56:171

Fall 2002

Operations Research

Homework Solutions

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Homework #1 Solutions –Fall 2002

1. The Keyesport Quarry has two different pits from which it obtains rock. The rock is run through a crusher to produce two products: concrete grade stone and road surface chat. Each ton of rock from the South pit converts into 0.75 tons of stone and 0.25 tons of chat when crushed. Rock from the North pit is of different quality. When it is crushed it produces a "50-50" split of stone and chat. The Quarry has contracts for 60 tons of stone and 40 tons of chat this planning period. The cost per ton of extracting and crushing rock from the South pit is 1.6 times as costly as from the North pit.

   a. What are the decision variables in the problem? *Be sure to give their definitions, not just their names!*
   Answer: S_ROCK = # of tons of rocks from the South pit.
   N_ROCK = # of tons of rocks from the North pit.

   b. There are two constraints for this problem.
   - State them in words.

   Answer:
   1. # of tons of concrete grade stone which is the sum of concrete grade stone from South pit and concrete grade stone from North pit is bigger than 60.
   2. # of tons of road surface chat which is the sum of road surface chat from South pit and road surface chat from North pit is bigger than 40.

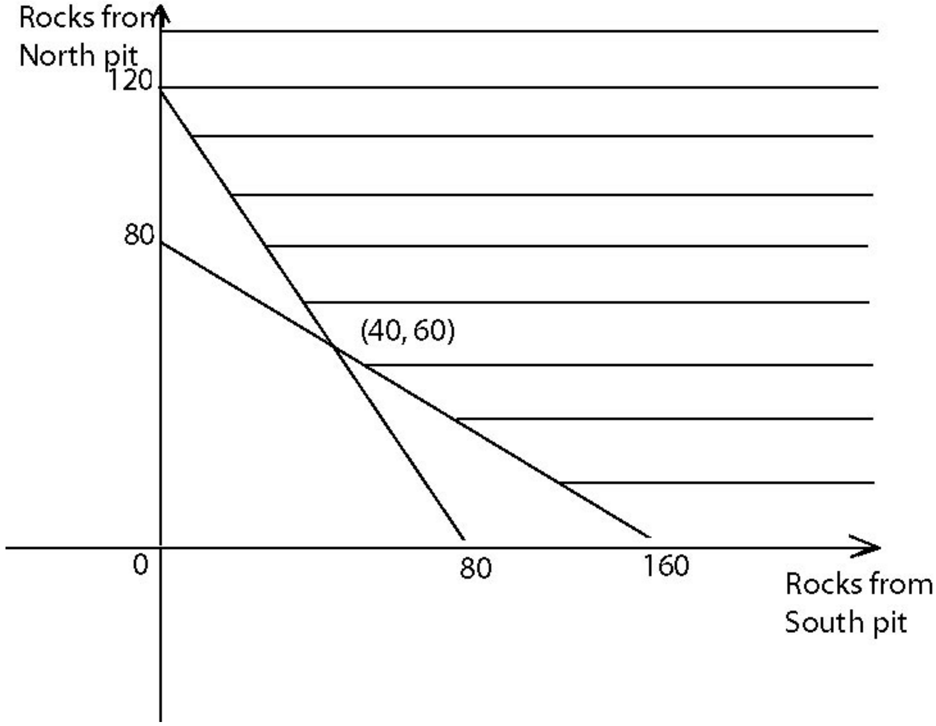
   - State them in equation or inequality form.

   Answer:
   ```
   0.75 S_ROCK + 0.5 N_ROCK ≥ 60
   0.25 S_ROCK + 0.5 N_ROCK ≥ 40
   ```

   c. State the objective function.
   Answer:
   Total cost of the processing rocks which is the sum of the cost of processing rocks from South pit and North pit in the unit of the cost of processing 1 ton's processing North pit (to be minimized):
   ```
   Min 1.6 S_ROCK + N_ROCK
   ```

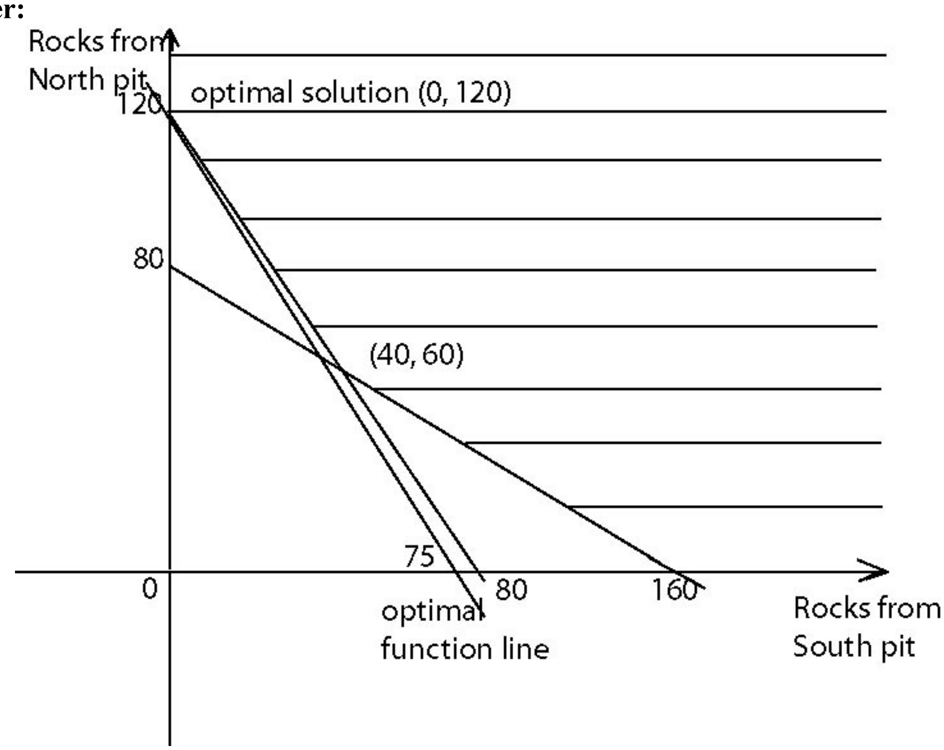
   d. Graph the feasible region (in 2 dimensions) for this problem.

Answer:



e. Draw an appropriate objective function line on the graph and indicate graphically and numerically the optimal solution.

Answer:



f. Use LINDO (or other appropriate LP solver) to compute the optimal solution.

Answer:

```
 LP OPTIMUM FOUND AT STEP      1

        OBJECTIVE FUNCTION VALUE

        1)      120.0000

  VARIABLE        VALUE          REDUCED COST
    S_ROCK         0.000000          0.100000
    N_ROCK       120.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -2.000000
        3)        20.000000          0.000000
```

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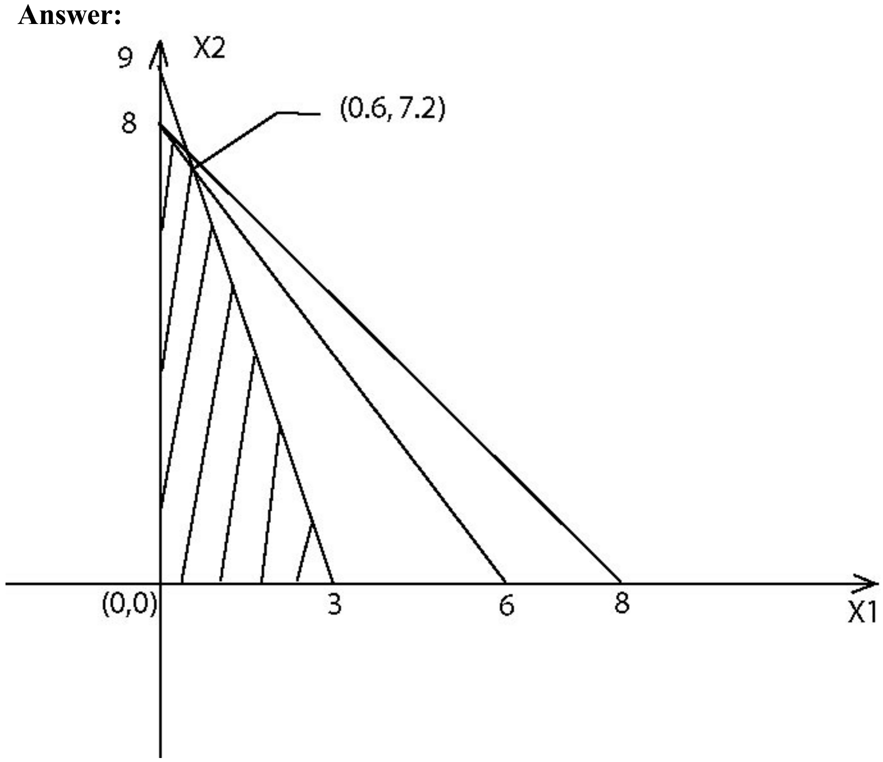
2\. a. Draw the feasible region of the following LP:

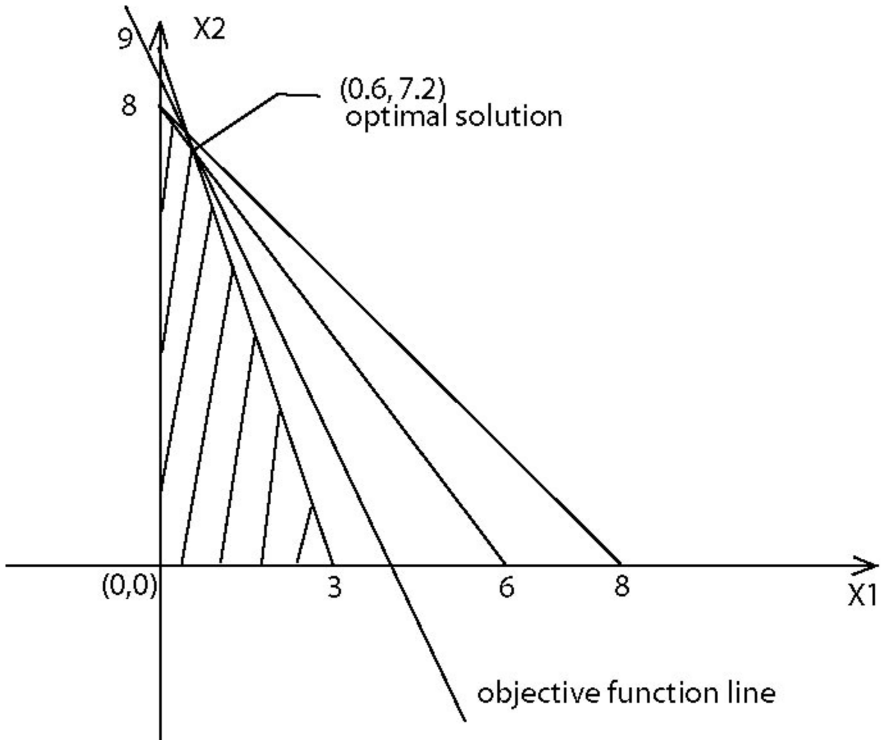
Maximize	$5X_1 + 2X_2$
subject to	$4X_1 + 3X_2 \leq 24$
	$X_1 + X_2 \leq 8$
	$3X_1 + X_2 \leq 9$
	$X_1 \geq 0, X_2 \geq 0$

Answer:



Note that the point (0,8) is the intersection of three boundary lines, indicating a degeneracy!

b. Indicate on the graph the optimal solution.
Answer:



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3. a. Compute the inverse of the matrix (showing your computational steps):

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 + 2R_1 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 = R_2 / 2 \\ R_3 = R_3 + R_2 / 2 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1/2 & 3/2 & 1/2 & 1 \end{bmatrix} \begin{matrix} R_1 = R_1 - 2R_3 \\ R_2 = R_2 + R_3 \\ R_3 = -2R_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ -3 & -1 & -2 \end{bmatrix}$$

b. Find a solution (if one exists) of the equations:

$$\begin{cases} X_1 + 2X_2 - X_3 = 4 \\ 2X_1 - X_2 + 2X_3 = 15 \\ 3X_2 - 2X_3 = -5 \end{cases}$$

Answer:

Using Gauss elimination, we reduce the augmented coefficient matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 2 & 15 \\ 0 & 3 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 4 & 7 \\ 0 & 3 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -4/5 & -7/5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\text{By back substitution } \begin{cases} X_1 = 4 - 2X_2 + X_3 = 8 \\ X_2 = -7/5 + (4/5)X_3 = -3 \\ X_3 = -2 \end{cases}$$

Note that this could also have been done by Gauss-Jordan elimination.

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Homework #2 Solutions – Fall 2002

1. (Exercise 3.4-18, page 98, of Hillier&Lieberman text, 7th edition)
"Oxbridge University maintains a powerful mainframe computer for research use by its faculty, Ph.D. students, and research associates. During all working hours, an operator must be able to operate and maintain the computer, as well as to perform some programming services. Beryl Ingram, the director of the computer facility, oversees the operation.
It is now the beginning of the fall semester, and Beryl is confronted with the problem of assigning different working hours to her operators. Because all the operators are currently enrolled in the university, they are available to work only a limited number of hours each day, as shown in the following table.

		Maximum # hours available				
Name	Wage $/hr	Mon	Tues	Wed	Thur	Fri
K.C.	10.00	6	0	6	0	6
D.H.	10.10	0	6	0	6	0
H.B.	9.90	4	8	4	0	4
S.C.	9.80	5	5	5	0	5
K.S.	10.80	3	0	3	8	0
N.K.	11.30	0	0	0	6	2

There are six operators (four undergraduate students and two graduate students). They all have different wage rates because of differences in their experience with computers and in their programming ability. The above table shows their wage rates, along with the maximum number of hours that each can work each day.
Each operator is guaranteed a certain minimum number of hours per week that will maintain an adequate knowledge of the operation. This level is set arbitrarily at 8 hours per week for the undergraduate students (K.C,, D.H, H.B, and S.C.) and 7 hours per week for the graduate students (K.S. and N.K.).
The computer facility is to be open for operation from 8 a.m. to 10 p.m. Monday through Friday with exactly one operator on duty during these hours. On Saturdays and Sundays, the computer is to be operated by other staff.
Because of a tight budget, Beryl has to minimize cost. She wishes to determine the number of hours she should assign to each operator on each day."

a. Formulate a linear programming model for this problem. Be sure to define your variables!

Answer:
Define decision variables
X_{ij} = # hours operater i is assigned to work on day j
for all *i=1… 6 (where 1=KC, 2=DH,… 6= NK); j=1,… 5 (where 1=MON, 2=TUE, … 5=FRI)*

Minimize $z= 10(X_{11} + X_{13} + X_{15}) + 10.1(X_{22} + X_{24}) + 9.9(X_{31} + X_{32} + X_{33}+X_{35}) + 9.8(X_{41} + X_{42} + X_{43} + X_{45}) + 10.8(X_{51} + X_{53} + X_{54}) + 11.3(X_{64} + X_{65})$

subject to

maximum number hours available each day:

$X_{11}\le 6$	$X_{22}\le 6$	$X_{31}\le 4$	$X_{41}\le 5$	$X_{51}\le 3$	$X_{64}\le 6$
$X_{13}\le 6$	$X_{24}\le 6$	$X_{32}\le 8$	$X_{42}\le 5$	$X_{53}\le 3$	$X_{65}\le 2$
$X_{15}\le 6$		$X_{33}\le 4$	$X_{43}\le 5$	$X_{54}\le 8$	
		$X_{35}\le 4$	$X_{45}\le 5$		

number of hours guaranteed for each operator:

$X_{11} + X_{13} + X_{15} \ge 8$

$X_{22} + X_{24} \ge 8$

$X_{31} + X_{32} + X_{33} + X_{35} \ge 8$

$X_{41} + X_{42} + X_{43} + X_{45} \ge 8$

$X_{51} + X_{53} + X_{54} \ge 7$

$X_{64} + X_{65} \ge 7$

total number hours worked each day is 14:

$X_{11} + X_{31} + X41 + X_{51} = 14$

$X_{22} + X_{23} + X_{42} = 14$

$X_{13} + X_{33} + X_{43} + X_{53} = 14$

$X_{24} + X_{44} + X_{54} + X_{64} = 14$

$X_{15} + X_{35} + X_{45} + X_{65} = 14$

nonnegativity:

$Xij \ge 0$ for all i & j

b. Use an LP solver (e.g. LINDO or LINGO) to find the optimal solution.

The **LINGO** model is as follows:

```
MODEL:     ! Oxbridge University Computer Center;

SETS:
      OPERATOR / KC, DH, HB, SC, KS, NK/: MINIMUM, PAYRATE;
      DAY /MON, TUE, WED, THU, FRI/: REQUIRED;
      ASSIGN(OPERATOR,DAY): AVAILABLE, X;
ENDSETS

DATA:
      MINIMUM = 8 8 8 8 7 7;
      PAYRATE = 10.00  10.10  9.90  9.80  10.80  11.30;
      REQUIRED = 14 14 14 14 14;
      AVAILABLE=  6  0  6  0  6
                  0  6  0  6  0
                  4  8  4  0  4
                  5  5  5  0  5
                  3  0  3  8  0
                  0  0  0  6  2;

ENDDATA

MIN = TOTALPAY;

!     total weekly payroll cost;
      TOTALPAY = @SUM(ASSIGN(I,J)|AVAILABLE(I,J) #NE# 0:
PAYRATE(I)*X(I,J) );
```

```
!     must schedule required hours each day;
      @FOR(DAY(J):
            @SUM(OPERATOR(I)|AVAILABLE(I,J) #NE# 0:  X(I,J) ) =
REQUIRED(J) );

!     must schedule each operator at least minimum number of hours;
      @FOR(OPERATOR(I):
            @SUM(DAY(J)|AVAILABLE(I,J) #NE# 0:  X(I,J) ) >= MINIMUM(I)
);

!     upper (& lower) bounds on variables;
      @FOR(ASSIGN(I,J)| AVAILABLE(I,J) #NE# 0:
            @BND(0, X(I,J), AVAILABLE(I,J) ); );
END
```

Note the use of the logical expression "`AVAILABLE(I,J) #NE# 0`" in order to avoid defining and referencing assignments in which the operator is not available.
Note also the use of @BND to impose the upper bounds instead of

```
      @FOR(ASSIGN(I,J)| AVAILABLE(I,J) #NE# 0:
            X(I,J) <= AVAILABLE(I,J)   );
```

The syntax is @BND(*lower_bound*, *variable_name*, *upper_bound*);

```
Global optimal solution found at step:          10
 Objective value:                         709.6000

                   Variable           Value        Reduced Cost
                   TOTALPAY        709.6000           0.0000000
                X( KC, MON)        3.000000           0.0000000
                X( KC, WED)        2.000000           0.0000000
                X( KC, FRI)        4.000000           0.0000000
                X( DH, TUE)        2.000000           0.0000000
                X( DH, THU)        6.000000          -0.1000000
                X( HB, MON)        4.000000          -0.1000000
                X( HB, TUE)        7.000000           0.0000000
                X( HB, WED)        4.000000          -0.1000000
                X( HB, FRI)        4.000000          -0.1000000
                X( SC, MON)        5.000000          -0.2000000
                X( SC, TUE)        5.000000          -0.1000000
                X( SC, WED)        5.000000          -0.2000000
                X( SC, FRI)        5.000000          -0.2000000
                X( KS, MON)        2.000000           0.0000000
                X( KS, WED)        3.000000           0.0000000
                X( KS, THU)        2.000000           0.0000000
                X( NK, THU)        6.000000           0.0000000
                X( NK, FRI)        1.000000           0.0000000
```

2. (Exercise 4.4-9, page 176, of Hillier&Lieberman text, 7th edition)
Work through the simplex method step by step (in tabular form) to solve the following problem:

Maximize $Z=2X_1 - X_2 + X_3$
subject to
$$3X_1 + X_2 + X_3 \le 6$$
$$X_1 - X_2 + 2X_3 \le 1$$
$$X_1 + X_2 - X_3 \le 2$$
and
$$X_1 \ge 0, X_2 \ge 0, X_3 \ge 0$$

Solution: Include a slack variable in each of the three inequality constraints, S_1, S_2, & S_3. Set up the initial tableau, and use (-Z) and the three slack variables for the initial basis.

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS
1	2	−1	1	0	0	0	0
0	3	1	1	1	0	0	6
0	1	−1	2	0	1	0	1
0	1	1	−1	0	0	1	2

We are maximizing, and so increasing either X_1 or X_3 (both of which have positive "relative profits") would improve, i.e., increase, the objective. We will arbitrarily choose X_1.

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS	*ratio*
1	2	−1	1	0	0	0	0	
0	3	1	1	1	0	0	6	6/3=2
0	**1**	−1	2	0	1	0	1	1/1=1
0	1	1	−1	0	0	1	2	2/1=2

As X_1 increases each of the basic variables S_1, S_2, & S_3 decrease (because of the positive *substitution rates*). The minimum ratio test indicates that the first to reach its lower bound of zero as X_1 increases is S_2, and hence X_1 replaces S_2 in the basis, and the pivot is to be done in the row in which the minimum ratio was computed.

The resulting tableau is:

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS	*ratio*
1	0	1	−3	0	−2	0	−2	
0	0	4	−5	1	−3	0	3	¾=0.75
0	1	−1	2	0	1	0	1	
0	0	**2**	−3	0	−1	1	1	½=0.5

The tableau is not optimal, since there is a positive relative profit in the X_2 column, which is therefore selected as the pivot column. As X_2 increases, S_1 and S_3 decrease (because of the positive substitution rates 4 & 2), and the minimum ratio test indicates that S3 is the first to reach zero. Hence the pivot is performed with the bottom row as the pivot row. The resulting tableau is:

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS
1	0	0	−1.5	0	−1.5	−0.5	−2.5
0	0	0	1	1	−1	−2	1
0	1	0	0.5	0	0.5	0.5	1.5
0	0	1	−1.5	0	0	−0.5	0.5

There is now no positive relative profit in the objective row, and therefore the current basis is optimal and the optimal solution is $X_1 = 1.5$, $X_2 = 0.5$ and $X_3 = 0$ with slack 1, 0, & 0, respectively, in the three constraints. The optimal objective value $Z = 2.5$.

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Homework #3 Solutions – Fall 2002

1. **Revised Simplex Method** Consider the LP problem

Maximize $z = 3x_1 - x_2 + 2x_3$
subject to

$$x_1 + x_2 + x_3 \le 15$$
$$2x_1 - x_2 + x_3 \le 2$$
$$-x_1 + x_2 + x_3 \le 4$$
$$x_j \ge 0, j = 1,2,3$$

a. Let $x_4, x_5, \&, x_6$ denote the slack variables for the three constraints, and write the LP with equality constraints.
***Answer*:**
Maximize $z = 3x_1 - x_2 + 2x_3$
subject to

$$x_1 + x_2 + x_3 + x_4 = 15$$
$$2x_1 - x_2 + x_3 + x_5 = 2$$
$$-x_1 + x_2 + x_3 + x_6 = 4$$
$$x_j \ge 0, j = 1,2,3,4,5,6$$

After several iterations of the revised simplex method,

the basis B={4,3,2} and the basis inverse matrix is $(A^B)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

b. Proceed with one iteration of the revised simplex method, by
i. Computing the simplex multiplier vector π
***Answer*:**

$$\pi = C_B (A^B)^{-1} = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0, & \frac{3}{2}, & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0, & 1.5, & 0.5 \end{bmatrix}$$

ii. "pricing", i.e., computing the "relative profits", of the non-basic columns.
***Answer*:**

$$C^N = \begin{bmatrix}3 & 0 & 0\end{bmatrix}, A^N = \begin{bmatrix}1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1\end{bmatrix}$$

$$\overline{C^N} = C^N - \pi A^N = \begin{bmatrix}\tfrac{1}{2} & -\tfrac{3}{2} & -\tfrac{1}{2}\end{bmatrix}$$

The relative profits for non-basic variables are $\overline{C_1} = 0.5$, $\overline{C_5} = -1.5$, $\overline{C_6} = -0.5$.

iii. Selecting the column to enter the basis.
Answer: Only the relative profit of X_1 is positive and the problem is Max problem, and so X_1 should enter the basic.

iv. Computing the substitution rates of the entering column.
Answer: The substitution rates of the entering variable X_1 is

$$\alpha = (A^B)^{-1} A_1 = \begin{bmatrix}1 & 0 & -1 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & \tfrac{1}{2}\end{bmatrix}\begin{bmatrix}1 \\ 2 \\ -1\end{bmatrix} = \begin{bmatrix}2 \\ \tfrac{1}{2} \\ -\tfrac{3}{2}\end{bmatrix}$$

v. Select the variable to leave the basis.
Answer:

The current right-hand-side is $\beta = X_B = (A^B)^{-1} b = \begin{bmatrix}11 \\ 3 \\ 1\end{bmatrix}$ and the ratios (right-hand-side over *positive* substitution rates) are $\begin{bmatrix}5.5 \\ 6 \\ \cdots\end{bmatrix}$. *(Note that the ratio is not computed for the last row.)*

So by the minimum ratio test, X_1 enters the basis, replacing the basic variable in the first row (the row in which the minimum ratio is found), namely X_4.

vi. Update the basis inverse matrix.
Answer: The new basis is B= {1, 3, 2}. The basis inverse can be updated by writing α, the column of substitution rates, alongside inverse matrix and pivoting in the first row, as shown:

$$\begin{bmatrix}1 & 0 & -1 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & \tfrac{1}{2}\end{bmatrix}\begin{bmatrix}\boxed{2} \\ \tfrac{1}{2} \\ -\tfrac{3}{2}\end{bmatrix} \rightarrow \begin{bmatrix}\tfrac{1}{2} & 0 & -\tfrac{1}{2} \\ -\tfrac{1}{4} & \tfrac{1}{2} & \tfrac{3}{4} \\ \tfrac{3}{4} & -\tfrac{1}{2} & -\tfrac{1}{4}\end{bmatrix}\begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}.$$

c. Write the dual of the above LP (i.e. with equality constraints & slack variables) in (a).
Answer: Minimize $z = 15y_1 + 2y_2 + 4y_3$
subject to
$y_1 + 2y_2 - y_3 \geq 3$

$$y_1 - y_2 + y_3 \geq -1$$
$$y_1 + y_2 + y_3 \geq 2$$
$$y_j \geq 0, j = 1,2,3$$

d. Substitute the vector π which you computed above in step (i) above to test whether it is feasible in the dual LP. Which constraint(s) if any are violated? How does this relate to the results in step (ii) above?

Answer: If we substitute $\pi = [0, \quad 1.5, \quad 0.5]$ for the dual variables y, the first constraint $y_1 + 2y_2 - y_3 \geq 3$ is violated.

Note: The simplex multipler vector π satisfies all the constraints in the dual problem *if & only if* the relative profits in (ii) are all non-positive (which implies that the solution is optimal).

2. LP formulation: *Staffing a Call Center* (Case 3.3, pages 106-108, *Intro. to O.R.* by Hillier & Lieberman) Answer parts (a), (b), & (c) on page 108, using LINGO with sets to enter the model.

For the following analysis, consider the labor cost for the time employees spent answering phones. The cost for paperwork time is charged to other cost centers.

a. How many Spanish-speaking operators and how many English-speaking operators does the hospital need to staff the call center during each 21-hour shift of the day in order to answer all calls? Please provide an integer number since half a human operator makes no sense.

Answer:

Work shift	Spanish Calls/hr	English Calls/hr	Spanish Speaking Operators Required	English Speaking Operators Required
7 a.m. – 9 a.m.	8	32	2	6
9 a.m. – 11 a.m.	17	68	3	12
11 a.m. – 1 p.m.	14	56	3	10
1 p.m. – 3 p.m.	19	76	4	13
3 p.m. – 5 p.m.	16	64	3	11
5 p.m. – 7 p.m.	7	28	2	5
7 p.m. – 9 p.m.	2	8	1	2

b. Formulate a linear programming model of this problem.

Answer: Define *decision variables*

E_i, S_i **:** the number of full-time English and Spanish speaking operators, respectively, whose starting shift is $i = 1,\ldots,5$, where starting shift means that the operator starts to answer calls at shift *i.*

P_1, P_2 **:** the number of part time employers beginning shift 3 p.m.- 5 p.m. and 5 p.m.-7 p.m, respectively.

There is a constraint for each language requirement and each 2-hour period:

Minimize $40E_1 + 40S_1 + 40E_2 + 40S_2 + 40E_3 + 40S_3 + 44E_4 + 44S_4 + 44E_5 + 44S_5 + 44P_1 + 48P_2$

subject to

$E_1 \geq 6$ *(English-speaking operator rqmts)*

$E_2 \geq 12$

$E_1 + E_3 \geq 10$

$E_2 + E_4 \geq 13$

$E_3 + E_5 + P_1 \geq 11$

$E_4 + P_1 + P_2 \geq 5$

$E_5 + P_2 \geq 2$

$S_1 \geq 2$ *(Spanish-speaking operator rqmts)*

$S_2 \geq 3$

$S_1 + S_3 \geq 3$

$S_2 + S_4 \geq 4$

$S_3 + S_5 \geq 3$

$S_4 \geq 2$

$S_5 \geq 1$

$E_i, S_i, P_j \geq 0$ for all i=1,…,7 and j=1,2.

c. Obtain an optimal solution for the LP model formulated in part (b)

***Answer*:**

```
     OBJECTIVE FUNCTION VALUE

     1)      1640.000

 VARIABLE        VALUE          REDUCED COST
       E1         6.000000          0.000000
       S1         2.000000          0.000000
       E2        12.000000          0.000000
       S2         3.000000          0.000000
       E3         5.000000          0.000000
       S3         2.000000          0.000000
       E4         1.000000          0.000000
       S4         2.000000          0.000000
       E5         2.000000          0.000000
       S5         1.000000          0.000000
       P1         4.000000          0.000000
       P2         0.000000         40.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000        -40.000000
       3)         0.000000          0.000000
       4)         1.000000          0.000000
       5)         0.000000        -40.000000
       6)         0.000000        -40.000000
       7)         0.000000         -4.000000
       8)         0.000000         -4.000000
```

9)	0.000000	-40.000000
10)	0.000000	-40.000000
11)	1.000000	0.000000
12)	1.000000	0.000000
13)	0.000000	-40.000000
14)	0.000000	-44.000000
15)	0.000000	-4.000000

LINGO model: This is a bigger challenge than most other models we've looked at. We define the sets LANGUAGE, PERIOD, & SHIFT, and then the derived sets which I've arbitrarily names A, B, & D. The attribute W (of set B) specifies which 2-hour period each shift is answering phones: W(i,j) = 1 if shift i is working in period j, and 0 otherwise. These binary values are then used to compute the pay for each shift and to impose the requirements for operators during each period. I have here defined the decision variables X(k,i) = # of operators speaking language k working shift i. Thus X(1,1) & X(2,1) are identical to the variables E_1 & S_1, respectively, in the model shown above. Because none of the part-time operators speak Spanish, the variables X(2,6) = X(2,7) = 0 (where Spanish is the 2nd language and the part-time shifts are #6&7).

```
MODEL:

SETS:
      LANGUAGE/E S/;
      PERIOD/1..7/: RATE  ;
      !  Shifts 1-5 are full-time, and 6&7 are part-time;
      SHIFT/1..7/: PAY ;
      A(LANGUAGE,PERIOD): REQMT;
      B(SHIFT,PERIOD): W;
      D(LANGUAGE,SHIFT): X;
ENDSETS

DATA:
      ! RATE is rate of pay for each 2-hour work period;
      RATE=20 20 20 20 20 24 24;
      ! REQMT(i,j) is requirement for operators speaking language i
            during 2-hour work period j;
      REQMT= 6 12 10 13 11  5  2
             2  3  3  4  3  2  1;
      ! W(j,k) indicates whether operator working shift j
            is answering phones during 2-hr work period k;
      W= 1 0 1 0 0 0 0
         0 1 0 1 0 0 0
         0 0 1 0 1 0 0
         0 0 0 1 0 1 0
         0 0 0 0 1 0 1
         0 0 0 0 1 1 0
         0 0 0 0 0 1 1;
ENDDATA
```

LINGO model

The LP model using these sets & data is

```
MIN = @SUM(LANGUAGE(I):
            @SUM(SHIFT(J): PAY(J) * X(I,J) ) );

!  Compute pay for each shift (both full-time & part-time);
@FOR(SHIFT(J):
      PAY(J) = @SUM(PERIOD(K): W(J,K)*RATE(K) ); );

!  No Spanish-speaking part-time operators (shifts 6&7) ;
      X(2,6)=0;
      X(2,7)=0;

!  For each language & work period, require needed # operators;
@FOR(LANGUAGE(I):
      @FOR(PERIOD(K):
            @SUM(SHIFT(J):
                   W(J,K)*X(I,J) )  >= REQMT(I,K); ); );

END
```

The solution found by LINGO is the same as that shown above.

```
 Objective value:                            1640.000

                      Variable           Value        Reduced Cost
                      X( E, 1)        6.000000           0.0000000
                      X( E, 2)        13.00000           0.0000000
                      X( E, 3)        4.000000           0.0000000
                      X( E, 5)        2.000000           0.0000000
                      X( E, 6)        5.000000           0.0000000
                      X( S, 1)        2.000000           0.0000000
                      X( S, 2)        3.000000           0.0000000
                      X( S, 3)        2.000000           0.0000000
                      X( S, 4)        2.000000           0.0000000
                      X( S, 5)        1.000000           0.0000000
```

3. Sensitivity Analysis

Ken and Larry, Inc., supplies its ice cream parlors with three flavors of ice cream: chocolate, vanilla, and banana. Because of extremely hot weather and a high demand for its products, the company has run short of its supply of ingredients: milk, sugar, & cream. Hence, they will not be able to fill all the orders received from their retail outlets, the ice cream parlors. Owing to these circumstances, the company has decided to choose the amount of each product to produce that will maximize total profit, given the constraints on supply of the basic ingredients. The chocolate, vanilla, and banana flavors generate, respectively, \$1.00, \$0.90, and \$0.95 per profit per gallon sold. The company has only 200 gallons of milk, 150 pounds of sugar, and 60 gallons of cream left in its inventory. The LP formulation for this problem has variables C, V, and B representing gallons of chocolate, vanilla, and banana ice cream produced, respectively.

LINDO model

```
!    Ken & Larry Ice Cream – from Intro to O.R. by
!    Hillier & Lieberman (7th ed) p. 296

MAXIMIZE  C+0.9V+0.95B
ST
      0.45C + 0.50V + 0.40B <= 200  ! milk resource
      0.50C + 0.40V + 0.40B <= 150  ! sugar resource
      0.10C + 0.15V + 0.20B <= 60   ! cream resource
END
```

```
        OBJECTIVE FUNCTION VALUE
        1)      341.2500

  VARIABLE        VALUE          REDUCED COST
         C          0.000000         0.037500
         V        300.000000         0.000000
         B         75.000000         0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         20.000000         0.000000
        3)          0.000000         1.875000
        4)          0.000000         1.000000
```

```
RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
        C        1.000000         0.037500         INFINITY
        V        0.900000         0.050000         0.012500
        B        0.950000         0.021429         0.050000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2      200.000000         INFINITY        20.000000
        3      150.000000        10.000000        30.000000
        4       60.000000        15.000000         3.750000
```

The LP formulation for this problem has variables C, V, and B representing gallons of chocolate, vanilla, and banana ice cream produced, respectively.

a. What is the optimal profit and the optimal solution?

Answer**:** The optimal profit is $*341.25*.

The optimal quantities of the products are 0 gallons of chocolate ice cream, 300 gallons of vanilla ice cream and 75 gallons of banana ice cream.

b. Suppose the profit per gallon of banana changes to $1.00. Will the optimal solution change, and what can be said about the effect on total profit?

***Answer*:** An increase of profit of the banana ice cream to $1.00 is an increase of $0.05. This exceeds the "Allowable Increase" (**0.021429**) in which the basis is unchanged. So the basis changes, changing the optimal solution and the total profit (which would of course increase.)

c. Suppose the profit per gallon of banana changes to 92 cents. Will the optimal solution changes, and what can be said about the effect on total profit?

***Answer*:** Because the decrease ($0.03) is less than the allowable decrease ($**0.05**) for which the basis is unchanged, the basic variables (& their values) are unchanged, but the total profit decreases by $0.03/gal. $\times$ 75 gal. = $2.25.

d. Suppose the company discovers that 3 gallons of cream have gone sour and so must be thrown out. Will the optimal solution change, and what can be said about the effect on the total profit?
***Answer*:** The optimal solution would be changed because the quantity of cream whose slack is 0 is changed. Because the decrease (3 gal.) is less than the allowable decrease (which is **3.75**), the total profit would decrease by $**3** (dual price of cream resource is $1.0/gal. so 3 gal.$\times$ 1.0 $/gal. = $3).

e. Suppose that the company has the opportunity to buy an additional 15 pounds of sugar at a total cost of $15. Should they buy it? Explain!

***Answer*:** Inside the allowable range, the dual price is **$1.875** so if 10 pounds of sugar is bought and used the profit increase by 10 $\times$ $1.875 = $18.75 which is more than the price of 15 pounds of sugar and brings more profit (if 15 pounds of sugar is available, there would not be *less* profit than when 10 pounds is used, and there possibly will be an additional increase in profit.) So the company *should* buy the 15 pounds of sugar at the stated price, since they would obtain *at least* $18.75−$15.00 = $3.75 in additional net profits.

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Homework #4 Solutions--Fall 2002

1. Ken & Larry's Ice Cream, *continued*. Refer to the problem description in last week's homework (HW#3). The optimal LP tableau provided by LINDO is as shown below.

```
THE TABLEAU

     ROW  (BASIS)        C         V         B      SLK 2      SLK 3
       1   ART         0.038     0.000     0.000     0.000      1.875
       2   SLK  2     -0.350     0.000     0.000     1.000     -2.000
       3         V     3.000     1.000     0.000     0.000     10.000
       4         B    -1.750     0.000     1.000     0.000     -7.500

     ROW     SLK 4
       1     1.000   341.250
       2     2.000    20.000
       3   -20.000   300.000
       4    20.000    75.000
```

a. Chocolate ice cream is not included in the optimal production plan. If one gallon of chocolate ice cream were to be produced, how would it change the quantity
   …of vanilla ice cream produced?
   …of banana ice cream produced?
   …of milk used?
   …of sugar used?
   …of cream used?

***Solution*:**

$$\begin{bmatrix} Profit \\ SLK2 \\ V \\ B \end{bmatrix} = \begin{bmatrix} 341.25 \\ 20 \\ 300 \\ 75 \end{bmatrix} - \begin{bmatrix} -0.038 \\ -0.35 \\ 3 \\ -1.75 \end{bmatrix} C$$

The change of quantity of the vanilla ice cream produced: <u>De</u>crease by 3 gallon ($-3\times1=-3$).
The change of quantity of the banana ice cream produced: <u>In</u>crease by 1.75 gallon ($-(-1.75)\times1=1.75$).
The change of quantity of the milk used: <u>De</u>crease by 0.35 (*increase of SLK2 by* 0.35).
The quantities of sugar or cream used are not changed.

b. In last week's homework, you were asked about the effect on profit of a reduction in the quantity of available cream due to spoilage. That is, the effect of an increase in the unused cream (slack in the available cream constraint). According to the substitution rates in the tableau, what would be the effect of this spoilage on the quantity
   …of vanilla ice cream produced?
   …of banana ice cream produced?
   …of milk used?
   …of sugar used?

***Solution*:**

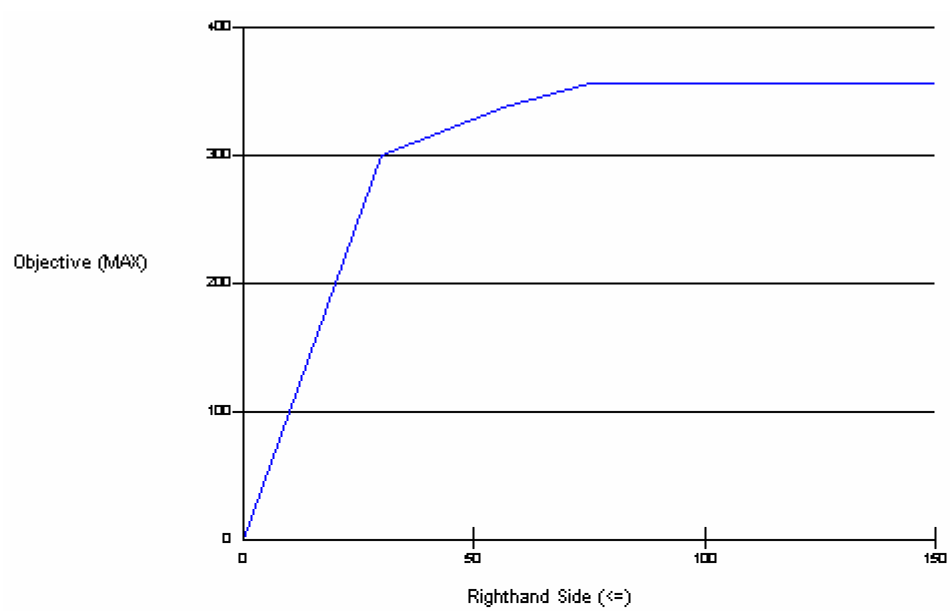
$$\begin{bmatrix} Profit \\ SLK2 \\ V \\ B \end{bmatrix} = \begin{bmatrix} 341.25 \\ 20 \\ 300 \\ 75 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -20 \\ 20 \end{bmatrix} SLK4$$

The spoilage implies that *SLK4* is <u>in</u>creased by 3 gallons.
The change of quantity of the vanilla ice cream produced: <u>In</u>crease by 60 gallons ($-(-20)\times 3 = 60$).
The change of quantity of the banana ice cream produced: Decrease by 60 gallons ($-20\times 3 = -60$).

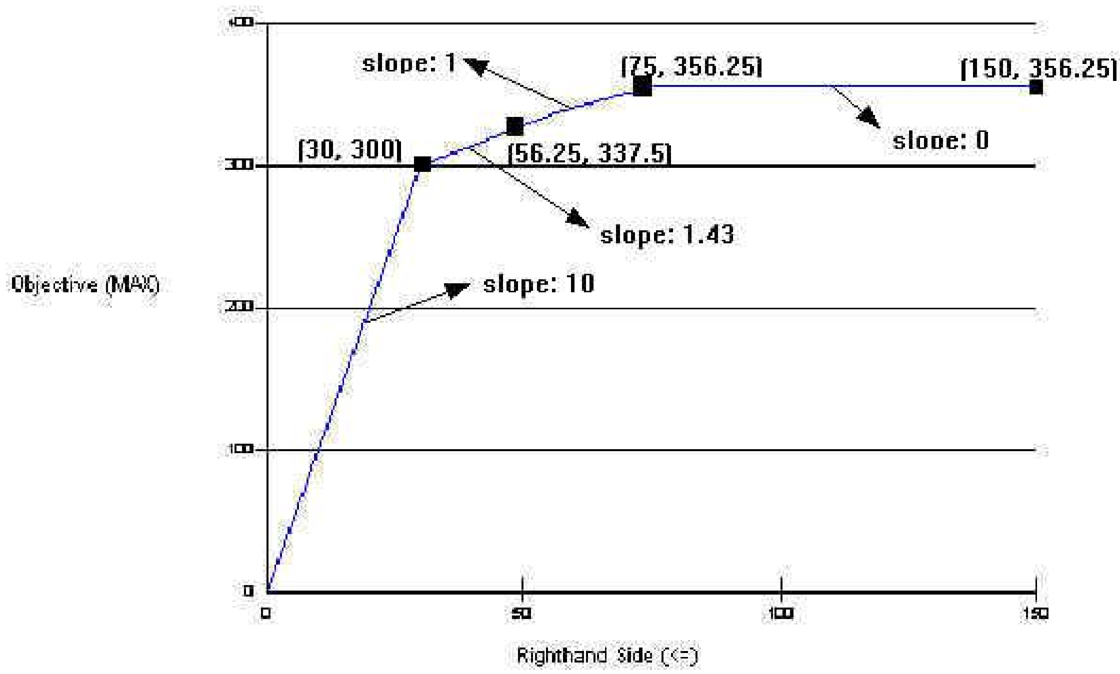
The right-hand-side of row #4 (available cream) was changed to zero, and then parametric analysis performed with the right-hand-side increasing to 150 gallons, with the results below.

```
RIGHTHANDSIDE PARAMETRICS REPORT FOR ROW: 4

   VAR    VAR  PIVOT   RHS    DUAL PRICE   OBJ
   OUT    IN    ROW    VAL   BEFORE PIVOT  VAL

                     0.0000   10.0000     0.000
 SLK 3        V   4  30.0000   10.0000   300.000
     C        B   3  56.2500    1.42857  337.500
     V   SLK 4    4  75.0000    1.00000  356.250
                    150.000     0.0000   356.250
```

The plot of optimal value vs gallons of cream available was also prepared by LINDO:



c. Using LINDO's report, indicate on the graph above the <u>slope</u> of each linear segment and the <u>coordinates</u> of each break-point (profit & gallons of cream).



2. LP model formulation. Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. the acreage and irrigation water available for the three farms are shown below:

Farm	Acreage	Water available (acre-ft)
1	400	1500
2	600	2000
3	300	900

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below. Any combination of crops may be grown on a farm.

Crop	Total harvesting capacity (in acres)	Water Reqmts (acre-ft per acre)	Expected profit ($/acre)
Milo	700	6	400
Cotton	800	4	300
Wheat	300	2	100

Using **LINGO**, the following sets were defined, with decision variables:

X_{ij} = # acreas of crop j planted on farm i.

```
MODEL:   ! MARKY DEE SOD'S RANCHES;

SETS:
      FARM/1..3/:ACREAGE, H20_AVAIL;
      CROP/MILO, COTTON, WHEAT/:CAPACITY, H20_RQMT, PROFIT;
      COMBO(FARM,CROP):X;
ENDSETS

DATA:
      ACREAGE   =  400  600 300;
      H20_AVAIL = 1500 2000 900;
      CAPACITY  =   700  800 300;
      H20_RQMT  =     6    4   2;
      PROFIT    =   400  300 100;
ENDDATA

!   INSERT OBJECTIVE & CONSTRAINTS HERE ;

END
```

a. Using LINGO, formulate the LP model to maximize the total expected profit of the three ranches.
Solution:

```
    MAX = @SUM(COMBO(I,J): PROFIT(J)*X(I,J) );
    @FOR(FARM(I):
          @SUM(COMBO(I,J): X(I,J)) <= ACREAGE(I) ;
          @SUM(COMBO(I,J): H20_RQMT(J)*X(I,J)) <= H20_AVAIL(I) ;
    );

    @FOR(CROP(J):
          @SUM(COMBO(I,J): X(I,J)) <= CAPACITY(J) ;
    );
```

b. Add the statements to the accompanying file (HW4_2.lg4) , and solve.
Solution: The primal solution:

Variable	Value	Reduced Cost
X(1, MILO)	0.0000000	0.0000000
X(1, COTTON)	375.0000	0.0000000
X(1, WHEAT)	0.0000000	33.33333
X(2, MILO)	50.00000	0.0000000
X(2, COTTON)	425.0000	0.0000000
X(2, WHEAT)	0.0000000	33.33333
X(3, MILO)	150.0000	0.0000000
X(3, COTTON)	0.0000000	0.0000000
X(3, WHEAT)	0.0000000	33.33333

The dual solution:

Row	Slack or Surplus	Dual Price
2	25.00000	0.0000000
3	0.0000000	66.66667
4	125.0000	0.0000000
5	0.0000000	66.66667
6	150.0000	0.0000000
7	0.0000000	66.66667
8	500.0000	0.0000000
9	0.0000000	33.33333
10	300.0000	0.0000000

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Homework #5 Solution– Fall 2002

1. Consider the transportation tableau:

dstn→ ↓source	1	2	3	4	5	Supply
A	12	8	9	15	11	9
B	10	11	12	11	14	7
C	9	7	11	14	8	4
D	13	12	13	12	12	7
E	8	9	10	9	10	3
Demand=	4	7	5	5	9	

a. Use the initial basic solution: X_{A3}=5, X_{A5}=4, X_{B1}=4, X_{B4}=3, X_{C4}=X_{C5}=2, X_{D2}=7, X_{E5}=3 & _____ = 0. *(Choose one more variable to complete the basis. Any choice is valid except one that would create a "cycle" of basic cells in the tableau!)*

Answer: Any cell *except* $X_{A4}, X_{B5}, X_{C1}, X_{C3}, X_{E3}$ or X_{E4}

dstn→ ↓source	1	2	3	4	5	Supply
A	12	8	9	15	11	9
B	10	11	12	11	14	7
C	9	7	11	14	8	4
D	13	12	13	12	12	7
E	8	9	10	9	10	3
Demand=	4	7	5	5	9	

Note that the diagonally shaded cells would create a cycle of basic cells if chosen to be basic.

b. Compute two different sets of values for the dual variables U & V (*simplex multipliers*) for this basis.

Answer: Let's choose X_{E2}=0 to be basic in (a) above.

If we arbitrarily choose $U_E = 0$ then $U_A = 1, U_B = -5, U_C = -2, U_D = 3$ and

$V_1 = 15, V_2 = 9, V_3 = 8, V_4 = 16, V_5 = 10$

		$V_1=$ 15	$V_2=$ 9	$V_3=$ 8	$V_4=$ 16	$V_5=$ 10	
		1	2	3	4	5	Supply
$U_A=$ 1	A	12	8	5 / 9	15	4 / 11	9
$U_B=$ -5	B	4 / 10	11	12	3 / 11	14	7
$U_C=$ -2	C	9	7	11	2 / 14	2 / 8	4
$U_D=$ 3	D	13	7 / 12	13	12	12	7
$U_E=$ 0	E	8	0 / 9	10	9	3 / 10	3
	Demand	4	7	5	5	9	

If we instead arbitrarily choose $U_A = 0$ then we will obtain different values:

$U_B = -6, U_C = -3, U_D = 2, U_E = -1$ and $V_1 = 16, V_2 = 10, V_3 = 9, V_4 = 17, V_5 = 11$

		$V_1=$ 16	$V_2=$ 10	$V_3=$ 9	$V_4=$ 17	$V_5=$ 11	
		1	2	3	4	5	Supply
$U_A=$ 0	A	12	8	5 / 9	15	4 / 11	9
$U_B=$ -6	B	4 / 10	11	12	3 / 11	14	7
$U_C=$ -3	C	9	7	11	2 / 14	2 / 8	4
$U_D=$ 2	D	13	7 / 12	13	12	12	7
$U_E=$ -1	E	8	0 / 9	10	9	3 / 10	3
	Demand	4	7	5	5	9	

c. Using each set of simplex multipliers, price all of the nonbasic cells. How do the reduced costs depend upon the choice of dual variables? Select the variable having the "most negative" reduced cost to enter the basis.

Answer: By calculating $\overline{C}_{ij} = C_{ij} - (U_i + V_j)$ for *i=A,B,C,D,E* and *j=1,2,3,4,5*

We can get the following reduced costs, when $U_E = 0$.

$\overline{C}_{A1} = -4, \overline{C}_{A2} = -2, \overline{C}_{A4} = -2,$

$\overline{C}_{B2} = 7, \overline{C}_{B3} = 9, \overline{C}_{B5} = 9,$

$$\overline{C}_{C1} = -4, \overline{C}_{C2} = 0, \overline{C}_{C3} = 5,$$
$$\overline{C}_{D1} = -5, \overline{C}_{D3} = 2, \overline{C}_{D4} = -7, \overline{C}_{D5} = -1,$$
$$\overline{C}_{E1} = -7, \overline{C}_{E3} == 2, \overline{C}_{E4} = -7$$

When $U_A = 0$, the results are *exactly* the same—the reduced costs depend on the sums ($U_i + V_j$), not on the values U_i & V_j individually!

The "most negative" (i.e., smallest) reduced cost is −7, which is that of each of the nonbasic variables X_{D4}, X_{E1}, X_{E4}.

d. What variable will leave the basis as the new variable enters the basis?

Answer: If, for example, we chose X_{E4} as a new basic variable then X_{C4} must leave the basis.

e. Complete the computation of the optimal solution, using the transportation simplex method.

Answer: The optimal solution is the following.

$$X_{A2} = 4, X_{A3} = 5,$$
$$X_{B1} = 4, X_{B4} = 3,$$
$$X_{C2} = 3, X_{C5} = 1,$$
$$X_{D5} = 7,$$
$$X_{E4} = 2, X_{E5} = 1$$

and all others are 0.

Cost = 291 (Solution is optimal!)

Next table is the following

		$V_1 =$ 9	$V_2 =$ 10	$V_3 =$ 9	$V_4 =$ 10	$V_5 =$ 11	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	3 12	-2 8	5 9	5 15	4 11	**9**
$U_B =$ 1	**B**	4 10	0 11	1 12	3 11	2 14	**7**
$U_C =$ -3	**C**	3 9	0 7	5 11	7 14	4 8	**4**
$U_D =$ 2	**D**	2 13	7 12	2 13	0 12	-1 12	**7**
$U_E =$ -1	**E**	0 8	0 9	2 10	2 9	1 10	**3**
	Demand	**4**	**7**	**5**	**5**	**9**	

The X_{A2} has the most negative reduced cost -2 and entering X_{A2} into the basis makes X_{E2} leave the basis:

		$V_1 =$ 7	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 8	$V_5 =$ 11	
		1	2	3	4	5	**Supply**
$U_A =$ 0	**A**	5 / 12	0 / 8	5 / 9	7 / 15	4 / 11	9
$U_B =$ 3	**B**	4 / 10	0 / 11	0 / 12	3 / 11	0 / 14	7
$U_C =$ -3	**C**	5 / 9	2 / 7	5 / 11	9 / 14	4 / 8	4
$U_D =$ 4	**D**	2 / 13	7 / 12	0 / 13	0 / 12	-3 / 12	7
$U_E =$ 1	**E**	0 / 8	0 / 9	0 / 10	2 / 9	1 / 10	3
	Demand	4	7	5	5	9	

The X_{D5} has most negative reduced cost -3 and entering X_{D5} into the basis makes X_{A5} leave the basis:

		$V_1 =$ 6	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 7	$V_5 =$ 8	
		1	2	3	4	5	**Supply**
$U_A =$ 0	**A**	6 / 12	4 / 8	5 / 9	8 / 15	3 / 11	9
$U_B =$ 4	**B**	4 / 10	-1 / 11	-1 / 12	3 / 11	2 / 14	7
$U_C =$ 0	**C**	3 / 9	-1 / 7	2 / 11	7 / 14	4 / 8	4
$U_D =$ 4	**D**	3 / 13	3 / 12	0 / 13	1 / 12	4 / 12	7
$U_E =$ 2	**E**	0 / 8	-1 / 9	-1 / 10	2 / 9	1 / 10	3
	Demand	4	7	5	5	9	

Continuing in the same way, we get the following table for which there is no variable having negative reduced cost—therefore it is the optimal solution.

		$V_1 =$ 7	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 8	$V_5 =$ 9	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	5 / **12**	4 / **8**	5 / **9**	7 / **15**	2 / **11**	**9**
$U_B =$ 3	**B**	4 / **10**	0 / **11**	0 / **12**	3 / **11**	2 / **14**	**7**
$U_C =$ -1	**C**	3 / **9**	3 / **7**	3 / **11**	7 / **14**	1 / **8**	**4**
$U_D =$ 3	**D**	3 / **13**	1 / **12**	1 / **13**	1 / **12**	7 / **12**	**7**
$U_E =$ 1	**E**	0 / **8**	0 / **9**	0 / **10**	2 / **9**	1 / **10**	**3**
	Demand	**4**	**7**	**5**	**5**	**9**	

2. Production scheduling (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 394) The MLK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced in Regular Time (RT) and in Overtime (OT). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced in regular time, (3) the cost (in thousands of dollars) per unit produced in overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for product 1 on the left and the number for product 2 on the right.

	Max combined production			Unit cost of production ($K)		
Month	RT	OT	Sales	RT	OT	Storage cost ($K)
1	10	3	5/3	15/16	18/20	1/2
2	8	2	3/5	17/15	20/18	2/1
3	10	3	4/4	19/17	22/22	

The production manager wants a schedule developed for the number of units of each of the two products to be produced in regular time and (if regular time production capacity is used up) in overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

a. Formulate this problem as a balanced transportation problem by constructing the appropriate transportation tableau.
b. Use the Northwest Corner Method to find an initial basic feasible solution. Is it degenerate?

Answer for a) and b): The solution is *not* degenerate.

	1A	1B	2A	2B	3A	3B	EXCESS	SUPPLY
R1	**5** 15	**3** 16	**2** 16	18	18	19	0	10
O1	18	20	**1** 19	**2** 22	21	23	0	3
R2	inf	inf	17	**3** 15	**4** 19	**1** 16	0	8
O2	inf	inf	20	18	22	**2** 19	0	2
R3	inf	inf	inf	inf	19	**1** 17	**9** 0	10
O3	inf	inf	inf	inf	22	22	**3** 0	3
	5	3	3	5	4	4	12	SUM=410

c. Use the transportation simplex algorithm to find the optimal solution. Is it degenerate? Are there multiple optima?

Answer: The optimal solution is the following.

	1A	1B	2A	2B	3A	3B	EXCESS	SUPPLY
R1	**5** 15	**3** 16	**2** 16	18	18	19	0	10
O1	18	20	19	22	21	23	**3** 0	3
R2	inf	inf	**1** 17	**5** 15	19	**2** 16	0	8
O2	inf	inf	20	18	22	19	**2** 0	2
R3	inf	inf	inf	inf	**4** 19	**2** 17	**4** 0	10
O3	inf	inf	inf	inf	22	22	**3** 0	3
Demand	5	3	3	5	4	4	12	SUM=389

3. **Assignment Problem.** (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 399.) Four cargo ships will be used for shipping goods from one port to four other ports (labeled 1, 2, 3, 4). Any ship can be used for making any one of these four trips. However, because of differences in the ships and cargoes, the total cost of loading, transporting, and unloading the goods for the different ship-port combinations varies considerably, as shown in the following table:

PORT→ ↓SHIP	1	2	3	4
1	$500	$400	$600	$700
2	$600	$600	$700	$500
3	$700	$500	$700	$600
4	$500	$400	$600	$600

The objective is to assign the four ships to four different ports in such a way as to minimize the total cost for all four shipments.

a. Use the Hungarian method to find an optimal solution.

Answer:

There are several optimal solutions:

After row reduction

PORT→ ↓ship	1	2	3	4
1	$100	$0	$200	$300
2	$100	$100	$200	$0
3	$200	$0	$200	$100
4	$100	$0	$200	$200

After column reduction

PORT→ ↓ship	1	2	3	4
1	$0	$0	$0	$300
2	$0	$100	$0	$0
3	$100	$0	$0	$100
4	$0	$0	$0	$200

For example, $X_{41}=X_{12}=X_{33}=X_{24}=1$ is optimal, as is $X_{11}=X_{32}=X_{43}=X_{24}=1$. (All optimal solutions have the assignment $X_{24}=1$.)

b. Reformulate this as an equivalent transportation problem.

Answer: Supplies & Demands are all 1!

dstn→ ↓source	1	2	3	4	Supply=
1	500	400	600	700	1
2	600	600	700	500	1
3	700	500	700	600	1
4	500	400	600	600	1
Demand=	1	1	1	1	

c. Use the Northwest Corner Method to obtain an initial basic feasible solution. (This will be a *degenerate* solution. Be sure to specify which variables are basic!)

Answer: Let the shaded cells form the initial basis.

	1	2	3	4	SUPPLY
1	1 / 500	400	600	700	1
2	600	1 / 600	700	500	1
3	700	500	1 / 700	600	1
4	500	400	600	1 / 600	1
Demand	1	1	1	1	

d. Use the transportation simplex method to find the optimal solution.

Answer:

		500	400	500	400	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	100 / 600	300 / 700	1
200	2	-100 / 600	1 / 600	0 / 700	-100 / 500	1
200	3	0 / 700	-100 / 500	1 / 700	0 / 600	1
200	4	-200 / 500	**-200** / **400**	-100 / 600	1 / 600	1
	Demand	1	1	1	1	

$X_{4,2}$ enters into the basis with value change and $X_{4,4}$ leaves the basis.

(Assignments, & therefore cost as well, have changed.)

		500	400	500	400	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	100 / 600	300 / 700	1
200	2	-100 / 600	0 / 600	1 / 700	**-100** / **500**	1
200	3	0 / 700	-100 / 500	0 / 700	1 / 600	1
0	4	0 / 500	1 / 400	100 / 600	200 / 600	1
	Demand	1	1	1	1	

$X_{2,4}$ has entered the basis with a value change and $X_{2,3}$ leaves the basis.

		500	400	400	300	
		1	2	3	4	SUPPLY
		1	0	200	400	
0	1	500	400	600	700	1
		-100	0	100	1	
200	2	600	600	700	500	1
		-100	**-200**	1	0	
300	3	700	**500**	700	600	1
		0	1	200	300	
0	4	500	400	600	600	1
	Demand	1	1	1	1	

$X_{3,2}$ enters into the basis *without* value change and $X_{2,2}$ leaves the basis.

		500	400	500	400	
		1	2	3	4	SUPPLY
		1	0	100	300	
0	1	500	400	600	700	1
		0	100	100	1	
100	2	600	600	700	500	1
		100	0	1	0	
100	3	700	500	700	600	1
		0	1	100	200	
0	4	500	400	600	600	1
	Demand	1	1	1	1	

There is no negative reduced cost, i.e., this is optimal.

e. In how many iterations was the solution degenerate?
Answer: <u>All</u> the solutions are degenerate.

f. How many iterations produce a change in the *values* of the variables?
Answer: 2 iterations produce a change in the value of the variables.

g. How many iterations leave the variables *unchanged in value* (although the basis changes)?
Answer: 1 iteration leaves all variables unchanged in value.

4. Return of Marky D. Sod Recall the LP model for this problem in HW#4:
Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. the acreage and irrigation water available for the three farms are shown below:

FARM	ACREAGE	WATER AVAILABLE (ACRE-FT)
1	400	1500
2	600	2000
3	300	900

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below. Any combination of crops may be grown on a farm.

CROP	TOTAL HARVESTING CAPACITY (IN ACRES)	WATER REQMTS (ACRE-FT PER ACRE)	EXPECTED PROFIT ($/ACRE)
Milo	700	6	400
Cotton	800	4	300
Wheat	300	2	100

Decision variables: X_{ij} = # acreas of crop j planted on farm i.
The LINDO model (generated by LINGO) is:

```
MAX     400 X1MILO + 300 X1COTTON + 100 X1WHEAT + 400 X2MILO
       + 300 X2COTTON + 100 X2WHEAT + 400 X3MILO + 300 X3COTTON + 100 X3WHEAT
SUBJECT TO
        2)   X1MILO + X1COTTON + X1WHEAT <=   400
        3)   6 X1MILO + 4 X1COTTON + 2 X1WHEAT <=   1500
        4)   X2MILO + X2COTTON + X2WHEAT <=   600
        5)   6 X2MILO + 4 X2COTTON + 2 X2WHEAT <=   2000
        6)   X3MILO + X3COTTON + X3WHEAT <=   300
        7)   6 X3MILO + 4 X3COTTON + 2 X3WHEAT <=   900
        8)   X1MILO + X2MILO + X3MILO <=   700
        9)   X1COTTON + X2COTTON + X3COTTON <=   800
       10)   X1WHEAT + X2WHEAT + X3WHEAT <=   300
  END

        1)      320000.0

  VARIABLE        VALUE          REDUCED COST
    X1MILO         0.000000          0.000000
  X1COTTON       375.000000          0.000000
   X1WHEAT         0.000000         33.333332
    X2MILO        50.000000          0.000000
  X2COTTON       425.000000          0.000000
   X2WHEAT         0.000000         33.333332
    X3MILO       150.000000          0.000000
  X3COTTON         0.000000          0.000000
   X3WHEAT         0.000000         33.333332

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        25.000000          0.000000
        3)         0.000000         66.666664
        4)       125.000000          0.000000
        5)         0.000000         66.666664
        6)       150.000000          0.000000
        7)         0.000000         66.666664
        8)       500.000000          0.000000
        9)         0.000000         33.333332
       10)       300.000000          0.000000
```

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1MILO	400.000000	0.000000	INFINITY
X1COTTON	300.000000	INFINITY	0.000000
X1WHEAT	100.000000	33.333328	INFINITY
X2MILO	400.000000	0.000000	0.000000
X2COTTON	300.000000	0.000000	0.000000
X2WHEAT	100.000000	33.333328	INFINITY
X3MILO	400.000000	INFINITY	0.000000
X3COTTON	300.000000	0.000000	INFINITY
X3WHEAT	100.000000	33.333328	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	400.000000	INFINITY	25.000000
3	1500.000000	100.000000	300.000000
4	600.000000	INFINITY	125.000000
5	2000.000000	750.000000	300.000000
6	300.000000	INFINITY	150.000000
7	900.000000	900.000000	900.000000
8	700.000000	INFINITY	500.000000
9	800.000000	75.000000	425.000000
10	300.000000	INFINITY	300.000000

THE TABLEAU:

ROW	(BASIS)	X1MILO	X1COTTON	X1WHEAT	X2MILO	X2COTTON	X2WHEAT
1	ART	0.000	0.000	33.333	0.000	0.000	33.333
2	SLK 2	-0.500	0.000	0.500	0.000	0.000	0.000
3	X1COTTON	1.500	1.000	0.500	0.000	0.000	0.000
4	SLK 4	0.500	0.000	0.167	0.000	0.000	0.667
5	X2MILO	1.000	0.000	0.333	1.000	0.000	0.333
6	SLK 6	0.000	0.000	0.000	0.000	0.000	0.000
7	X3MILO	0.000	0.000	0.000	0.000	0.000	0.000
8	SLK 8	0.000	0.000	-0.333	0.000	0.000	-0.333
9	X2COTTON	-1.500	0.000	-0.500	0.000	1.000	0.000
10	SLK 10	0.000	0.000	1.000	0.000	0.000	1.000

ROW	X3MILO	X3COTTON	X3WHEAT	SLK 2	SLK 3	SLK 4	SLK 5
1	0.000	0.000	33.333	0.000	66.667	0.000	66.667
2	0.000	0.000	0.000	1.000	-0.250	0.000	0.000
3	0.000	0.000	0.000	0.000	0.250	0.000	0.000
4	0.000	-0.333	0.000	0.000	0.083	1.000	-0.167
5	0.000	-0.667	0.000	0.000	0.167	0.000	0.167
6	0.000	0.333	0.667	0.000	0.000	0.000	0.000
7	1.000	0.667	0.333	0.000	0.000	0.000	0.000
8	0.000	0.000	-0.333	0.000	-0.167	0.000	-0.167
9	0.000	1.000	0.000	0.000	-0.250	0.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000

ROW	SLK 6	SLK 7	SLK 8	SLK 9	SLK 10	
1	0.00E+00	67.	0.00E+00	33.	0.00E+00	0.32E+06
2	0.000	0.000	0.000	0.000	0.000	25.000
3	0.000	0.000	0.000	0.000	0.000	375.000
4	0.000	0.000	0.000	-0.333	0.000	125.000
5	0.000	0.000	0.000	-0.667	0.000	50.000
6	1.000	-0.167	0.000	0.000	0.000	150.000

7	0.000	0.167	0.000	0.000	0.000	150.000
8	0.000	-0.167	1.000	0.667	0.000	500.000
9	0.000	0.000	0.000	1.000	0.000	425.000
10	0.000	0.000	0.000	0.000	1.000	300.000

a. Another farmer whose farm adjoins Sod Farm #3 might be willing to sell Marky a portion of his water rights. How much should Marky offer, and for how many acre-feet?

Answer: If the price is strictly less than \$66.67 per acre-feet, he can buy up to 900 acre-feet.

b. What increase in the profit per acre for wheat is required in order for it to be profitable for Marky to plant any?

Answer: The profit per acre for wheat must increase by more than \$33.33 for it to be profitable for Marky to plant any wheat on *any* farm.

c. If Marky were to plant 100 acres of wheat on Farm #1, how should he best adjust the optimal plan above?

Answer:

$$\begin{bmatrix} ART \\ SLK2 \\ X1COTTON \\ SLK4 \\ X2MILO \\ SKL6 \\ X3MILLO \\ SKL8 \\ X2COTTON \\ SLK10 \end{bmatrix} = \begin{bmatrix} 0.32E+06 \\ 25 \\ 375 \\ 125 \\ 50 \\ 150 \\ 150 \\ 500 \\ 425 \\ 300 \end{bmatrix} - \begin{bmatrix} 33.333 \\ 0.5 \\ 0.5 \\ 0.167 \\ 0.333 \\ 0 \\ 0 \\ -0.333 \\ -0.5 \\ 1 \end{bmatrix} X1WHEAT = \begin{bmatrix} 316670 \\ -25 \\ 325 \\ 108.3 \\ 16.7 \\ 150 \\ 150 \\ 533.3 \\ 475 \\ 200 \end{bmatrix}$$

An increase in X1WHEAT of 100 is impossible because *SLK2* would become negative. By performing the "minimum ratio test", we discover that for up to an increase of 50 acres of wheat he could adjust the optimal plan with this equation, but after that he would need to solve the problem again (adding the constraint X1WHEAT=100).

d. Is there another optimal basic solution, besides the one given above? If so, how does it differ from that given above?

Answer: Because there are non-basic variables with reduced cost 0 (namesly X3COTTON and X1MILO), increasing either of these variables up to its allowable limit does not change the objective value, and is therefore also an optimal solution.

56:171 Operations Research
Homework #7 Solutions -- Fall 2002

1. **Decision Analysis** (adapted from Exercise 15.2-7, page 784, *Operations Research*, 7th edition, by Hillier & Lieberman.)
Dwight Moody is the manager of a large farm with 1,000 acres of arable land. For greater efficiency, Dwight always devotes the farm to growing one crop at a time. He now needs to make a decision on which one of four crops to grow during the upcoming growing season. For each of these crops, Dwight has obtained the following estimates of crop yields and net incomes per bushel under various weather conditions.

Weather	Crop 1	Crop 2	Crop 3	Crop 4
Dry	20	15	30	40
Moderate	35	20	25	40
Damp	40	30	25	40
Net income/bushel	\$1.00	\$1.50	\$1.00	\$0.50

After referring to historical meteorological records, Dwight also estimated the following probabilities for the weather during the growing season:

Dry	0.3
Moderate	0.5
Damp	0.2

Using the criterion of "Maximize expected payoff", determine which crop to grow.

Solution: Expected payoffs

- Crop 1: $(20\times0.3+35\times0.5+40\times0.2)\times\$1.00=\$31.50$
- Crop 2: $(15\times0.3+20\times0.5+30\times0.2)\times\$1.50=\$30.75$
- Crop 3: $(30\times0.3+25\times0.5+25\times0.2)\times\$1.10=\$26.50$
- Crop 4: $(40\times0.3+40\times0.5+40\times0.2)\times\$0.50=\$20.00$

Dwight Moody should choose crop 1 with \$31.50 payoff.

2. **Bayes' Rule** (Exercise 15.3-15, pp. 788-789, *Operations Research*, 7th edition, by Hillier & Lieberman)

There are two biased coins, coin A with probability of landing heads equal to 0.8 and the coin B with probability of heads equal to 0.4. One coin is chosen at random (each with probability 50%) to be tossed twice. You are to receive \$100 if you correctly predict how many heads will occur in two tosses of this coin.

a. Using the "Maximum Expected Payoff" criterion, what is the optimal prediction, and what is the corresponding expected payoff?

Solution: We are given $P(H|A) = 0.8$ and $P(H|B) = 0.4$

$P(2H \mid A) = (0.8)^2 = 0.64$ $\qquad$ $P(2H \mid B) = (0.4)^2 = 0.16$

$P(1H \mid A) = 1-0.64-0.04 = 0.32$ $\qquad$ $P(2H \mid B) = 1-0.16-0.36 = 0.48$

$P(0H \mid A) = (0.2)^2 = 0.04$ $\qquad$ $P(2H \mid B) = (0.6)^2 = 0.36$

$P(2H) = P(2H \mid A)\times P(A) + P(2H \mid B)\times P(B) = 0.5\times0.64 + 0.5\times0.16 =$ **0.4**

$P(1H) = P(1H \mid A)\times P(A) + P(1H \mid B)\times P(B) = 0.5\times0.32 + 0.5\times0.48 =$ **0.4**

$P(0H) = P(0H \mid A)\times P(A) + P(0H \mid B)\times P(B) = 0.5\times0.04 + 0.5\times0.36 = 0.2$

Should predict either 1 *or* 2 heads, each with expected payoff \$40.00

Suppose now that you may observe a preliminary toss of the chosen coin before predicting.

b. Determine your optimal prediction after observing a head in the preliminary toss.

Solution: Let H_0 denote the event that the outcome of the preliminary toss is heads, and T_0 if tails. By the "law of total probability",

$$P(H_0) = P(H_0 \mid A)P(A) + P(H_0 \mid B)P(B) = 0.8\times 0.5 + 0.4\times 0.5 = 0.6$$

and $P(T_0) = 1 - P(H_0) = 0.4$

According to Bayes' Rule,

$$P(A \mid H_0) = \frac{P(H_0 \mid A)\times P(A)}{P(H_0)} = \frac{0.8\times 0.5}{0.6} = \frac{2}{3} \Rightarrow P(B \mid H_0) = 1 - \frac{2}{3} = \frac{1}{3}$$

Then the probabilities of the outcomes of the following tosses (given H_0) are

P(0H|H_0) = P(0H|when coin is A) ×P(A|H_0) + P(0H|when coin is B) ×P(B|H_0)
= 0.04×2/3 + 0.36×1/3 = 0.1467

P(1H|H_0) = P(1H|when coin is A) ×P(A|H_0) + P(1H|when coin is B) ×P(B|H_0)
= 0.32×2/3 + 0.48×1/3 = 0.3733

P(2H|H_0) = P(2H|when coin is A) ×P(A|H_0) + P(2H|when coin is B)×P(B|H_0)
= 0.64×2/3 + 0.16×1/3 = 0.48

Expected maximal payoff, given H_0, is $48.00, obtained if one predicts two heads.

…after observing a tail in the preliminary toss.

Solution: According to Bayes' Rule,

$$P(A \mid T_0) = \frac{P(T_0 \mid A)\times P(A)}{P(T_0)} = \frac{0.2\times 0.5}{0.4} = \frac{1}{4} \Rightarrow P(B \mid T_0) = 1 - \frac{1}{4} = \frac{3}{4}$$

Then the probabilities of the outcomes of the following tosses (given T_0) are

P(0H|T_0) = P(0H|when coin is A)×P(A|T_0) + P(0H|when coin is B) ×P(B|T_0)
= 0.04×1/4 + 0.36×3/4 = 0.28

P(1H|T_0) = P(1H|when coin is A) ×P(A|T_0) + P(1H|when coin is B))×P(B|T_0)
= 0.32×1/4 + 0.48×3/4 = 0.44

P(2H|T_0) = P(2H|when coin is A))×P(A|T_0) + P(2H|when coin is B))×P(B|T_0)
= 0.64×1/4 + 0.16×3/4 = 0.28

Expected maximal payoff, given T_0, is now $44.00, again by predicting two heads.

c. What is the expected value of the preliminary toss?

Solution: The expected payoff if there is a preliminary toss is

EVWSI =expected value with sample information
= E(payoff|H_0)P(H_0) + E(payoff|T_0)P(T_0)
= $48 ×0.6 + $44 ×0.4 = $46.40

EVWOI = expected value without information
= $40.00 (from part (a).)

EVSI = expected value of sample information
= **EVWSI – EVWOI**
(i.e., expected value with sample info minus expected value w/o info)
= $46.40 – $40 = $6.40

3. Integer Programming Model (based upon Case 12.3, page 649-653 of Operations Research, 7th edition, by Hillier & Lieberman. See the text for the complete case description. What follows is a condensed version.)

Brenda Sims, the saleswoman on the floor at Furniture City, understood that Furniture City required a new inventory policy. Not only was the megastore losing money by making customers unhappy with delivery delays, but it was also losing money by wasting warehouse space. By changing the inventory policy to stock only popular items and replenish them immediately when they are sold, Furniture City would ensure that the majority of customers receive their furniture immediately and that the valuable warehouse space was utilized effectively.

She decided… to use her kitchen department as a model for the new inventory policy. She would identify all kitchen sets comprising 85% of customers orders. Given the fixed amount of warehouse space allocated to the kitchen department, she would identify the items Furniture City should stock in order to satisfy the greatest number of customer orders.

Brenda analyzed her records over the past three years and determined that 20 kitchen sets were responsible for 85% of customer orders. These 20 kitchen sets were composed of up to eight features, usually with four styles of each feature (except for the dishwashers, with two styles.)

- Floor tile: styles T1, T2, T3, T4
- Wallpaper: styles W1, W2, W3, W4
- Light fixtures: styles L1, L2, L3, L4
- Cabinets: styles C1, C2, C3, C4
- Countertops: styles O1, O2, O3, O4
- Dishwashers: styles D1, D2
- Sinks: styles S1, S2, S3, S4
- Ranges: styles R1, R2, R3, R4

(Sets, 14 through 20, however, do not include dishwashers.)

The warehouse could hold 50 ft^2 of tile and 12 rolls of wallpaper in the inventory bins. the inventory shelves could hold two light fixtures, two cabinets, three countertops, and two sinks. Dishwashers and ranges are similar in size, so Furniture City stored them in similar locations. The warehouse floor could hold a total of four dishwashers and ranges.

Every kitchen set includes exactly 20 ft^2 of tile and exactly 5 rolls of wallpaper. Therefore, 20 ft^2 of a particular style of tile and five rolls of a particular style of wallpaper are required for the styles to be in stock.

a. Formulate and use LINGO to solve a binary integer programming model which will maximize the total number of kitchen sets (and thus the number of customer orders) Furniture City stocks in the local warehouse. Assume that when a customer orders a kitchen set, all the particular items composing that kitchen set are replenished at the local warehouse immediately. (The sets and data section of a LINGO model may be downloaded with this homework assignment.)

Solution

```
MODEL: ! Case: Stocking Kitchen Sets ;
! Solution provided by Grant Mast, Bart Sorensen, Dan Mullen;
SETS:
 KITCHSET/1..20/: s;
FEATURE/1..30/: X;
 BELONG(KITCHSET,FEATURE): A;
 FGROUP/1..7/:CAPACITY;
 ENDSETS
 DATA: ! A(i,j) = 1 if kitchen set i includes feature j ;
 !  T T T T W W W W L L L L C C C C O O O O S S S S D D R R R R;
 A= 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0 1 0 0
 0 1 0 0 1 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 0
 1 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0
 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0
 0 0 0 1 0 0 0 1 1 0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 1 0 1 0 0 0
 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1
 1 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0
 0 1 0 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0 0 1 0 0 0 1
 0 1 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0
 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 1 0 0 0 1 0
 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
 0 1 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 1 0 0
 0 0 0 1 0 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0 0 1 0 0 1 0 0 0 1 0
 0 0 0 1 0 0 0 1 0 0 0 1 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0
 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0
 0 0 1 0 0 0 1 0 0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0
 1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0
 0 1 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0
 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 1
 0 1 0 0 0 0 1 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 1;
CAPACITY = 2 2 2 2 3 2 4;
ENDDATA
 MAX = @SUM(KITCHSET(i):s(i));
@FOR(KITCHSET(i)| i #LE# 13:
 @SUM(FEATURE(j):X(j)*A(i,j))>=8*s(i));
 @FOR(KITCHSET(i) | I #GT# 13:
@SUM(FEATURE(j):X(j)*A(i,j))>=7*s(i));
@FOR(FGROUP(K)| K #LT# 7:
@SUM(FEATURE(J)| J #GE# 4*(K-1)+1 #AND# J #LE# 4*K : X(J) ) <= CAPACITY(K));
 @FOR(FGROUP(K)| K #GE# 7:
@SUM(FEATURE(J)| J #GE# 4*K-3 #AND# J #LE# 4*K+2 : X(J) ) <= CAPACITY(K));

@FOR(KITCHSET(I):  @BIN(s(I)););
@FOR(FEATURE(J):   @BIN(X(J)););

END
```

The number of binary integer variables(20+30=50) exceeds the maximum number which is allowed by the student version of LINGO. The solution shown below was found by using LINGO to create a file in "MPS" format which can be read by most solvers—in this case CPLEX.

b. How many of each feature and style should Furniture City stock in the local warehouse? How many different kitchen sets are in stock?

Solution: Four kitchen sets (#8, 15, 18, and 20) are kept in stock in the solution which was found:

```
CPLEX> display solution variables 1-50
Variable Name          Solution Value
X(2                   1.000000
X(3                   1.000000
X(5                   1.000000
X(7                   1.000000
X(9                   1.000000
X(11                  1.000000
X(13                  1.000000
X(14                  1.000000
X(17                  1.000000
X(18                  1.000000
X(20                  1.000000
X(21                  1.000000
X(23                  1.000000
X(23                  1.000000
X(26                  1.000000
X(28                  1.000000
X(29                  1.000000
X(30                  1.000000
S(8                   1.000000
S(15                  1.000000
S(18                  1.000000
S(20                  1.000000
All other variables in the range 1-50 are zero.
```

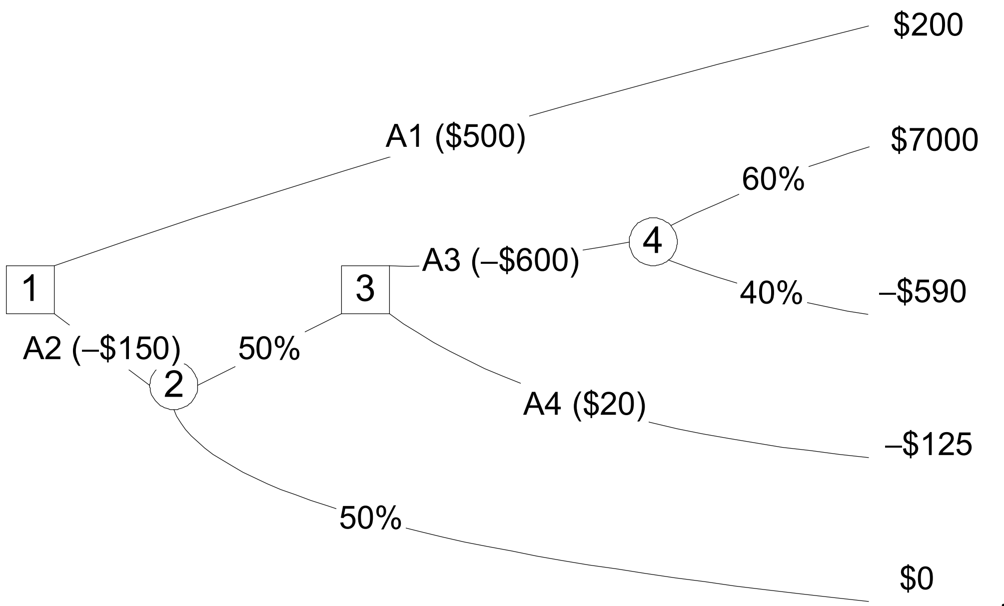
Furniture City decides to discontinue carrying nursery sets, and the warehouse space previously allocated to the nursery department is divided between the existing departments at Furniture City. The kitchen department receives enough additional space to allow it to stock both styles of dishwashers and three of the four styles of ranges.

c. How does the optimal inventory policy for the kitchen department change with this additional warehouse space?

	T1	T2	T3	T4	W1	W2	W3	W4	L1	L2	L3	L4	C1	C2	C3	C4	O1	O2	O3	O4	D1	D2	S1	S2	S3	S4	R1	R2	R3	R4
Set 1		X				X						X		X						X		X		X				X		
Set 2		X			X				X							X				X		X				X		X		
Set 3	X						X			X			X				X				X				X				X	
Set 4			X				X				X				X				X		X		X				X			
Set 5				X				X	X				X					X			X			X			X			
Set 6		X				X				X						X				X		X			X					X
Set 7	X						X					X			X			X			X		X				X			
Set 8		X			X						X		X				X					X			X					X
Set 9		X			X					X					X			X				X		X				X		
Set10	X				X				X				X						X		X					X			X	
Set11			X		X						X				X		X				X		X						X	
Set12		X				X			X					X				X				X				X		X		
Set13				X				X			X				X		X				X			X					X	
Set14				X				X				X	X						X				X				X			
Set15			X				X		X				X				X								X				X	
Set16			X				X					X	X						X					X			X			
Set17	X							X		X					X				X							X			X	
Set18		X					X				X			X						X			X					X		
Set19		X						X				X				X				X				X						X
Set20		X					X		X				X					X							X					X

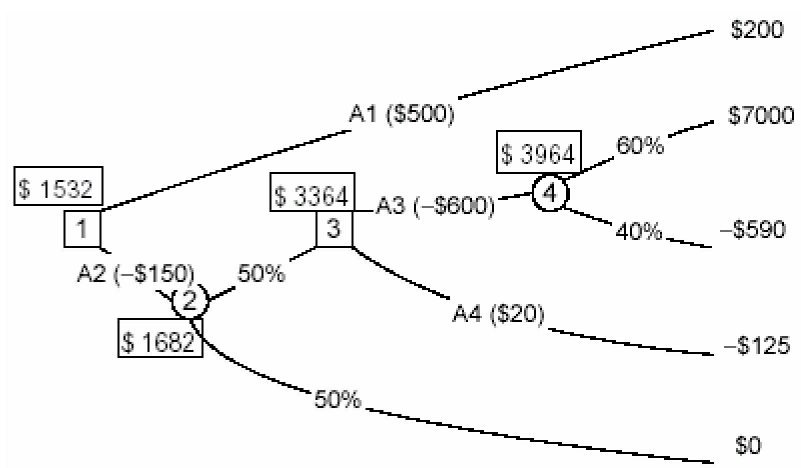
Table: Features composing each of twenty kitchen sets.

4. Decision Trees. Consider the decision tree below: On each decision branch, the immediate payoff (if +) or cost (if –) is shown. The probability is shown on each random branch. On the far right is the final payoff or cost



a. Fold back" the decision tree.

Solution:



b. What is the *expected payoff* at node 1?
   Solution: $1532
c. What is the *optimal decision* at node 1?
   Solution: Select alternative A2.

56:171 Operations Research
Homework #8 Solution -- Fall 2002

1. **Decision Analysis** (an exercise from *Operations Research: a Practical Introduction*, by M. Carter & C. Price)
Suppose that you are in the position of having to buy a used car, and you have narrowed down your choices to two possible models: one car is a private sale and the other is from a dealer. You must now choose between them. The cars are similar, and the only criterion is to minimize expected cost. The dealer car is more expensive, but it comes with a one-year warranty which would cover all costs of repairs. You decide that, if the car will last for 1 year, you can sell it again and recover a large part of your investment. If it falls apart, it will not be worth fixing. After test driving both cars and checking for obvious flaws, you make the following evaluation of probably resale value:

Car	Purchase price	Probability of lasting one year	Estimated resale price
A: Private	\$800	0.3	\$600
B: Dealer	\$1500	0.9	\$1000

a. Which car would you buy?

b. What is the expected value of perfect information (EVPI)?

Suppose you have the opportunity to take car A to an independent mechanic, who will charge you \$50 to do a complete inspection and offer you an opinion as to whether the car will last 1 year. For various subjective reasons, you assign the following probabiliities to the accuracy of the mechanic's opinion:

Given:	Mechanic says Yes	Mechanic says No
A car that will last 1 year	70%	30%
A car that will not last 1 year	10%	90%

(For example, if a car that will last 1 year is taken to the mechanic, there is 70% probability that he will give you the opinion that it will last a year.)

c. Assuming that you must buy one of these two cars, formulate this problem as a decision tree problem.

First we use Bayes' Rule to compute the posterior probabilities of survival & failure of car A, given the mechanic's report::

$$P\{S_i \mid O_j\} = \frac{P\{O_j \mid S_i\} \times P\{S_i\}}{P\{O_j\}}$$

where $P\{O_j\} = \sum_i P\{O_j \mid S_i\} \times P\{S_i\}$

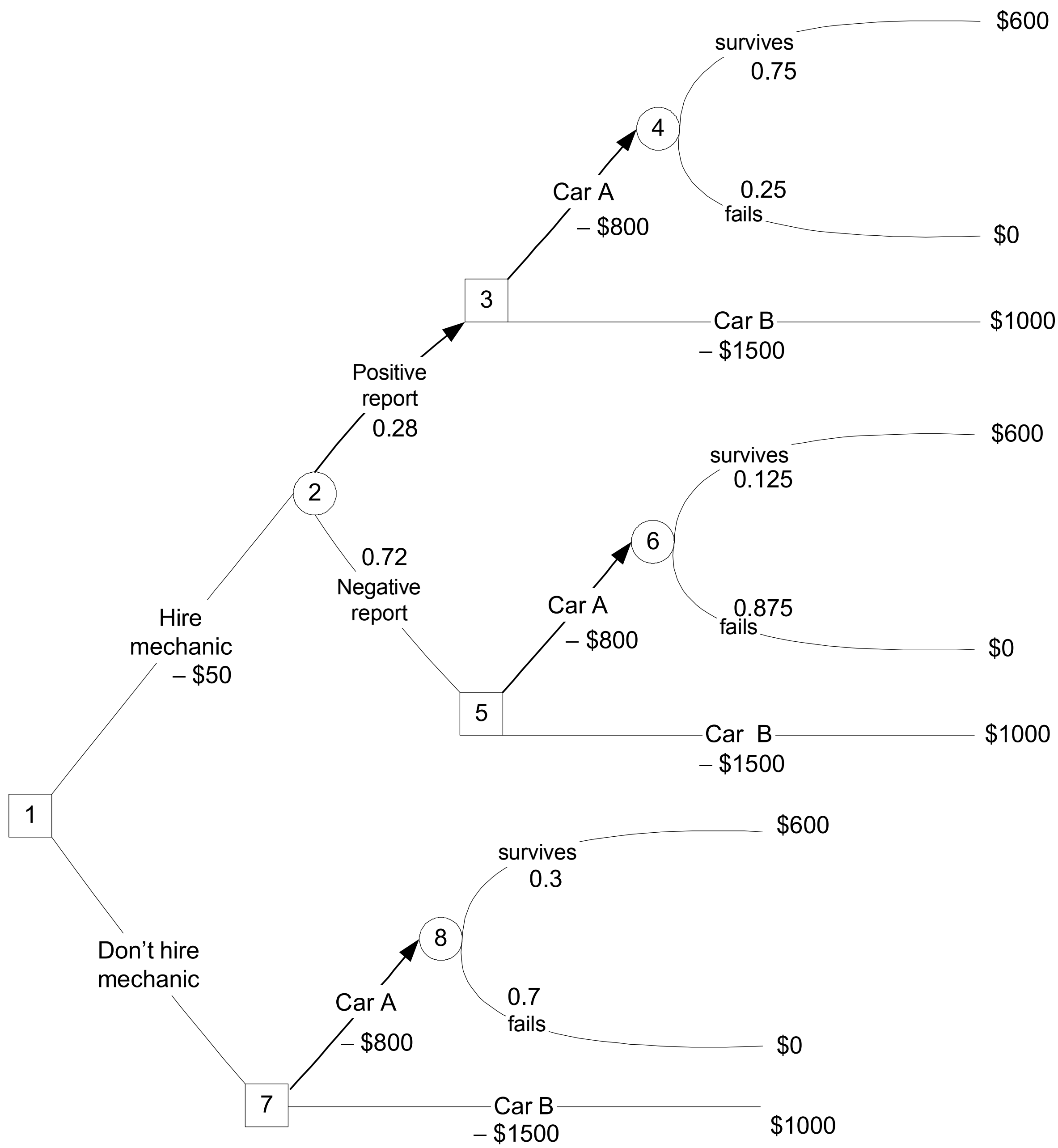
Thus, for example, the probability that the mechanic will give a positive report is 28%.

If he does, car A is 75% likely to survive. If, on the other hand, he gives a negative report (with probability 72%) the care is 87.5% likely to fail.

Template for Posterior Probabilities

Data:		P(Finding | State)	
State of Nature	Prior Probability	Finding	
		PR	NR
Survive	0.3	0.7	0.3
Fail	0.7	0.1	0.9
Posterior Probabilities:		P(State | Finding) State of Nature	
Finding	P(Finding)	Survive	Fail
PR	0.28	0.75	0.25
NR	0.72	0.125	0.875

SOLUTION



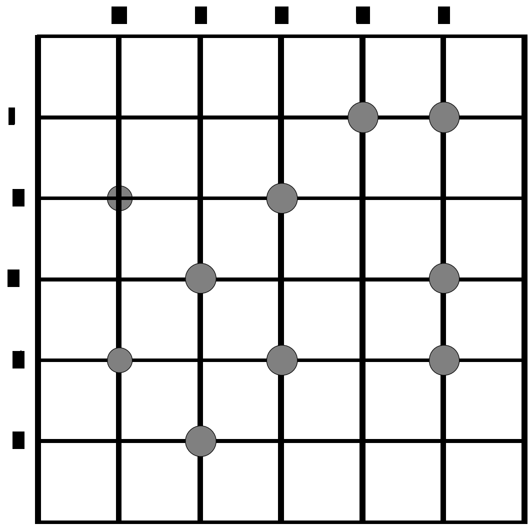
d. What is the expected value of the mechanic's advice?

Is it worth asking for the mechanic's opinion?

What is your optimal decision strategy?

Note: it is not necessary to ask for advice on car B because its problems could be repaired under the warranty!

2. Integer Programming A convenience store chain is planning to enter a growing market and must determine where to open several new stores. The map shows the major streets in the area being considered. (Adjacent streets are 1 mile apart. A Avenue, B Avenue, etc. are N-S streets (with A Ave. being the westernmost) while 1st Street, 2nd Street, etc. are E-W streets (with 1st Street being the furthest north.) . The symbol ● indicates possible store locations. All travel must follow the street network, so distance is determined with a rectilinear metric. For instance, the distance between corners A1 and C2 is 3 miles.



- The costs of purchasing property & constructing stores at the various locations are as follows:

Location	A2	A4	B3	B5	C2	C4	D1	E1	E3	E4
Cost	100	80	90	50	80	90	100	70	90	80

- No two stores can be on the same street (either north-south or east-west).
- Sttores must be at least 3 miles apart.
- Every grid point (A1, B2, etc.) must be no more than 3 miles from a store.

a. Set up an integer programming model that can be used to find the optimal store locations.
b. Find the optimal locations and the minimum cost..

LINDO model:

```
MIN 100XA2 + 80XA4 + 90XB3 + 50XB5 + 80XC2 + 90XC4 + 100XD1 + 70XE1 + 90XE3
    + 80XE4
ST

   !No two stores on same street(vertical)
   XA2 + XA4 <= 1
   XB3 + XB5 <= 1
   XC2 + XC4 <= 1
   XD1         <= 1
   XE1 + XE3 + XE4 <=1

   !No two stores on same street (horizontal)
   XD1 + XE1 <= 1
   XA2 + XC2 <= 1
   XB3 + XE3 <= 1
```

```
XA4 + XC4 + XE4 <= 1
XB5       <= 1

!Store A2 3 mile constraint
XA2 + XA4 <= 1
XA2 + XB3 <= 1
XA2 + XC2 <= 1

!Store A4 3 mile constraint
XA4 + XB3 <= 1
XA4 + XB5 <= 1
XA4 + XC4 <= 1

!Store B3 3 mile constraint
XB3 + XB5 <= 1
XB3 + XC2 <= 1
XB3 + XC4 <= 1

!Store B5 3 mile constraint
XB5 + XC4 <= 1

!Store C2 3 mile constraint
XC2 + XC4 <= 1
XC2 + XD1 <= 1
!XC2 + XE1 <= 1
!XC2 + XE3 <= 1

!Store C4 3 mile constraint
XC4 + XE3 <= 1
XC4 + XE4 <= 1

!Store D1 3 mile constraint
XD1 + XE1 <= 1
!XD1 + XE3 <= 1

!Store E1 3 mile constraint
XE1 + XE3 <= 1
!XE1 + XE4 <= 1

!Store E3 3 mile constraint
XE3 + XE4 <= 1

!Grid Point 3 mile constraint

!A
XA2 + XA4 + XB3 + XC2 >= 1
XA2 + XA4 + XB3 + XC2 >= 1
XA2 + XA4 + XB3 + XB5 + XC2 + XC4 >= 1
XA2 + XA4 + XB3 + XB5 + XC4 >= 1
XA2 + XA4 + XB3 + XB5 + XC4 >= 1

!B
XA2 + XB3 + XC2 + XD1 + XE1 >= 1
XA2 + XA4 + XB3 + XB5 + XC2 + XC4 + XD1 >= 1
XA2 + XA4 + XB3 + XB5 + XC2 + XC4 + XE3 >= 1
XA2 + XA4 + XB3 + XB5 + XC2 + XC4 + XE4 >= 1
XA4 + XB3 + XB5 + XC4 >= 1
```

```
  !C
  XA2 + XB3 + XC2 + XC4 + XD1 + XE1 >= 1
  XA2 + XB3 + XC2 + XC4 + XD1 + XE1 + XE3 >= 1
  XA2 + XA4 + XB3 + XB5 + XC2 + XC4 + XD1 + XE3 + XE4 >= 1
  XA4 + XB3 + XB5 + XC2 + XC4 + XE3 + XE4 >= 1
  XA4 + XB3 + XB5 + XC2 + XC4 + XE4 >= 1

  !D
  XC2 + XD1 + XE1 + XE3 >= 1
  XD2 + XA2 + XB3 + XC2 + XC4 + XD1 + XE1 + XE3 + XE4 >= 1
  XB3 + XC2 + XC4 + XD1 + XE1 + XE3 + XE5 >= 1
  XA4 + XB3 + XB5 + XC2 + XC4 + XD1 + XE3 + XE4 >= 1
  XB5 + XC4 + XE3 + XE4 >= 1

  !E
  XC2 + XD1 + XE1 + XE3 + XE4 >= 1
  XE2 + XC2 + XD1 + XE1 + XE3 + XE4 >= 1
  XB3 + XC2 + XC4 + XB1 + XE1 + XE3 + XE4 >= 1
  XC4 + XE1 + XE3 + XE4 >= 1
  XB5 + XC4 + XE3 + XE4 >= 1

END

INT 10
```

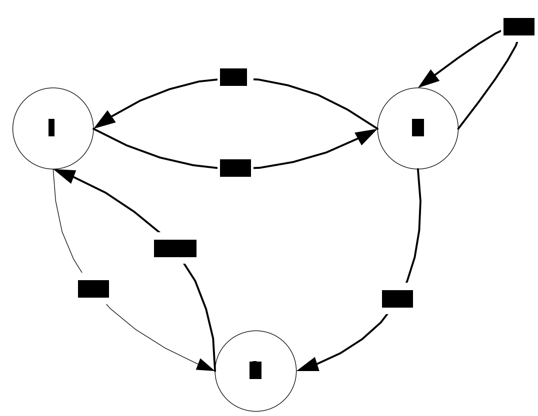
Solution:

```
        OBJECTIVE FUNCTION VALUE

         1)      200.0000

  VARIABLE        VALUE          REDUCED COST
       XB5         1.000000         50.000000
       XC2         1.000000         80.000000
       XE1         1.000000         70.000000
```

3. Discrete-time Markov Chains A stochastic process with three states has the transition probabilities shown below:



a. Write the transition probability matrix P.

Suppose that the system begins in state 1, and is in state 3 after two steps.

b. What are the possible sequences of two transitions that might have occurred?

c. What are the probabilities of each of these sequences?

d. What is the probability $p_{13}^{(2)}$?

c. Write the equations which determine π, the steadystate probability distribution.

d. Compute the steadystate probability distribution π.