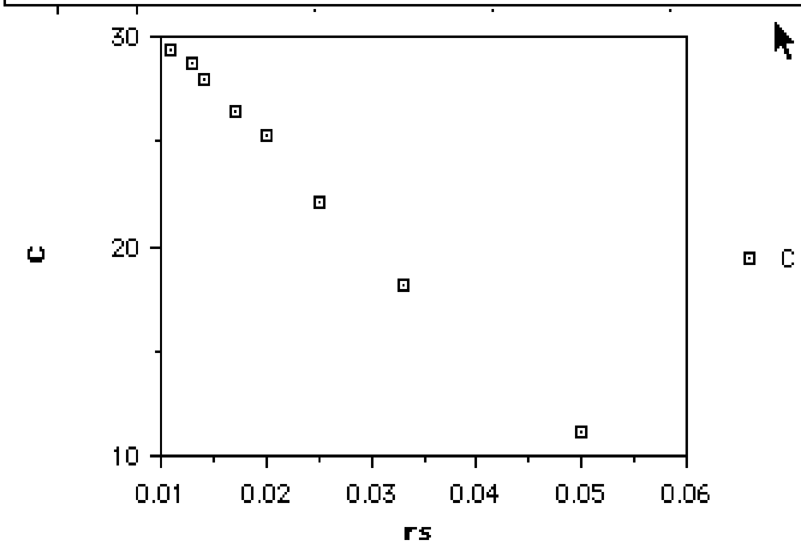
c. Choose "transform" from the "data" menu to create a new variable rs which is the reciprocal of s. (Put this new variable into Column 3.)

Untitled Data

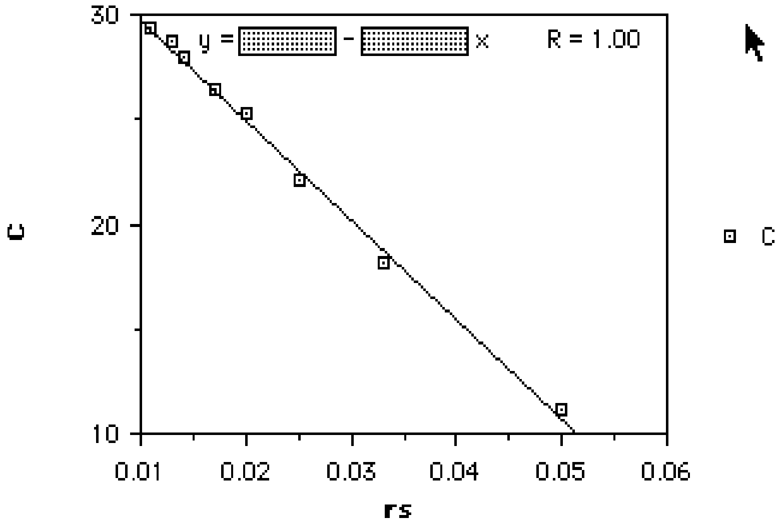
	1	2	3
	s	C	rs
1	20	11.2	0.050
2	30	18.1	0.033
3	40	22.1	0.025
4	50	25.3	0.020
5	60	26.4	0.017
6	70	27.9	0.014
7	80	28.7	0.013
8	90	29.4	0.011

d. Plot the "scatter plot" of rs (horizontal axis) versus C (vertical axis). Does the plot appear to be linear? _______





e. After plotting C versus rs (=1/s), select "Simple" from the "Curve Fit" menu, in order to fit a simple linear relationship between C and 1/s, i.e., to determine a and b such that $C \approx a + b(^1/_s)$. What is the value of a? __________ of b? _____________



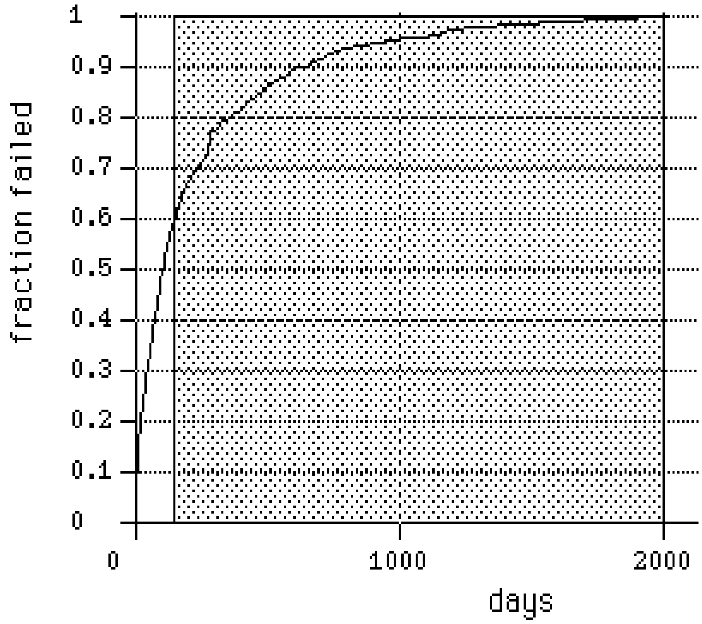
(Note: the above data is complete fictitious!)

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #4 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. Suppose that your company wishes to estimate the reliability of an electric motor. Two hundred units are tested simultaneously, and the time(in days) of the first 120 failures is recorded.

0.1	0.2	0.2	0.2	0.5	0.6	0.7	1.0	1.2	1.2
1.8	2.1	3.2	3.3	3.7	4.6	5.4	8.6	8.6	9.7
10.7	10.8	11.6	11.8	11.9	11.9	12.5	13.0	13.0	15.9
16.0	16.1	16.8	17.1	17.3	20.2	21.8	21.8	22.2	23.0
23.5	24.3	25.1	25.1	27.7	28.1	28.3	29.0	30.0	33.9
35.1	37.2	37.6	39.1	40.6	41.0	43.7	44.8	44.9	46.5
46.8	47.5	51.0	52.4	54.1	57.1	61.2	61.9	62.0	63.2
63.4	63.9	65.1	65.7	66.5	69.1	71.1	72.3	73.5	77.5
78.2	78.6	80.5	81.3	82.4	83.3	84.8	86.1	87.4	87.6
91.0	91.1	91.5	91.8	93.3	94.0	94.6	101.2	102.5	105.1
106.7	110.9	113.9	114.2	116.2	117.8	118.1	118.5	119.8	121.0
126.3	127.7	128.5	129.4	135.3	139.8	140.1	141.5	146.3	149.8

(The experiment was terminated after 150 days, giving us the unshaded curve below.If we had continued until the last motor had failed, the experiment would have lasted over five years!)



To simplify the computations, the data was aggregated, giving the table below showing the failure times of the tenth, twentieth, thirtieth, etc. motor:

a. Plot the value of ($\ln \ln {}^{1}/_{R}$) on the vertical axis and ln T on the horizontal axis of ordinary graph paper.

b. By "eyeballing it", draw a straight line which seems best to fit the data point.

c. What is the slope of this line?

d. What is the y-intercept of this line?

e. What is therefore your estimate of the parameters k and u of the Weibull distribution for the lifetimes of these motors?

failed Time %Surviving

NF	T	R	Ln(T)	Ln Ln 1/R
10	1.2	0.95	0.182322	¯2.9702
20	9.7	0.9	2.27213	¯2.25037
30	15.9	0.85	2.76632	¯1.81696
40	23	0.8	3.13549	¯1.49994
50	33.9	0.75	3.52342	¯1.2459
60	46.5	0.7	3.83945	¯1.03093
70	63.2	0.65	4.1463	¯0.842151
80	77.5	0.6	4.35028	¯0.671727
90	87.6	0.55	4.47278	¯0.514437
100	105.1	0.5	4.65491	¯0.366513
110	121	0.45	4.79579	¯0.225011
120	149.8	0.4	5.0093	¯0.0874216

f. What is the expected lifetime of the motors, according to your Weibull probability model? (You may use the table below for the gamma function in the computation of μ. Values of $\Gamma(1+{}^{1}/_{k})$ are given for k=0.1, 0.2, ... 3.9)

$$\Gamma\left(1+\frac{1}{k}\right)$$

k	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	∞	3628800	120	9.2605	3.3234	2.0000	1.5046	1.2658	1.1330	1.0522
1	1.0000	0.9649	0.9407	0.9236	0.9114	0.9027	0.8966	0.8922	0.8893	0.8874
2	0.8862	0.8857	0.8856	0.8859	0.8865	0.8873	0.8882	0.8893	0.8905	0.8917
3	0.8930	0.8943	0.8957	0.8970	0.8984	0.8997	0.9011	0.9025	0.9038	0.9051

g. Perform a Chi-Square goodness of fit test to decide whether the Weibull probability distribution model which you have found is a "good" fit of the data. Complete the table:

Interval	O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
0 - 1.2	10	_____	______
1.2-9.7	10	_____	______
9.7-15.9	10	_____	______
15.9-23	10	_____	______
23-33.9	10	_____	______
33.9-46.5	10	_____	______
46.5-63.2	10	_____	______
63.2-77.5	10	_____	______
77.5-87.6	10	_____	______
87.6-105.1	10	_____	______
105.1-121	10	_____	______
121-149.8	10	_____	______
		Total: D =	______

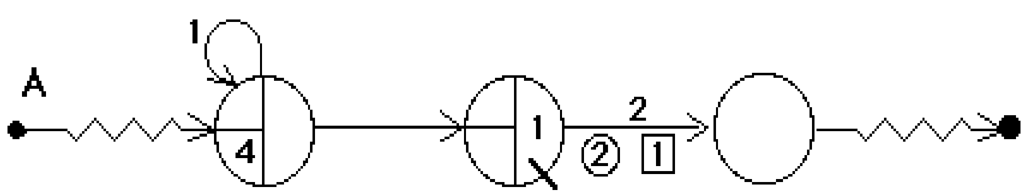
h. What is the number of "degrees of freedom"? ______ (Keep in mind that two parameters, u & k, were estimated based upon the data!)

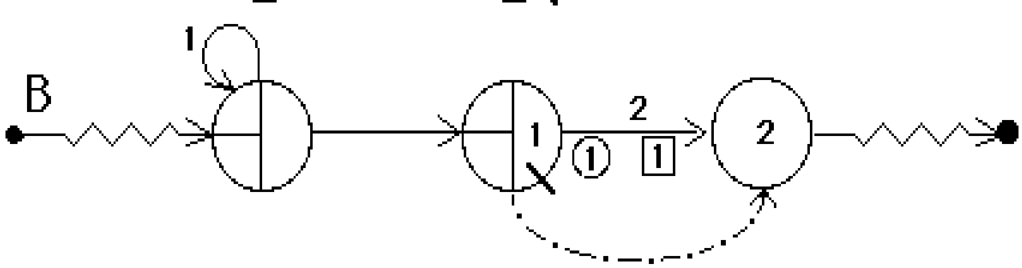
i. Using α = 5%, should the probability distribution be accepted or rejected?

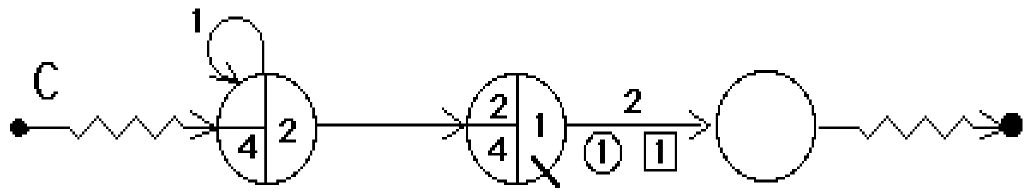
«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #5 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. For each SLAM network below, state
   - the time at which the first entity leaves the system
   - the time at which the simulation ends (or the last entity leaves the system, whichever is first)
   - the number of entities which have left the system when the simulation ends.

Note that all activity durations are constants, and none random!

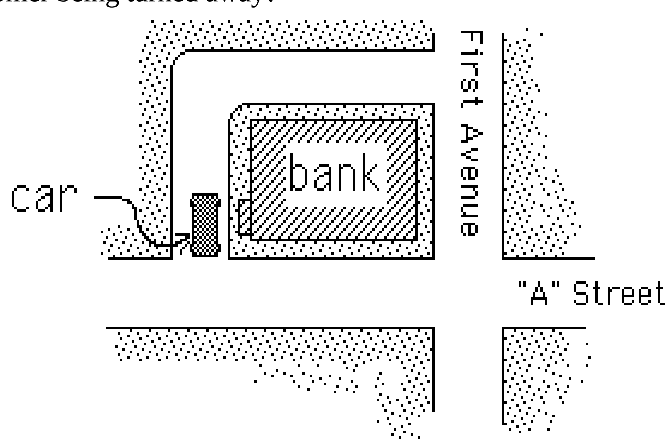






Network	Time first entity leaves	Time second entity leaves	Time Simulation Ends	Number of entities which leave system
A	________	________	________	________
B	________	________	________	________
C	________	________	________	________

2. Consider again the example of the drive-up bank teller window. Arrival of customers forms a Poisson process, with an average of one arrival every 5 minutes. Time to serve each customer has exponential distribution with average of two minutes. There is sufficient space in the drive-up lane for four cars to wait behind the car currently being served; the first car not able to enter the drive-up lane when it is filled will cause the simulation to terminate. The systems analyst believes that the time spent in the system (both waiting time and service time) for the customers will have an exponential distribution. He has prepared the following SLAM model, and has included COLCT ("collect") statements to collect statistics on both the time in the system (with histogram) AND the time that the first customer is turned away. The system is to be simulated for 480 minutes (8 hours), unless it is terminated because of a customer's being turned away.
   (a.) How many customers were served during the simulation?
   (b.) What fraction of the customers spend more than 5 minutes at the bank?
   (c.) What was the longest time spent by a customer at the bank during this simulation?
   (d.) What is the mean (average) time spent by customers in the system?
   (e.) Test the "goodness-of-fit" of the exponential probability distribution having this mean value. (Use alpha = 10%, and group the histogram cells as necessary so that there are at least 5 observations in each cell.)
   (f.) At what time does the simulation end? Is it because of the maximum time (480 minutes) or because of a customer being turned away?




```
GEN,BRICKER,BANKTELLER,2/23/1995,,,,,,,72;
LIM,2,1,50;
INIT,0,480;
NETWORK;
      CREATE,EXPON(5.0),,1;
      QUE(1),0,4,BALK(OVFLO);
      ACT(1)/1,EXPON(2.0);
      COLCT,INTVL(1),CUSTOMER_TIME,20/.5/.5;
      TERM;
```

```
OVFLO COLCT,FIRST;
      TERM,1;
      END;
FIN;
```

```
              S L A M   I I   S U M M A R Y   R E P O R T
 SIMULATION PROJECT BANKTELLER              BY BRICKER


 CURRENT TIME   0.4081E+03
 STATISTICAL ARRAYS CLEARED AT TIME  0.0000E+00

           **STATISTICS FOR VARIABLES BASED ON OBSERVATION**

                  MEAN    STANDARD  COEFF. OF  MINIMUM   MAXIMUM  NO.OF
                  VALUE   DEVIATION VARIATION   VALUE     VALUE    OBS

CUSTOMER_TIME   0.303E+01 0.286E+01 0.944E+00 0.345E-01 0.110E+02    88
                0.408E+03 0.000E+00 0.000E+00 0.408E+03 0.408E+03     1

           **FILE STATISTICS**

FILE                   AVERAGE  STANDARD   MAXIMUM  CURRENT AVERAGE
NUMBER  LABEL/TYPE     LENGTH   DEVIATION  LENGTH   LENGTH  WAIT TIME

   1        QUEUE        0.300     0.724       4        4      1.317
   2                     0.000     0.000       0        0      0.000
   3        CALENDAR     1.439     0.496       3        2      2.669

           **SERVICE ACTIVITY STATISTICS**

ACT ACT LABEL OR  SER AVERAGE    STD  CUR AVERAGE MAX IDL MAX BSY  ENT
NUM START NODE    CAP   UTIL     DEV  UTIL BLOCK  TME/SER TME/SER  CNT

  1      QUEUE      1   0.439   0.50    1    0.00   17.35   29.23   88
                        **HISTOGRAM NUMBER 1**
                             CUSTOMER_TIME

OBS  RELA   UPPER
FREQ FREQ  CELL LIM 0        20        40        60        80       100
                    +    +    +    +    +    +    +    +    +    +    +
 10 0.114 0.500E+00 +******                                           +
  9 0.102 0.100E+01 +*****     C                                      +
 16 0.182 0.150E+01 +*********          C                             +
 13 0.148 0.200E+01 +*******                   C                      +
  4 0.045 0.250E+01 +**                           C                   +
  7 0.080 0.300E+01 +****                             C               +
  5 0.057 0.350E+01 +***                                C             +
  1 0.011 0.400E+01 +*                                   C            +
  3 0.034 0.450E+01 +**                                    C          +
  0 0.000 0.500E+01 +                                      C          +
  2 0.023 0.550E+01 +*                                      C         +
  1 0.011 0.600E+01 +*                                      C         +
  3 0.034 0.650E+01 +**                                       C       +
  0 0.000 0.700E+01 +                                         C       +
  4 0.045 0.750E+01 +**                                         C     +
  1 0.011 0.800E+01 +*                                           C    +
  2 0.023 0.850E+01 +*                                            C   +
  2 0.023 0.900E+01 +*                                             C  +
```

```
  2 0.023 0.950E+01 +*                                             C +
  2 0.023 0.100E+02 +*                                              C+
  0 0.000 0.105E+02 +                                               C+
  1 0.011    INF    +*                                               C
---                 +    +    +    +    +    +    +    +    +    +    +
 88                 0         20        40        60        80       100

            **STATISTICS FOR VARIABLES BASED ON OBSERVATION**

                   MEAN    STANDARD  COEFF. OF  MINIMUM   MAXIMUM  NO.OF
                   VALUE   DEVIATION VARIATION   VALUE     VALUE    OBS

 CUSTOMER_TIME   0.303E+01 0.286E+01 0.944E+00 0.345E-01 0.110E+02    88
Fortran STOP
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«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #6 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

Project Scheduling. An equipment maintenance building is to be erected near a large construction site. An electric generator and a large water storage tank are to be installed a short distance away and connected to the building. The activity descriptions and estimated durations for the project are:

Activity	Description	Predecessor(s)	Duration optimistic	most likely	pessimistic
A	Clear & level site	none	1	2	4
B	Erect building	A	4	6	9
C	Install generator	A	1	3	4
D	Install water tank	A	1	2	4
E	Install maintenance equipment	B	2	4	6
F	Connect generator & tank to bldg	B,C,D	2	5	7
G	Paint & finish work on building	B	2	3	4
H	Facility test & checkout	E,F	1	2	3

a. Draw the AON (activity-on-node) network representing this project.

b. Draw the AOA (activity-on-arrow) network representing this project. Explain the necessity for any "dummy" activities which you have included.

c. Label the nodes of the AOA network, so that i<j if there is an activity with node i as its start and node j as its end node.

In questions (d) through (h), use the "most likely" as the duration:

d. Perform the forward pass through the AOA network to obtain for each node i, ET(i) = earliest possible time for event i.

e. What is the earliest completion time for this project?

f. Perform the backward pass through the AOA network to obtain, for each node i, LT(i) = latest possible time for event i (assuming the project is to be completed in the time which you have specified in (e).)

g. For each activity, compute:

ES = earliest start time   EF = earliest finish time
LS = latest start time   LF = latest finish time
TF = total float (slack)

h. Which activities are "critical", i.e., have zero float ("slack")?

i. Schedule this project by entering the AON network into MacProject II (found on several of the Macintosh II computers in the computer lab on 3rd floor of the Engineering Building.) Specify that the start time for the project will be November 15, 1995. What is the earliest completion time for the project? (Note that 5-day work weeks are assumed by default.)

Now, assume that the durations are random, with triangular distribution.

j. What is the expected duration of the critical path found in (h)?

k. What is the standard duration of the critical path found in (h)?

m. Using the PERT technique, what is the probability that the project duration will not exceed the expected duration by more than 3 days?

n. Draw a SLAM network to simulate this project, and simulate it 1000 times.

o. What is the average duration of the project according to SLAM? (Compare it to your answer in (j).)

p. What is the standard deviation of the project duration according to SLAM? (Compare it to your answer in (k).)

q. According to the SLAM simulation, what is the probability that the project duration will not exceed the expected duration by more than three day?

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #7 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

1. Markov Chain Model of a Reservoir: A city's water supply comes from a reservoir. Careful study of this reservoir over the past twenty years has shown that, if the reservoir was full at the beginning of one summer, then the probability that it would be full at the beginning of the next summer is 80%; however, if the reservoir was not full at the beginning of one summer, the probability that it would be full at the beginning of the next summer is only 40%.

The below computer output may be consulted to help answer some of the following questions. Note that state #1 represents the condition "full" and state #2 represents the condition "not full"

Powers of P

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.688 & 0.312 \\ 0.624 & 0.376 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.6752 & 0.3248 \\ 0.6496 & 0.3504 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.67008 & 0.32992 \\ 0.65984 & 0.34016 \end{bmatrix}$$

Steady State Distribution

i	Pi
1	0.66666667
2	0.33333333

Mean First Passage Times

from \ to	1	2
1	1.5	5
2	2.5	3

Expected no. of visits during first 5 stages

from \ to	1	2
1	3.55328	1.44672
2	2.89344	2.10656

First Passage Probabilities

stage 1: $\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$

stage 2: $\begin{bmatrix} 0.08 & 0.16 \\ 0.24 & 0.08 \end{bmatrix}$

stage 3: $\begin{bmatrix} 0.048 & 0.128 \\ 0.144 & 0.064 \end{bmatrix}$

stage 4: $\begin{bmatrix} 0.0288 & 0.1024 \\ 0.0864 & 0.0512 \end{bmatrix}$

stage 5: $\begin{bmatrix} 0.01728 & 0.08192 \\ 0.05184 & 0.04096 \end{bmatrix}$

a. Draw a diagram of a Markov Chain model of this reservoir.

b. Why are we guaranteed that this system has a steady-state probability distribution?

c. Write equations which could be solved to compute the steady-state distribution. (You need not solve them!)

d. Over a 100-year period, how many summers can the reservoir be expected to be full?

e. If the reservoir was full at the beginning of summer 1994, what is the probability that

- it will be full at the beginning of summer 1995?
- it will be full at the beginning of summer 1996?

f. If the reservoir was full at the beginning of summer 1994, what is the expected number of summers during the next 5 years that the reservoir will not be full?

g. If the reservoir was full at the beginning of summer 1994, what is the expected number of years before the reservoir will not be full at the beginning of the summer?

2. Absorption Analysis of Markov Chain. In response to pressure from the Board of Regents to increase the number of students who complete their degrees within four years, the Engineering College admissions office has modeled the academic career of a student as a Markov chain:

Each student's state is observed at the beginning of each *fall* semester. For example, if a student is a junior at the beginning of the current fall semester, there is an 80% chance that he/she will be a senior at the beginning of the next fall semester, a 15% chance that he/she will still be a junior, and a 5% chance that he/she will have quit. (For simplicity we will assume that once a student quits, he/she never re-enrolls.)

State	Description
1	Freshman
2	Sophomore
3	Junior
4	Senior
5	Drop-out
6	Graduate

a. Draw a diagram for this Markov chain.

b. Which states are transient?

c. Which states are recurrent?

d. Which states are absorbing?

e. Does this system have a steady-state probability distribution? *Justify your answer*.

Consult the computer output below to answer the questions that follow.

f. If a student enters the college as a freshman, how many years can he or she expect to spend as a student in the college?

g. What is the probability that, at the beginning of the fourth year in the college, he or she is classified as a senior?

h. What is the probability that he or she eventually will graduate?

i. If a student has survived to the point that he or she has been classified as a junior, what is then the probability that he or she eventually graduates?

Transition probabilities (P)

from \ to	1	2	3	4	5	6
1	0.1	0.8	0	0	0.1	0
2	0	0.1	0.85	0	0.05	0
3	0	0	0.15	0.8	0.05	0
4	0	0	0	0.1	0.05	0.85
5	0	0	0	0	1	0
6	0	0	0	0	0	1

E = Expected No. Visits to Transient States

from \ to	1	2	3	4
1	1.1111111	0.98765432	0.98765432	0.87791495
2	0	1.1111111	1.1111111	0.98765432
3	0	0	1.1764706	1.0457516
4	0	0	0	1.1111111

P^2

from \ to	1	2	3	4	5	6
1	0.01	0.16	0.68	0	0.15	0
2	0	0.01	0.2125	0.68	0.0975	0
3	0	0	0.0225	0.2	0.0975	0.68
4	0	0	0	0.01	0.055	0.935
5	0	0	0	0	1	0
6	0	0	0	0	0	1

P^3

from \ to	1	2	3	4	5	6
1	0.001	0.024	0.238	0.544	0.193	0
2	0	0.001	0.040375	0.238	0.142625	0.578
3	0	0	0.003375	0.038	0.108625	0.85
4	0	0	0	0.001	0.0555	0.9435
5	0	0	0	0	1	0
6	0	0	0	0	0	1

A = Absorption Probabilities

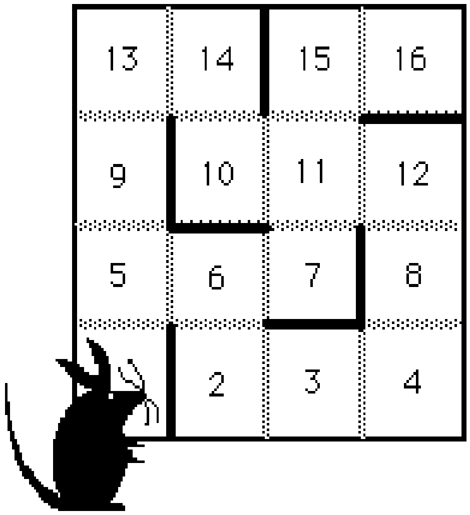
from \ to	5	6
1	0.25377229	0.74622771
2	0.16049383	0.83950617
3	0.11111111	0.88888889
4	0.055555556	0.94444444

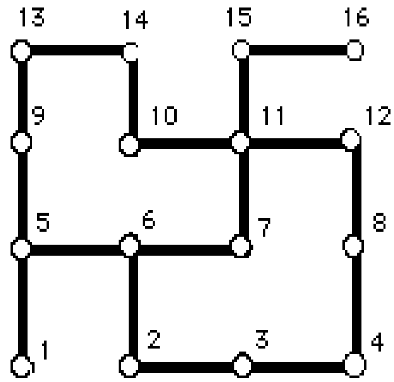
P^4

from \ to	1	2	3	4	5	6
1	0.0001	0.0032	0.0561	0.2448	0.2334	0.4624
2	0	0.0001	0.00690625	0.0561	0.15659375	0.7803
3	0	0	0.00050625	0.0065	0.11069375	0.8823
4	0	0	0	0.0001	0.05555	0.94435
5	0	0	0	0	1	0
6	0	0	0	0	0	1

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #8 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

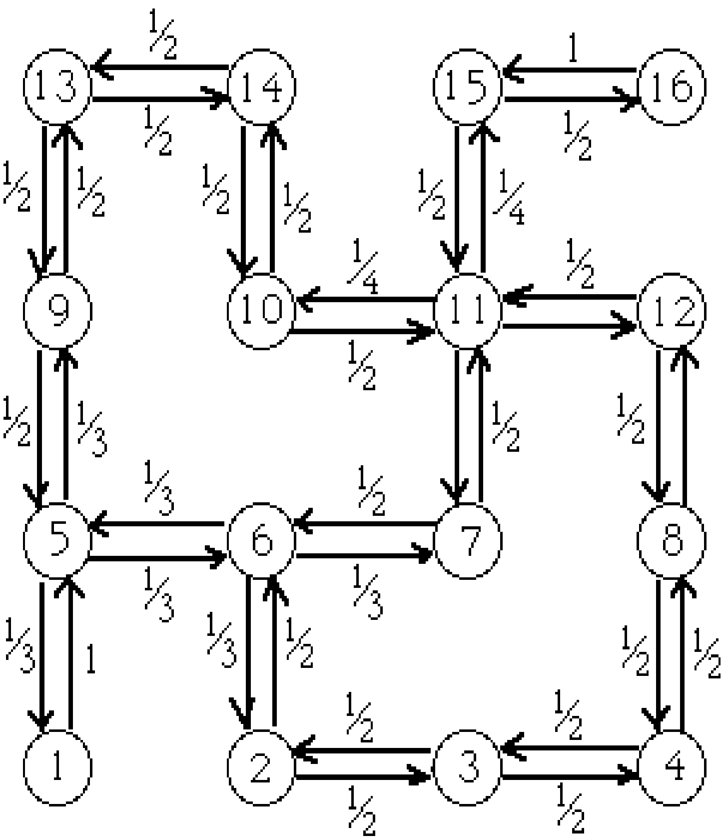
1. We wish to model the passage of a rat through a maze, in the form of a 4x4 array of boxes, such as the one below on the left:





The solid lines represent walls, the shaded lines represent doors. We will assume that a rat is placed into box #1. While in any box, the rat is assumed to be equally likely to choose each of the doors leaving the box (including the one by which he entered the box). For example, when in box #2 above, the probability of going next to boxes 3 and 6 are each $^{1}/_{2}$, regardless of the door by which he entered the box. *This assumption implies that no learning takes place if the rat tries the maze several times. (Note that the assumptions imply that the mouse is as equally likely to exit a box by the door through which he entered as any of the other exiting doors.)*

Based upon this "memorylessness" assumption, the movement of the rat through the maze can be modeled as a discrete-time Markov chain:



The steadystate probability distribution exists because the chain is regular, and is:

Steady State Distribution

i	P{i}
1	0.0294
2	0.0588
3	0.0588
4	0.0588
5	0.0882
6	0.0882
7	0.0588
8	0.0588
9	0.0588
10	0.0588
11	0.118
12	0.0588
13	0.0588
14	0.0588
15	0.0588
16	0.0294

The mean first passage time matrix (M) is

M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	34	31	43.6	49.1	1	11.3	23.1	47.6	19.6	33	23.3	39	31.1	35.6	54.3	87.3
2	59.7	17	18.3	29.5	26.7	8.67	20.5	33.6	39.6	36	20.7	30.7	45.5	44.3	51.7	84.7
3	65.2	11.2	17	15.7	32.2	15.3	23.7	24.4	44	37	20.5	26	48.7	46.4	51.5	84.5
4	68.7	20.4	13.7	17	35.7	20	25	13.2	46.4	36	18.4	19.3	50	46.5	49.4	82.4
5	33	30	42.6	48.1	11.3	10.3	22.1	46.6	18.6	32	22.3	38	30.1	34.6	53.3	86.3
6	52.1	20.8	34.5	41.2	19.1	11.3	15.2	40.8	33.2	33	18.8	33.3	40.2	40.1	49.8	82.8
7	60.7	29.4	39.7	43	27.7	12	17	39.2	38.4	28	10.4	28.3	42	38.5	41.4	74.4
8	70.3	27.6	25.5	16.3	37.3	22.7	24.3	17	46.8	33	14.3	10.7	49.3	44.7	45.3	78.3
9	43.9	35.2	46.7	51.1	10.9	16.7	25.1	48.4	17	27	21.9	38.7	16.1	25.1	52.9	85.9
10	64.5	38.8	46.9	47.9	31.5	23.7	21.9	41.8	34.2	17	8.47	28.7	29.9	18.5	39.5	72.5
11	67.3	36	42.9	42.8	34.3	22	16.8	35.6	41.6	21	8.5	21.3	41.8	34.9	31	64
12	69.8	32.8	35.2	30.5	36.8	23.3	21.5	18.8	45.2	28	8.13	17	46.5	40.8	39.1	72.1
13	52.7	38.4	48.7	52	19.7	21	26	48.2	13.4	20	19.4	37.3	17	13.5	50.4	83.4
14	59.6	39.6	48.8	50.9	26.6	23.3	24.9	46	24.8	11	14.9	34	15.9	17	45.9	78.9
15	70.3	39	45.9	45.8	37.3	25	19.8	38.6	44.6	24	3	24.3	44.8	37.9	17	33
16	71.3	40	46.9	46.8	38.3	26	20.8	39.6	45.6	25	4	25.3	45.8	38.9	1	34

The first-visit probabilities from box #1 to the reward in box #16 are:

First Visit Probabilities to State 16 from State 1

$f_{1,16}^{(n)}$

n	P
1	0
2	0
3	0
4	0
5	0
6	0.00694
7	0
8	0.0126
9	0
10	0.0165

The shortest path from box 1 to box 16 is: 1->5->6->7->11->15->16, or 6 moves. The matrix P^6 is

6-th Power

	1	2	3	4	5	6	7	8
1	0.151	0	0.0802	0	0	0.276	0	0.0208
2	0	0.22	0	0.206	0.198	0	0.125	0
3	0.0401	0	0.279	0	0	0.22	0	0.227
4	0	0.206	0	0.299	0.0758	0	0.0992	0
5	0	0.132	0	0.0505	0.365	0	0.117	0
6	0.0922	0	0.147	0	0	0.256	0	0.0903
7	0	0.125	0	0.0992	0.176	0	0.141	0
8	0.0104	0	0.227	0	0	0.135	0	0.266
9	0.122	0	0.0471	0	0	0.206	0	0.0182
10	0	0.0448	0	0.0521	0.128	0	0.113	0
11	0.0253	0	0.077	0	0	0.127	0	0.125
12	0	0.111	0	0.193	0.0535	0	0.116	0
13	0	0.0523	0	0.0148	0.273	0	0.0879	0
14	0.0602	0	0.02	0	0	0.116	0	0.0469
15	0	0.0431	0	0.0599	0.0483	0	0.142	0
16	0.00694	0	0.026	0	0	0.0903	0	0.0938

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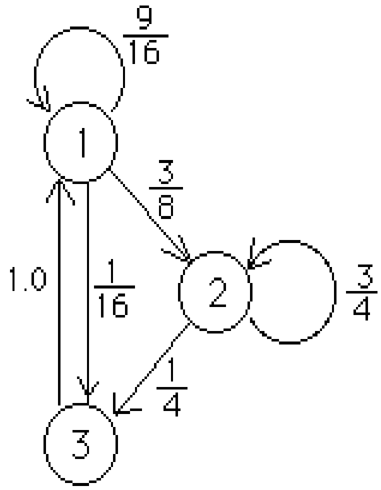
9	10	11	12	13	14	15	16
0.243	0	0.101	0	0	0.12	0	0.00694
0	0.0448	0	0.111	0.0523	0	0.0431	0
0.0471	0	0.154	0	0	0.02	0	0.013
0	0.0521	0	0.193	0.0148	0	0.0599	0
0	0.0855	0	0.0357	0.182	0	0.0322	0
0.137	0	0.169	0	0	0.0772	0	0.0301
0	0.113	0	0.116	0.0879	0	0.142	0
0.0182	0	0.25	0	0	0.0469	0	0.0469
0.256	0	0.136	0	0	0.203	0	0.0113
0	0.198	0	0.12	0.186	0	0.159	0
0.0678	0	0.343	0	0	0.123	0	0.112
0	0.12	0	0.198	0.0503	0	0.159	0
0	0.186	0	0.0503	0.278	0	0.0582	0
0.203	0	0.247	0	0	0.26	0	0.0469
0	0.159	0	0.159	0.0582	0	0.331	0
0.0226	0	0.448	0	0	0.0938	0	0.219

a. Which box will be visited most frequently by the rat?
b. A reward (e.g. food) is placed in box #16 for the rat. What is the expected number of moves of the rat required to reach this reward?
c. The minimum number of moves required to reach the reward is six. What is the probability that the rat reaches the reward in **exactly** this number of moves?
d. What is the probability that the rat reaches the reward with no more than 4 unnecessary moves?
e. Briefly discuss the utility of this model in testing a hypothesis that a *real* rat is able to learn, thereby finding the reward in relatively few moves after he has made several trial runs through the maze.
f. Briefly discuss how you might modify your Markov chain model if the rat will never exit a box by the same door through which he entered (i.e., he is no longer completely "memoryless", in that he remembers the door through which he entered), unless he has reached a "dead end", in which case he reverses his path.

2. A machine has two critical parts that are subject to failure. The machine can continue to operate if one part has failed. Only in the case where *both* parts are no longer intact does a repair need to be done. A repair takes exactly one day, and after a repair both parts are intact again. At the beginning of each day, the machine is examined to determine whether or not a repair is required. If at the beginning of a day a part is intact, then it will fail during the day with probability 0.25. Each repair costs $50. For each day the machine is running, it generates $100 in profit. (For the sake of simplicity, assume that all failures occur very late in the day, so that if the machine is operating at the beginning of the day, it will generate the full $100 in profit, and repairs will not begin until the following morning.) Below is a discrete-time Markov chain model of this system, with three states:
(1) Both parts intact
(2) One part intact
(3) Both parts failed
The transition probabilities are found as follows:
p_{12} = P{one part of two fails} = 2(0.25) (0.75)
p_{13} = P{two parts fail} = (0.25)(0.25)
$p_{11} = 1 - p_{12} - p_{13}$
etc.
Note that if the morning inspection finds that both parts have failed, that day is spent in repairing the machine, so that at the beginning of the next day it will be restored to its original condition, so that $p_{31} = 1.0$



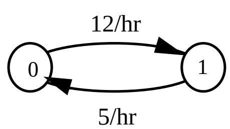
$$P = \begin{bmatrix} \frac{9}{16} & \frac{3}{8} & \frac{1}{16} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix}$$

a. Write the system of linear equations which must be solved to compute the steady-state distribution.
b. Find the steady-state distribution for your Markov chain.
c. Compute the average profit per day for this machine.

3. (Exercise 6, page 1083-1084 of text by Winston) Bectol, Inc. is building a dam. A total of 10,000,000 cu ft of dirt is needed to construct the dam. A bulldozer is used to collect dirt for the dam. Then the dirt is moved via dumpers to the dam site. Only one bulldozer is available, and it rents for $100 per hour. Bectol can rent, at $40 per hour, as many dumpers as desired. Each dumper can hold 1000 cu ft of dirt. It takes an average of 12 minutes for the bulldozer to load a dumper with dirt, and it takes each dumper an average of five minutes to deliver the dirt to the dam and return to the bulldozer. Making appropriate assumptions about exponentiality so as to obtain a birth/death model, determine the optimal number of dumpers and the minimum total expected cost of moving the dirt needed to build the dam.

Hint: We have to use 10,000,000/1000=10,000 loads of dumper to deliver all the dirt.

Case 1 : One dumper :
Define state 0 : no dumper in the system,
state 1 : one dumper in the system.
Then we obtain a birth/death model:



Steady-state Distribution

i	Pi	CDF
0	0.294118	0.294118
1	0.705882	1.000000

where the steady-state distribution is found by

$$\frac{1}{\pi_0} = 1 + \frac{12/\text{hr}}{5/\text{hr}} = \frac{17}{5} \qquad \pi_0 = \frac{5}{17}\text{, etc.}$$

The average departure rate of dumper is $(1-\pi_0)5=0.705882(5)=3.52941$(times/hr)
The total cost = (10,000/3.52941)($100+$40)=396667.

Use trial & error to find the optimal number of dumpers.

«»«»«»«»«»«»«»«»«»«»«»«»«»«»«» HW #9 «»«»«»«»«»«»«»«»«»«»«»«»«»«»«»

The following exercises should be done assuming that the queueing system operates in steady state.

1. Each airline passenger and his/her luggage must be checked to determine if he/she is carrying weapons onto the airplane. Suppose that at C.R. Airport an average of 10 passengers/minute arrive (with exponentially-distributed times between arrivals). To check passengers for weapons, the airport must have a checkpoint consisting of a metal detector and baggage X-ray machine. Whenever a checkpoint is in operation, two employees are required. A checkpoint can check an average of 12 passengers/minute (one at a time), with the time for each having an exponential distribution. Assume that the airport has only one checkpoint.

a. What is the probability that a passenger will have to wait before being checked?

b. On the average, how many passengers are waiting in line to enter the checkpoint?

c. On the average, how long will a passenger spend at the checkpoint?

d. Draw a SLAM network model of the above system, with a COLCT to collect the statistics which you enable you to answer the questions above.

2. An average of 100 customers arrive each hour at a bank, forming a Poisson process. The service time per customer has exponential distribution, with mean 1 minute. The manager wants to ensure that the *average* time which customers will have to wait in line is no more than 0.5 minute.

a. If the bank follows the policy of having all customers join a single queue to wait for a teller, how many tellers should the bank hire?

b. Draw a SLAM network model of the above system, with a COLCT to collect the statistics which you enable you to answer the question above.

3. An average of 60 cars per hour arrive (forming a Poisson process) arrive at the MacBurger's drive-in window. However, if four or more cars are in line (including the car at the window), an arriving car will not enter the line (i.e., "balk"). It takes an average of 3 minutes (exponentially distributed) to serve a car.

a. What is the average number of cars waiting for the drive-in window (not including a car at the window)?

b. On the average, how many cars will be served per hour?

c. If you have just joined the line, how many minutes will you expect to pass before you receive your food?

d. Draw a SLAM network model of the above system, with a COLCT to collect the statistics which you enable you to answer the questions above.

4. Choose one of the three SLAM networks above, and simulate it on the computer for an 8-hour period. Compare the simulation results with your previous answers based upon steadystate queueing theory.

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