56:171
Operations Research
Fall 1999

Quiz Solutions

Dept of Mechanical & Industrial Engineering
University of Iowa

56:171 Operations Research
Quiz #6 Solutions – Fall 1999

Part A. The Cubs are trying to determine which of the following free agent pitchers should be signed: Rick Sutcliffe (RS), Bruce Sutter (BS), Dennis Eckersley (DE), Steve Trout (ST), Tim Stoddard (TS). The cost of signing each pitcher and the number of victories each pitcher will add to the Cubs are shown below.

Pitcher	Cost of signing ($million)	Right- or Left-handed?	Victories added to Cubs
RS	$6	Right	6
BS	$4	Right	5
DE	$3	Left	3
ST	$2	Left	3
TS	$2	Right	2

Define binary decision variables RS, BS, etc., e.g.,
RS = 1 if Rick Sutcliffe is signed, and 0 otherwise.

From the list below, select the linear inequality which imposes each of the following restrictions:

1. If RS is signed, then TS cannot be signed: $\boxed{RS + TS \leq 1}$
2. At most two right-handed pitchers can be signed: $\boxed{RS + BS + TS \leq 2}$
3. If DE is signed, then ST must be signed: $\boxed{ST \geq DE}$
4. At least one left-handed pitcher must be signed: $\boxed{DE + ST \geq 1}$
5. The Cubs cannot sign both RS and BS: $\boxed{RS + BS \leq 1}$

a. $ST \geq DE$
b. $DE + ST \leq 1$
c. $RS + BS + TS \geq 2$
d. $RS + BS + TS \leq 2$
e. $RS + BS + TS \geq 1$
f. $RS + BS = 1$
g. $RS + BS = 0$
h. $ST \leq DE$
i. $RS + BS \leq 1$
j. $RS + BS \geq 1$
k. $ST + DE = 1$
l. $RS \leq TS$
m. $DE + ST \geq 1$
n. $RS + TS \leq 1$
o. $DE + ST \leq 1$
p. $RS + TS = 1$
q. *None of the above*

Part B. A court decision has stated that the enrollment of each high school in Metropolis must be at least 20% black. The numbers of black and white high school students in each of the city's five school districts are shown in the table below.

District	White students	Black students	Total
1	80	30	110
2	70	5	75
3	90	10	100
4	50	40	90
5	60	30	90

School board policy requires that all the students in a given district must attend the same school.

Define the decision variables:

X_{ij} = 1 if all students in district *i* are assigned to school *j*
= 0 otherwise

For each of the following restrictions, select the corresponding linear constraint from the list below:

___ 6. Students in district 1 must be assigned to a school: $\boxed{X_{11} + X_{12} = 1}$

___ 7. The enrollment of school 1 must be at least 150: $\boxed{110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51} \geq 150}$

___ 8. The enrollment of black students in school 1 must be at least 20% of its total enrollment:

$$\boxed{30X_{11} + 5X_{21} + 10X_{31} + 40X_{41} + 30X_{51} \geq 0.2\,(110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51})}$$

___ 9. Districts 2 and 5 cannot be assigned to the same school: $\boxed{X_{21} + X_{51} = 1 \;\&\; X_{22} + X_{52} = 1}$

___ 10. At least three districts must be assigned to school #1: $\boxed{X_{11} + X_{21} + X_{31} + X_{41} + X_{51} \geq 3}$

a. $110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51} \geq 150$

b. $110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51} \leq 150$

c. $X_{11} + X_{21} + X_{31} + X_{41} + X_{51} = 3$

d. $X_{11} + X_{12} \leq 1$

e. $X_{11} + X_{12} = 1$

f. $X_{11} + X_{12} \geq 1$

g. $X_{21} + X_{51} = 1 \;\&\; X_{22} + X_{52} = 1$

h. $X_{21} \leq X_{51} \;\&\; X_{22} \leq X_{52}$

i. $X_{11} + X_{21} + X_{31} + X_{41} + X_{51} \geq 3$

j. $30X_{11} + 5X_{21} + 10X_{31} + 40X_{41} + 30X_{51} \geq 20$

k. $X_{21} \times X_{51} = 1$

l. $X_{11} + X_{12} \geq 1$

m. $X_{11} + X_{12} = 1$

n. $X_{11} + X_{21} = 1$

o. $X_{21} + X_{51} \geq 1 \;\&\; X_{22} + X_{52} \geq 1$

p. $30X_{11} + 5X_{21} + 10X_{31} + 40X_{41} + 30X_{51} \geq 0.2\,(110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51})$

q. $110X_{11} + 75X_{21} + 100X_{31} + 90X_{41} + 90X_{51} \geq 0.2\,(30X_{11} + 5X_{21} + 10X_{31} + 40X_{41} + 30X_{51})$

r. $30X_{11} + 5X_{21} + 10X_{31} + 40X_{41} + 30X_{51} \geq 30$

s. *None of the above*

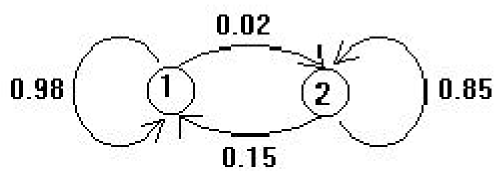
56:171 Operations Research
Quiz #7 Solutions – Fall 1999

Discrete-time Markov chains Let X_n denote the quality of the n^{th} item produced by a production system, with X_n=1 meaning "good" and X_n=2 meaning "defective". Suppose that $\{X_n: n=0,1,2,...\}$ is a Markov chain whose transition probability matrix P (and P^2 and P^3) are

$$P = \begin{bmatrix} .98 & .02 \\ .15 & .85 \end{bmatrix},\ P^2 = \begin{bmatrix} .963 & .037 \\ .275 & .725 \end{bmatrix},\ P^3 = \begin{bmatrix} .95 & .05 \\ .378 & .622 \end{bmatrix}$$

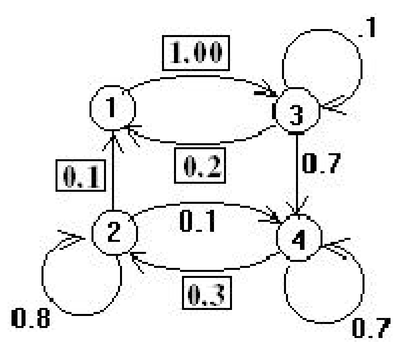
That is, if the previous item was "good", the probability of producing a defective item is 2%, but if the previous item was defective, there is an 85% probability that the next item will also be defective.

1. Sketch the diagram showing the states and transitions (with transition probabilities):



2. What's the probability that, if the 1st item is good, the next one (i.e., the 2nd) is defective? $\boxed{p_{12}=2\%}$
3. What is the probability that, if the first item is defective, the second is defective? $\boxed{p_{22}=85\%}$
4. What is the probability that, if the *first two* items are defective, the third is defective? $\boxed{p_{22}=85\%}$
   Note: The condition of the first item is irrelevant, because a Markov chain is "memoryless".
5. What is the probability that, if the first item is good, the third is defective? $\boxed{p^2_{12} = 3.7\%}$
6. What is the probability that, if the first item is defective, the third is also defective? $\boxed{p^2_{22} = 72.5\%}$

7. Write the transition probability matrix for the following Markov chain diagram:



(Note: some probabilities have not been specified in the diagram, but may be determined by the probabilities which are specified.)

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.1 & 0.8 & 0 & 0.1 \\ 0.2 & 0 & 0.1 & 0.7 \\ 0 & 0.3 & 0.0 & 0.7 \end{bmatrix}$$

Auth: Katherine Pryor, Chelsea White, III, James L. Bander
Titl: A Dial-a-Ride Paratransit Service that Integrates Three Modes of Transportation
Mtng: INFORMS Atlanta, Nov, 1996
Locn: Contribute, Henry
Time: Wednesday 14:45-16:1Transportation
ID : WD24.3
Addr: Univ. of MI, Dept. of Ind. & Op. Eng., 1205 Beal Ave., Ann Arbor, MI 48109-2140,
Abst: Some Dial-a-Ride paratransit services for persons with disabilitiesì schedule
vans, taxis and buses; however, the rider assignments toì each of the fleets
are made independently, resulting in someì inefficiencies. We discuss a
software program that integrates theì rider assignments to these fleets
so as to minimize the total systemì cost.

Auth: James L. Bander, Chelsea C. White, III
Titl: Information Technology & Trucking
Mtng: INFORMS San Diego, May, 1997
Locn: Contribute,
Time: Sunday 16:30-18:00
ID : SE14.3
Addr: Univ. of MI, 200 Eng. Programs Bldg., 2609 Draper St., Ann Arbor, MI 48109-2140,
Abst:

Auth: James L. Bander, Chelsea C. White, III
Titl: A Shortest Path Problem with Stochastic & Dynamic Arc Lengths
Mtng: INFORMS Montreal, April, 1998
Locn: Contribute, St. Leonard
Time: Wednesday 12:30-14:0Networks & Graphs 2
ID : WC35.1
Addr: Univ. of MI, 1205 Beal Ave., IOE Dept., Ann Arbor, MI 48105 ,
Abst: We present computational experience with a variation on the shortest path
problem in which the lengths of network arcs are random variables whose
distribution depends on both the state of the network and the time that
the arc is traversed.

Auth: James L. Bander, Chelsea C. White, III
Titl: Solution Techniques for a Stochastic Shortest Path Problem
Mtng: INFORMS Dallas, October, 1997
Locn: Contribute, Colonnade E
Time: Wednesday 09:45-11:1Stochastic Modeling & Analysis
ID : WB29.1
Addr: Univ. of MI, IOE Bldg., 1205 Beal Ave., Ann Arbor, MI 48109-2140,
Abst: We consider a variant of the shortest path problem in which the travel
times along each arc are random variables whose distribution depends on
the state of the network. The problem is formulated as a SMDP. We compare
exact solution techniques to suboptimal designs.

Auth: Chelsea White, III, James L. Bander
Titl: Inbound Logistics at the Michigan-Ontario Border (R)
Mtng: INFORMS Atlanta, Nov, 1996
Locn: Contribute, Rockdale
Time: Wednesday 13:00-14:3Inbound Logistics
ID : WC16.3
Addr: Univ. of MI, 1205 Beal Ave., Ann Arbor, MI 48109 ,
Abst: Trucks delivering freight across the US-Canadian border to US autoì assembly
plants may find it optimal to change the initiallyì determined route (and
hence tour, if applicable) enroute as trafficì and weather conditions change.

We examine the role of optimizationì and communication technology in this highly competitive environment.

ID : TC32.4
Auth: Norton, J. T., Bander, J. L.; Sulzberg, J. D.; Schlactus, L.
Titl: Expert Diagnostic System
Mtng: ORSA/TIMS Philadelphia; October, 1990
Time: Tuesday 2:15pm-3:35pm
Locn: Contributed Session/Suite 701
Addr: University of Virginia, Department of Systems Engineering, Thornton Hall, Charlottesville, VA 22903; University of Virginia; University of Virginia; University of Virginia
Abst: We describe the knowledge acquisition, conceptual analysis and implementation of an Expert Diagnostic System for textile dye ranges. The Expert System improved productivitity and formalized troubleshooting when implemented at a textile plant. We pay particular attention to changes in the knowledge base during the Expert System Development Life Cycle.

Auth: James L. Bander
Titl: The Money Pump: When Not to Use Utility Models
Mtng: INFORMS Atlanta, Nov, 1996
Locn: Contribute, Douglas
Time: Wednesday 13:00-14:3Multicriteria Decision Making
ID : WC13.1
Addr: Univ. of MI, Dept. of Ind. & Op. Eng., 1205 Beal Ave., Ann Arbor, MI 48109-2140,
Abst: Utility and expectancy models were developed for normative modeling of decision-making behavior, but they have been widely used for both normative and descriptive purposes. The point of this presentation is to raise questions about the appropriateness of utility and expectancy models for descriptive modeling of human behavior.

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Quiz #9 – Fall 1999

A. ***Manufacturing System with Inspection & Rework***: Consider a system in which there are three machining operations, each followed by an inspection. Relevant data are:

OPERATION	TIME RQMT. (man-hrs)	OPERATING COST ($/hr.)	SCRAP RATE %	%SENT BACK FOR REWORK
Machine A	1.5	20.00	15	
Inspection A	0.25	8.00	4	8
Machine B	1.0	16.00	6	
Inspection B	0.25	8.00	5	4
Machine C	1.5	20.00	5	
Inspection C	.5	8.00	9	7
Pack & Ship	0.25	8.00		

Consult the computer output below to answer the questions:

1. What percent of the parts which are started are successfully completed? *Choose nearest value.*
   a. 50% b. 55% c. 60% $\boxed{\text{d. 65\%}}$ *(62.8%)*
   e. 70% f. 75% g. 80% h. 85%
2. What is the **expected** number of blanks which are required to fill the order for 10 parts? *Choose nearest value*
   a. 11 b. 12 c. 13 d. 14
   e. 15 $\boxed{\text{f. 16}}$ *(10/0.628=15.9)* g. 17 h. 18
3. What is the probability that a part which passes inspection B will ultimately be scrapped? *Choose nearest value.*
   a. 5% b. 10% $\boxed{\text{c. 15\%}}$ *(12.7%)* d. 20%
   e. 25% f. 30% g. 35% h. 40%
4. What is the expected number of times that a part is inspected? *Choose nearest value*
   a. 1 b. 1.5 c. 2 $\boxed{\text{d. 2.5}}$ *(0.912+ .791+ 0.748=2.452)*
   e. 3 f. 3.5 g. 4 h. 4.5
5. If a part reaches Machine C, what is the probability that it will be successfully completed? *Choose nearest value.*
   a. 60% b. 65% c. 70% d. 75%
   e. 80% $\boxed{\text{f. 85\%}}$ (87.26%) g. 90% h. 95%

Transition Probability Matrix

from	1	2	3	4	5	6	7	8
1	0	0.85	0	0	0	0	0	0.15
2	0.08	0	0.88	0	0	0	0	0.04
3	0	0	0	0.94	0	0	0	0.06
4	0	0	0.05	0	0.91	0	0	0.04
5	0	0	0	0	0	0.95	0	0.05
6	0	0	0	0	0.09	0	0.84	0.07
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

A = Absorption Probabilities

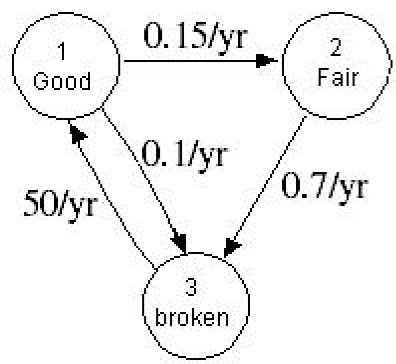
from	7	8
1	0.62861	0.37139
2	0.739541	0.260459
3	0.783241	0.216759
4	0.833235	0.166765
5	0.872608	0.127392
6	0.918535	0.0814653

E = Expected No. Visits to Transient States

from	1	2	3	4	5	6
1	1.07296	0.912017	0.842156	0.791627	0.787732	0.748345
2	0.0858369	1.07296	0.990772	0.931326	0.926743	0.880406
3	0	0	1.04932	0.986359	0.981505	0.93243
4	0	0	0.0524659	1.04932	1.04415	0.991947
5	0	0	0	0	1.09349	1.03882
6	0	0	0	0	0.0984144	1.09349

B. Continuous-time Markov Chains. Consider the vehicle replacement problem:

I own one car. At any time, my current car is in good, fair, or broken-down condition. My policy is to drive my car until it breaks down, at which time I replace it. I have modeled the process as a continuous-time Markov chain, with the transition diagram below. (Transition rates are shown.) It costs me \$9000 to purchase a good car; a broken-down car has no trade-in. It costs me \$1000/yr to operate a good car and \$1500/yr to operate a fair car.



1. What is the value of the matrix L of transition rates?

$$\Lambda = \begin{bmatrix} -0.25 & 0.15 & 0.1 \\ 0 & -0.7 & 0.7 \\ 50 & 0 & -50 \end{bmatrix}$$

2. The probability distribution of the length of time between purchase of a car and when it has deteriorated to a "fair" car is

   a. Uniform b. Normal $\boxed{\text{c. Exponential}}$
   d. Markov e. Gamma f. *None of the above*

3. Suppose that I have just purchased a car. What is the probability that this (good) car will change its state within the next year?

   a. $1 - e^{0.25}$ $\boxed{\text{b. } 1 - e^{-0.25}}$
   c. $e^{0.25}$ d. $e^{-0.25}$ e. *None of the above*

4. Suppose that I purchased my current car one year ago. Then the probability that one year from now my car will not have deteriorated into a "fair" car is

   a. $1 - e^{0.25}$ b. $1 - e^{-0.25}$ c. $1 - e^{0.5}$ d. $1 - e^{-0.5}$
   e. $e^{0.25}$ $\boxed{\text{f. } e^{-0.25}}$ g. $e^{0.5}$ h. $e^{-0.5}$ i.. *None of the above*

5. Which (one or more) of the following equations describe the steadystate probability distribution?

   a. $\pi_1 + \pi_2 + \pi_3 = 0$ $\boxed{\text{d. } \pi_1 + \pi_2 + \pi_3 = 1}$ g. $0.15\pi_1 = 0.7\pi_2 + 0.1\pi_3$
   b. $\pi_1 = 0.15\pi_2 + 0.1\pi_3$ e. $\pi_1 = 0.15\pi_1 + 0.7\pi_2 + 50\pi_3$ $\boxed{\text{h. } 0.25\pi_1 = 50\pi_3}$
   $\boxed{\text{c. } 0.15\pi_1 = 0.7\pi_2}$ f. $0.25\pi_1 = 0.7\pi_2 + 50\pi_3$ i. $0.15\pi_1 = 50\pi_3$

6. Suppose that the steadystate probabilities are $\pi = (0.8, 0.195, 0.005)$. *(Not the actual values!)* Then the expected time T between replacements, measured in years, is *(choose nearest value):*

   a. 1 b. 1.5 c. 2
   d. 2.5 e. 3 f. 3.5
   $\boxed{\text{g. } 4}$ h. 4.5 i. 5

Note:

$$\pi_3 = 0.005 = \frac{\text{average time per visit to state 3}}{\text{average cycle time}} = \frac{0.02 \text{ years}}{T}$$

$$\Rightarrow T = \frac{0.02 \text{ years}}{0.005} = 4 \text{ years}$$

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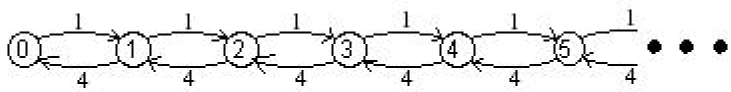
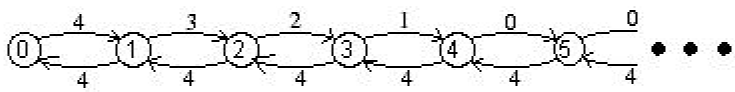
For each diagram of a Markov model of a queue in (1) through (5) below, indicate the correct Kendall's classification from among the following choices :

(a) M/M/1	(b) M/M/2	(c) M/M/1/4
(d) M/M/4	(e) M/M/2/4	(f) M/M/2/4/4
(g) M/M/1/4/4	(h) M/M/4/2	(i) M/M/4/4
(j) M/M/2/2/4	(k) M/M/1/4/2	(l) *none of the above*

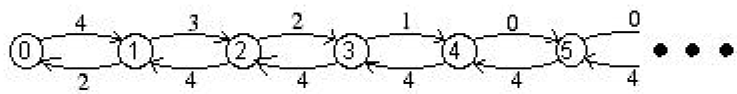
<ins>M/M/1</ins> (1.)



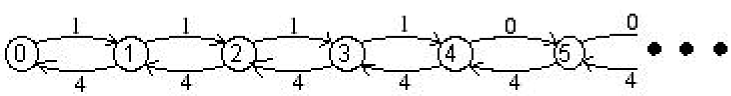
<ins>M/M/1/4/4</ins> (2.)



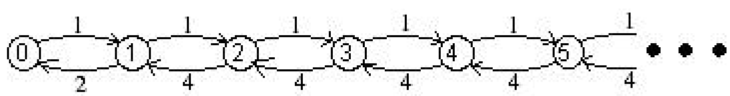
<ins>M/M/2/4/4</ins> (3.)



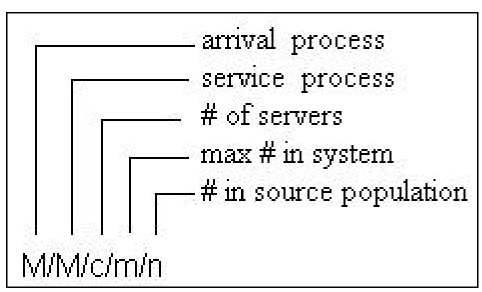
<ins>M/M/1/4</ins> (4.)



<ins>M/M/2</ins> (5.)



Note: Kendall's notation:

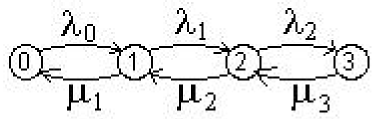


A machine operator has the task of keeping three machines running. Each machine runs for an average of 1 hour before it becomes jammed or otherwise needs the operator's attention. He then spends an average of twelve minutes restoring the machine to running condition. Define a continuous-time Markov chain, the state of the system being the number of machines <u>not</u> running.

6. $\boxed{\text{True}}$ or False (circle): This Markov chain is a birth/death process.

7. Specify the letter for each of the transition rates:

λ_0= 3/hr	λ_1 = 2/hr	λ_2 = 1/hr
μ_1 = 5/hr	μ_2 = 5/hr	μ_3 = 5/hr



8. Which equation is used to compute the steady-state probability π_0?

(a.) $\dfrac{1}{\pi_0} = 1 + \dfrac{\lambda_0}{\mu_1} + \dfrac{\lambda_1}{\mu_2} + \dfrac{\lambda_2}{\mu_3}$

$\boxed{\text{(e.)}}$ $\dfrac{1}{\pi_0} = 1 + \dfrac{\lambda_0}{\mu_1} + \dfrac{\lambda_0}{\mu_1}\dfrac{\lambda_1}{\mu_2} + \dfrac{\lambda_0}{\mu_1}\dfrac{\lambda_1}{\mu_2}\dfrac{\lambda_2}{\mu_3}$

(b.) $\pi_0 = 1 + \dfrac{\lambda_0}{\mu_1} + \dfrac{\lambda_0}{\mu_1}\dfrac{\lambda_1}{\mu_2} + \dfrac{\lambda_0}{\mu_1}\dfrac{\lambda_1}{\mu_2}\dfrac{\lambda_2}{\mu_3}$

(f.) $\pi_0 = 1 + \dfrac{\lambda_0}{\mu_1} + \dfrac{\lambda_1}{\mu_2} + \dfrac{\lambda_2}{\mu_3}$

(c.) $\pi_0 = 1 \times \dfrac{\lambda_0}{\mu_1} \times \dfrac{\lambda_1}{\mu_2} \times \dfrac{\lambda_2}{\mu_3}$

(g.) $\dfrac{1}{\pi_0} = 1 \times \dfrac{\lambda_0}{\mu_1} \times \dfrac{\lambda_1}{\mu_2} \times \dfrac{\lambda_2}{\mu_3}$

(d.) $\pi_0 = \dfrac{\lambda_0}{\mu_1} \times \dfrac{\lambda_1}{\mu_2} \times \dfrac{\lambda_2}{\mu_3}$

(h) None of the above

____9. What is the relationship between π_0 and π_1 *for this system?*

a. $\pi_1 = \pi_0$ | b. $\pi_1 = 0.1\ \pi_0$ | $\boxed{\text{c.}}$ $\pi_1 = 0.6\ \pi_0$

d. $\pi_1 = 1/6 \times \pi_0$ | e. $\pi_1 = 3\ \pi_0$ | f. *None of the above*

____10. <u>If</u> the average number of machines <u>not</u> running were 0.5 and the average time between machine jams were 0.4 hr, what is the average turnaround time (waiting plus service time) to restore a machine to running condition? *(Choose nearest answer)*

a. 0.1 hour | $\boxed{\text{c.}}$ 0.2 hour | e. 0.3 hour

b. 0.4 hour | d. 0.5 hour | f. 0.6 hour

Note: Little's Law: L = λW where L = 0.5 and λ = 1/(0.4hr) = 2.5/hr so that W = L/λ = 0.2 hr.

56:171 Operations Research
Quiz #11 Solutions -- Fall 1999

Part I: Suppose that a new car costs \$10,000 and that the annual operating cost & trade-in value are as follows

Age of car (years)	Trade-in value	Operating cost in previous year
1	\$7000	\$300
2	\$6000	\$500
3	\$4000	\$800
4	\$3000	\$1200
5	\$2000	\$1600
6 or more	\$1000	\$2200

I wish to determine the replacement policy that, starting with a new car, minimizes my net cost of owning and operating a car for the next ten years (from t=0 until t=10)? (Do not include the cost of the initial car.)

As in the class notes, define:
G(t) = minimum total cost incurred from time t until the end of the planning period, if a new car has just been purchased. *(Note: this does not include the cost of purchasing this initial new car.)*
X*(t) = optimal replacement time for a car which has been purchased at the beginning of period t.

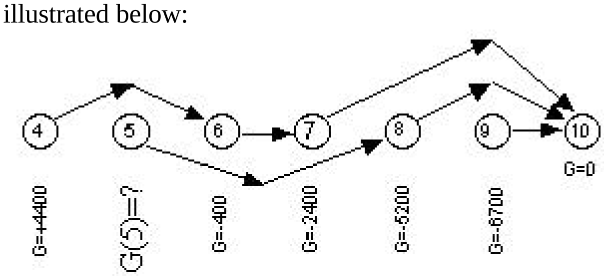
The optimal value function G(t) is defined recursively by

$$G(t) = \underset{t+1 \le x \le T}{\text{minimum}} \left\{ \sum_{i=1}^{x-t} C_i - S_{x-t} + P_x + G(x) \right\}$$

where

$G(10) = 0$
P_t = purchase price of a new car at time t
C_i = cost of operation & maintenance of a car in its i^{th} year.
S_j = trade-in value of a car of age j

The computation of G(4) through G(10), i.e., for the final 6 years, has already been done in the example presented in class, and is illustrated below:



1. What is the value of G(5)? <u>\$ 2400</u>

Note: Since the diagram indicates that a car which is new at the beginning of year 5 should be traded in at year 8, G(5) will be

- the cost of operating the car for 3 years (300+500+800),
- minus the trade-in value of a 3-year-old car (– 4000)
- plus the cost of the replacement car (10000)
- plus the cost from year 8 until the end of the planning period (G(8) = –5200).

2. If I purchase a new car at the *end* of year 4, i.e., at time t=4, how many additional cars should I purchase until the end of the planning period? __2__

Note: *the diagram indicates that trade-ins will occur at the beginning of years 6, 7, and 10. No new car is to be purchased at the end of the planning period*

3. If I purchase a new car at the *end* of year 4, i.e., at time t=4, what is my average cost/year until the end of the planning period? $ 733.33/year *Note: $4400/6 years = $733.33/year*

Part II. *Optimal Reliability by means of redundancy.* A system consists of three components, each of which is necessary for the operation of the system. The weight and the reliability of each component, i.e., the probability that the component survives for the system's intended lifetime, is shown in the table below:

Component	Weight (kg)	Reliability (%)
1	1	70
2	2	80
3	1	75

The total weight of the system is to be no more than 7 kg. We will use dynamic programming to determine how many redundant units of each component should be included in order to maximize the reliability of the system.

The stages correspond to the three types of components. We will perform a backward recursion, in which we imagine that we are deciding first how many units of type 3 are to be included, then type 2, and finally type 1. The state ***s*** of the system at stage ***n*** is the number of kg remaining to be filled with components n, n-1, … 1, and the optimal value $V_n(s)$ is the maximum reliability that can be attained for the subsystem consisting of components of type n, n-1, …1 if s kg are available. The computations are done first for stage 1, then stage 2, and finally stage 3.

Optimal System Reliability Using Redundancy

Recursion type: backward

---Stage 1---

s \ x:	1	2	3
1	0.7000	-∞	-∞
2	0.7000	0.9100	-∞
3	0.7000	0.9100	0.9730
4	0.7000	0.9100	0.9730
5	0.7000	0.9100	0.9730
6	0.7000	0.9100	0.9730
7	0.7000	0.9100	0.9730

State s	Optimal Values $V_1(s)$	Optimal Decisions	Resulting State
1	0.7000	1	0
2	0.9100	2	0
3	0.9730	3	0
4	0.9730	3	1
5	0.9730	3	2
6	0.9730	3	3
7	0.9730	3	4

---Stage 2---			
s \ x:	1	2	3
3	0.5600	-∞	-∞
4	0.7280	-∞	-∞
5	0.7784	0.6720	-∞
6	0.7784	0.8736	-∞
7	0.7784	0.9341	0.6944

State s	Optimal Values $V_2(s)$	Optimal Decisions	Resulting State
3	0.5600	1	1
4	0.7280	1	2
5	0.7784	1	3
6	0.8736	2	2
7	0.9341	2	3

---Stage 3---			
s \ x:	1	2	3
4	0.4200	-∞	-∞
5	0.5460	0.5250	-∞
6	0.5838	0.6825	0.5512
7	**??????**	0.7298	0.7166

State s	Optimal Values $V_3(s)$	Optimal Decisions	Resulting State
4	0.4200	1	3
5	0.5460	1	4
6	0.6825	2	4
7	0.7298	2	5

_e_ 4. What is the reliability of a subsystem consisting of 2 units of component #1?

a. 0.7^2 b. 0.3^2 c. $1-0.7^2$
d. 2×0.7 $\boxed{\text{e. } 1-0.3^2}$ f. *None of the above*

5. What is the missing value in the table at stage 3? 0.6552
Note: if the state of the system is 7 at stage 3, and 1 unit of component 3 is included, then the reliability of the resulting system will be 75% · $V_2(7-1) = 0.75 \cdot 0.8736 = 0.6552$

6. What is the maximum reliability that can be obtained using redundant units with a weight restriction of 7 kg.? 72.98% *Note: this is $V_3(7)$ above*.

7. If only six kg were available, the maximum reliability that could be achieved is 68.25%

8. If only six kg. were available, the optimal design would include:
_2_ units of component #1
_1_ units of component #2
_2_ units of component #3

56:171 Operations Research
Quiz #12 Solutions -- Fall 1999

Part I: ***Production Planning*** We wish to plan production of an expensive, low-demand item for the next three months (January, February, & March).

- the cost of production is \$10 for setup, plus \$5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is \$2 per unit, based upon the level at the beginning of the month.
- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand D is random, with the same probability distribution each month:

demand d	0	1	2
P{D=d}	0.2	0.5	0.3

- there is a penalty of \$25 per unit for any demand which cannot be satisfied. Backorders are not allowed.
- the initial inventory (i.e., the inventory at the end of December) is 1.
- a salvage value of \$4 per unit is received for any inventory remaining at the end of the last month (March)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 3 = January, stage 2 = February, etc.** (i.e., n = # months remaining in planning period.)

a. What is the optimal production quantity for January? __0____

b. What is the total expected cost for the three months? __\$36.364 = $f_3(1)$

c. If, during January, the demand is 1 unit, what should be produced in February? __3_= $X^*_2(0)$

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ __\$24.30__

ii. the optimal decision $x_2^*(1)$ ____0_____

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ___\$37.36____

Note: (storage cost)+(prod'n cost)+Expected future cost

$$= 0 + 15 + [\ (0.2)f_1\ (1) + (0.5)f_1(0)+(0.3)(25+f_1\ (0))]$$
$$= 15 + (0.2)(8.7) + (0.5)(16.4) + (0.3)(25+16.4) = 37.36$$

```
==========================================================
                     ---Stage 1---
   s \ x:   0        1        2        3        4

   0 |   27.5000  21.7000  16.4000  17.4000  19.2000
   1 |    8.7000  13.4000  14.4000  16.2000  20.0000
   2 |   -0.4000  11.4000  13.2000  17.0000  22.0000
   3 |   -1.6000  10.2000  14.0000  19.0000  24.0000

        Optimal  Optimal
 State  Values   Decision

   0 |   16.4000 |    2
   1 |    8.7000 |    0
   2 |   -0.4000 |    0
   3 |   -1.6000 |    0

==========================================================
                    ---Stage 2---
   s \ x:   0        1        2        3        4

   0 |   43.9000  ???????  29.3500  27.4900  29.0000
   1 |   24.3600  26.3500  24.4900  26.0000  30.4000
   2 |   13.3500  21.4900  23.0000  27.4000  32.4000
   3 |    8.4900  20.0000  24.4000  29.4000  34.4000
```

State	Optimal Values	Optimal Decision
0	27.4900	3
1	$\boxed{???????}$	$\boxed{?}$
2	13.3500	0
3	8.4900	0

===

---Stage 3---

s \ x:	0	1	2	3	4
0	54.9900	49.3640	43.0970	40.6810	39.9480
1	36.3640	40.0970	37.6810	36.9480	40.4900
2	27.0970	34.6810	33.9480	37.4900	42.4900
3	21.6810	30.9480	34.4900	39.4900	44.4900

State	Optimal Values	Optimal Decision
0	39.9480	4
1	36.3640	0
2	27.0970	0
3	21.6810	0

===

Part II. ***Markov chains*** The Green Valley Christmas Tree Farm owns a plot of land with 5000 evergreen trees. Each year they allow individuals to select and cut Christmas trees. However, they protect small trees (usually less than 4 feet tall) so that they will grow and be available for sale in future years. Currently 2000 trees are classified as protected trees, while the remaining 3000 are available for cutting. However, even though a tree is available for cutting in a given year, it possibly might not be selected for cutting until future years. While most trees not cut in a given year live until the next year (protected or unprotected), approximately 15% are lost to disease. Each year, approximately 50% of the unprotected trees are cut, and 40% of the protected trees surviving from the previous year have matured sufficiently to be made available for cutting.

Define a Markov chain model of the system consisting of a **single** tree, with states (1) protected, (2) unprotected, (3) dead, (4) cut & sold. The transition probability matrix is

$$P = \begin{bmatrix} 0.51 & 0.34 & 0.15 & 0 \\ 0 & 0.425 & 0.075 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following computations were performed:

$$(I\text{-}Q)^{-1} = \begin{bmatrix} 0.49 & -0.34 \\ 0 & 0.575 \end{bmatrix}^{-1} = \begin{bmatrix} 2.0408 & 1.2067 \\ 0 & 1.7391 \end{bmatrix} = E \text{ \textit{(expected stages in transitive states)}}$$

$$ER = \begin{bmatrix} 2.0408 & 1.2067 \\ 0 & 1.7391 \end{bmatrix} \begin{bmatrix} 0.15 & 0 \\ 0.075 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.3966 & 0.6034 \\ 0.1304 & 0.8696 \end{bmatrix} = A \text{ \textit{(absorption probabilities)}}$$

1. What are the absorbing states of this model? <u>3&4</u>
2. What is the probability that a tree which is protected is eventually sold? a_{13} = <u>39.66%</u>
3. What is the probability that a protected tree eventually dies of disease? a_{14} = <u>60.34%</u>
4. How many of the farm's 5000 trees are expected to be sold eventually? <u>2315.6</u>
   Note: $2000a_{14} + 3000a_{24} = 2000(0.6034) + 3000(0.3696) = 2315.6$
5. If a tree is initially protected, what is the expected number of years until it is either sold or dies? <u>2.2478 years</u>
   Note: $e_{11} + e_{12} = 2.0408 + 1.207 = 3.2478$. *That is, the tree will visit the protected state an expected 2.0408 times (including its initial visit) and the unprotected state 1.2067 times. Because this includes the initial visit to state 1, so the number is actually one less, i.e., 2.2478 years*