56:171
Operations Research
Fall 2001

Homework Solutions

Dept of Mechanical & Industrial Engineering
University of Iowa

56:171 Operations Research
Homework #1 Solution – Fall 2001

1. A company makes two products in a single plant. It runs this plant for 100 hours each week. Each unit of product A that the company produces consumes two hours of plant capacity, earns the company a profit of $1000, and causes, as an undesirable side effect, the emission of 4 ounces of particulates. Each unit of product B that the company produces consumes one hour of capacity, earns the company a profit of $2000, and causes the emission of 3 ounces of particulates and 1 ounce of chemicals. The EPA (environmental Protection Agency) requires the company to limit particulate emission to at most 240 ounces per week and chemical emission to at most 60 ounces per week.

a. Write down the linear programming model for maximizing the company's profits subject to the restrictions on production capacity and emissions.

Solution:

Decision variables:
A = # units of product A that the company produces per week
B = # units of product B that the company produces per week

Objective Function:
`Max 1000 A + 2000 B` *(profit $/week)*

Constraints:
- Restrictions on production
  $2A + B \le 100$
- Restrictions on emission
  $4A + 3B \le 240$
  $B \le 60$
- Nonnegativity constraint on each of the two variables.
  $A \ge 0$, $B \ge 0$

b. What is the optimal solution of the LP?

Solution: (from LINDO)

```
        OBJECTIVE FUNCTION VALUE

        1)      135000.0

  VARIABLE        VALUE          REDUCED COST
         A        15.000000          0.000000
         B        60.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        10.000000          0.000000
        3)         0.000000        250.000000
        4)         0.000000       1250.000000

 NO. ITERATIONS=       2
```

The optimal plan is to produce each week 15 units of product A and 60 units of product B, which earns the company a profit of $135,000/week.

2. Cattle feed can be mixed from oats, corn, alfalfa, and peanut hulls. The following table shows the current cost per ton (in dollars) of each of these ingredients, together with the percentage of recommended daily allowances for protein, fat, and fiber that a serving of it fulfills.

	Oats	Corn	Alfalfa	Peanut hulls
% protein	60	80	55	40
% fat	50	70	40	100
% fiber	90	30	60	80
Cost $/ton	200	150	100	75

We want to find a minimum cost way to produce feed that satisfies at least 60% of the daily allowance for protein and fiber while not exceeding 60% of the fat allowance.

Solution:

Decision variables:

Define the variables OATS, CORN, ALFALFA, and HULLS to be the quantity (in tons) mixed to obtain a ton of cattle feed.

Complete LP Formulation :

```
MIN    Z = 200 OATS + 150 CORN + 100 ALFALFA + 75 HULLS
SUBJECT TO
       60 OATS + 80 CORN + 55 ALFALFA +  40 HULLS >=   60
       50 OATS + 70 CORN + 40 ALFALFA + 100 HULLS <=   60
       90 OATS + 30 CORN + 60 ALFALFA +  80 HULLS >=   60
       OATS + CORN + ALFALFA + HULLS =    1
       OATS >= 0, CORN >= 0, ALFALFA >= 0, HULLS >= 0
```

Solution from LINDO :

```
LP OPTIMUM FOUND AT STEP      4

      OBJECTIVE FUNCTION VALUE

      1)      125.0000

VARIABLE        VALUE          REDUCED COST
    OATS         0.157143         0.000000
    CORN         0.271429         0.000000
 ALFALFA         0.400000         0.000000
   HULLS         0.171429         0.000000


     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)         0.000000        -5.000000
      3)         0.000000         0.000000
      4)         0.000000        -2.500000
      5)         0.000000       325.000000

NO. ITERATIONS=       4
```

The optimal solution is to mix 0.16 tons of oats, 0.27 tons of corns, 0.4 tons of alfalfa, and 0.17 tons of peanut hulls to obtain a ton of feed. The cost of a ton of feed is $125.

3. "Mama's Kitchen" serves from 5:30 a.m. each morning until 1:30 p.m. in the afternoon. Tables are set and cleared by busers working 4-hour shifts beginning on the hour from 5:00 a.m. through 10:00 a.m. Most are college students who hate to get up in the morning, so Mama's pays $9 per hour for the 5:00, 6:00, and 7:00 a.m. shifts, and $7.50 per hour for the others. (That is, a person works a shift consisting of 4 consecutive hours, with the wages equal to 4×$9 for the three early shifts, and 4×$7.50 for the 3 later shifts.) The manager seeks a minimum cost staffing plan that will have at least the number of busers on duty each hour as specified below:

	5 am	6 am	7 am	8 am	9 am	10am	11am	Noon	1 pm
#reqd	2	3	5	5	3	2	4	6	3

Solution:

Decision variables:

Xi = the # of employees who start to work on ith shift. (i = 1, 2, ... , 6)

LP Formulation :

```
MIN 36 X1 + 36 X2 + 36 X3 + 30 X4 + 30 X5 + 30 X6
SUBJECT TO
 X1                             >=   2  (Restriction of # of busers on duty at  5am)
 X1 + X2                        >=   3  (Restriction of # of busers on duty at  6am)
 X1 + X2 + X3                   >=   5  (Restriction of # of busers on duty at  7am)
 X1 + X2 + X3 + X4              >=   5  (Restriction of # of busers on duty at  8am)
      X2 + X3 + X4 + X5         >=   3  (Restriction of # of busers on duty at  9am)
           X3 + X4 + X5 + X6    >=   2  (Restriction of # of busers on duty at 10am)
                X4 + X5 + X6    >=   4  (Restriction of # of busers on duty at 11am)
                     X5 + X6    >=   6  (Restriction of # of busers on duty at 12pm)
                          X6    >=   3  (Restriction of # of busers on duty at  1pm)
 Xi >= 0  (for i = 1,2,3,4,5,6)         (Sign restrictions)
```

Solution from LINDO :

```
     LP OPTIMUM FOUND AT STEP      9

            OBJECTIVE FUNCTION VALUE
            1)      360.0000

      VARIABLE        VALUE          REDUCED COST
            X1         3.000000          0.000000
            X3         2.000000          0.000000
            X5         3.000000          0.000000
            X6         3.000000          0.000000

           ROW   SLACK OR SURPLUS     DUAL PRICES
            2)         1.000000          0.000000
            3)         0.000000          0.000000
            4)         0.000000         -6.000000
            5)         0.000000        -30.000000
            6)         2.000000          0.000000
            7)         6.000000          0.000000
            8)         2.000000          0.000000
```

```
   9)          0.000000          -30.000000
  10)          0.000000            0.000000
```

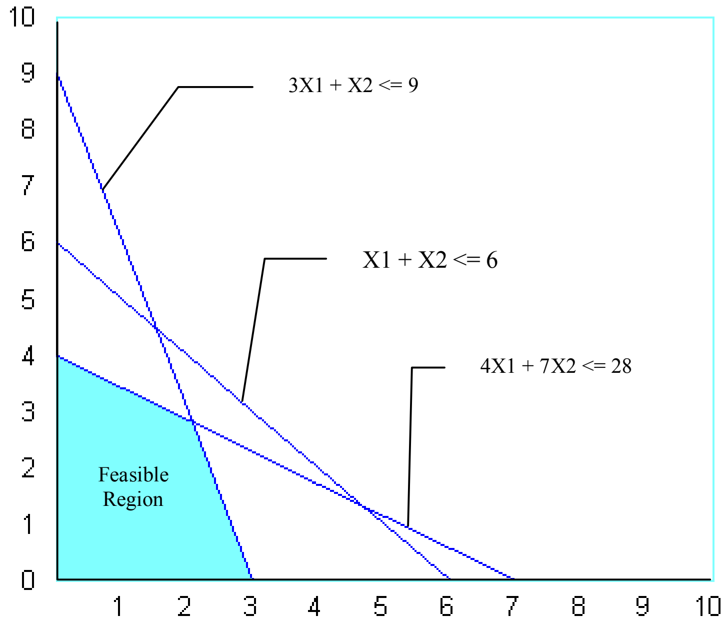
That is, the optimal staffing plan is to employ

3 busers for the 1st shift(4-hour shifts begins at 5:00a.m.),
2 busers for the 3rd shift(4-hour shifts begins at 7:00a.m.),
3 busers for the 5th shift(4-hour shifts begins at 9:00a.m.), and
3 busers for the 6th shift(4-hour shifts begins at 10:00a.m.).

Note that the solution of the LP (with continuous variables) is actually integer-valued!

4. a. Draw the feasible region of the following LP:

$$\begin{aligned} \text{Maximize} \quad & 3X_1 + 2X_2 \\ \text{subject to} \quad & 4X_1 + 7X_2 \le 28 \\ & X_1 + X_2 \le 6 \\ & 3X_1 + X_2 \le 9 \\ & X_1 \ge 0, X_2 \ge 0 \end{aligned}$$



b. Use the simplex algorithm to find the optimal solution of the above LP. *(Show the initial and each succeeding tableau.)*

Solution: After adding slack variables X3, X4, and X5 to the three constraints, we obtain the initial tableau as follows :

Basis	– Z	X1	X2	X3	X4	X5		RHS
	1	3	2	0	0	0		0
X3	0	4	7	1	0	0		28
X4	0	1	1	0	1	0		6
X5	0	***3***	1	0	0	1		9

Entering variable : X1 ; Leaving variable : X5

Either X1 or X2 might be selected to enter the basis – both have positive relative profits in row 0. Because it has the larger relative profit, we here enter X1 into the basis. The minimum ratio test

$\left(\text{i.e., Min}\left\{\frac{28}{4},\frac{6}{1},\frac{9}{3}\right\}=3\right)$ indicates that the pivot should be in the bottom row, i.e., X5 should leave the basis. The resulting tableau is shown below :

Basis	– Z	X1	X2	X3	X4	X5		RHS
	1	0	1	0	0	-1		-9
X3	0	0	5.666667	1	0	-1.33333		16
X4	0	0	0.666667	0	1	-0.33333		3
X1	0	1	0.333333	0	0	0.333333		3

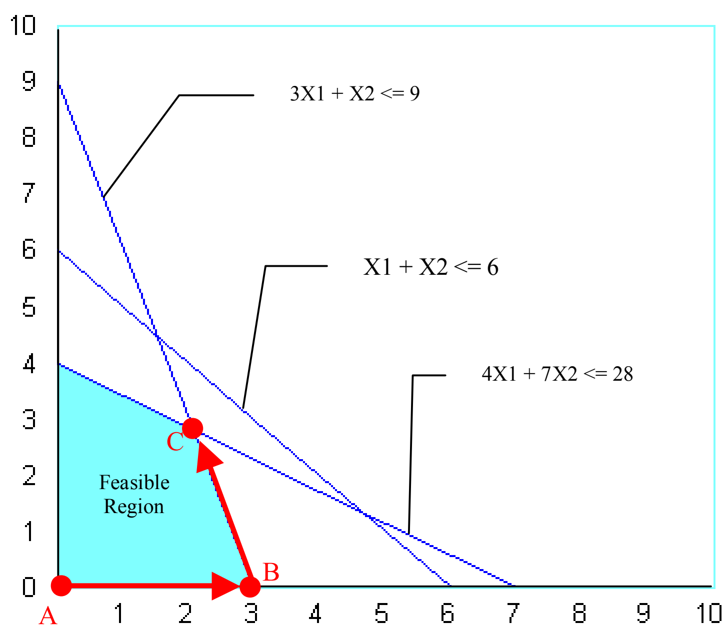
Entering variable : X2 ; Leaving variable : X3

Since X2 is the only variable with a positive relative profit in row0, we enter X2 into the basis. The minimum ratio test $\left(\text{i.e., Min}\left\{\frac{16}{5.6667},\frac{3}{0.6667},\frac{3}{0.3333}\right\}=2.8235\right)$ indicates that X3 should leave the basis, i.e., the pivot should be in row 1. The resulting tableau is shown below :

Basis	– Z	X1	X2	X3	X4	X5		RHS
	1	0	0	-0.17647	0	-0.76471		-11.8235
X2	0	0	1	0.176471	0	-0.23529		2.823529
X4	0	1	0	0	1	0		6
X1	0	1	0	-0.05882	0	0.411765		2.058824

Since each variable has a nonpositive relative profit in row 0, this is an optimal tableau. Thus, the optimal solution to LP is
Z = 11.8235 , X2 = 2.8235 , X4 = 6 , X1 = 2.0588 , X3 = X5 = 0.

c. On the sketch of the feasible region in (a), indicate the initial basic solution and the basic solution at each succeeding iteration.



Extreme point A : Initial basic solution
Extreme point B : Second basic solution
Extreme point C : New basic solution

5a. What is INFORMS?

Institute **f**or **O**perations **R**esearch and the **M**anagement **S**ciences

5b. Find (on the INFORMS website at http://www.informs.org) a definition of "Operations Research".

Operations Research (OR) and the Management Sciences (MS) are the professional disciplines that deal with the application of information technology for informed decision-making.
OR/MS Professionals aim to provide rational bases for decision making by seeking to understand and structure complex situations and to use this understanding to predict system behavior and improve system performance. Much of this work is done using analytical and numerical techniques to develop and manipulate mathematical and computer models of organizational systems composed of people, machines, and procedures.

56:171 Operations Research
Homework #2 Solutions -- Fall 2001

The Diet Problem. "The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person." Go to the URL:

http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies/diet/index.html

a. What are the restrictions on calories in the default set of requirements?

Solution: $2,000 \leq \text{calories} \leq 2,250$

☐	Frozen Broccoli	$0.16	10 Oz Pkg	☐	Carrots,Raw	$0.07	1/2 Cup Shredded
☐	Celery, Raw	$0.04	1 Stalk	☐	Frozen Corn	$0.18	1/2 Cup
☐	Lettuce,Iceberg,Raw	$0.02	1 Leaf	☐	Peppers, Sweet, Raw	$0.53	1 Pepper
☐	Potatoes, Baked	$0.06	1/2 Cup	☐	Tofu	$0.31	1/4 block
☐	Roasted Chicken	$0.84	1 lb chicken	☐	Spaghetti W/ Sauce	$0.78	1 1/2 Cup
☐	Tomato,Red,Ripe,Raw	$0.27	1 Tomato, 2-3/5 In	☐	Apple,Raw,W/Skin	$0.24	1 Fruit,3/Lb,Wo/Rf
☐	Banana	$0.15	1 Fruit,Wo/Skn&Seeds	☐	Grapes	$0.32	10 Fruits,Wo/Rf
☐	Kiwifruit,Raw,Fresh	$0.49	1 Med Frt,Wo/Skin	☐	Oranges	$0.15	1 Frt,2-5/8 Diam
☐	Bagels	$0.16	1 Oz	☐	Wheat Bread	$0.05	1 Sl
☐	White Bread	$0.06	1 Sl	☐	Oatmeal Cookies	$0.09	1 Cookie
☐	Apple Pie	$0.16	1 Oz	☐	Chocolate Chip Cookies	$0.03	1 Cookie
☐	Butter,Regular	$0.05	1 Pat	☐	Cheddar Cheese	$0.25	1 Oz
☐	3.3% Fat,Whole Milk	$0.16	1 C	☐	2% Lowfat Milk	$0.23	1 C
☐	Skim Milk	$0.13	1 C	☐	Poached Eggs	$0.08	Lrg Egg
☐	Scrambled Eggs	$0.11	1 Egg	☐	Bologna,Turkey	$0.15	1 Oz
☐	Frankfurter, Beef	$0.27	1 Frankfurter	☐	Ham,Sliced,Extralean	$0.33	1 Sl,6-1/4x4x1/16 In
☐	Kielbasa,Prk	$0.15	1 Sl,6x3-3/4x1/16 In	☐	Cap'N Crunch	$0.31	1 Oz
☐	Cheerios	$0.28	1 Oz	☐	Corn Flks, Kellogg'S	$0.28	1 Oz
☐	Raisin Brn, Kellg'S	$0.34	1.3 Oz	☐	Rice Krispies	$0.32	1 Oz
☐	Special K	$0.38	1 Oz	☐	Oatmeal	$0.82	1 C
☐	Malt-O-Meal,Choc	$0.52	1 C	☐	Pizza W/Pepperoni	$0.44	1 Slice
☐	Taco	$0.59	1 Small Taco	☐	Hamburger W/Toppings	$0.83	1 Burger
☐	Hotdog, Plain	$0.31	1 Hotdog	☐	Couscous	$0.39	1/2 Cup
☐	White Rice	$0.08	1/2 Cup	☐	Macaroni,Ckd	$0.17	1/2 Cup
☐	Peanut Butter	$0.07	2 Tbsp	☐	Pork	$0.81	4 Oz
☐	Sardines in Oil	$0.45	2 Sardines	☐	White Tuna in Water	$0.69	3 Oz
☐	Popcorn,Air-Popped	$0.04	1 Oz	☐	Potato Chips,Bbqflvr	$0.22	1 Oz
☐	Pretzels	$0.12	1 Oz	☐	Tortilla Chip	$0.19	1 Oz
☐	Chicknoodl Soup	$0.39	1 C (8 Fl Oz)	☐	Splt Pea&Hamsoup	$0.67	1 C (8 Fl Oz)
☐	Vegetbeef Soup	$0.71	1 C (8 Fl Oz)	☐	Neweng Clamchwd	$0.75	1 C (8 Fl Oz)
☐	Tomato Soup	$0.39	1 C (8 Fl Oz)	☐	New E Clamchwd,W/Mlk	$0.99	1 C (8 Fl Oz)
☐	Crm Mshrm Soup,W/Mlk	$0.65	1 C (8 Fl Oz)	☐	Beanbacn Soup,W/Watr	$0.67	1 C (8 Fl Oz)

b. What is the minimum-cost menu meeting the nutritional requirements using the foods you indicated? Indicate the solution in the left 2 columns of the table below.

Change the default upper limit on calories to 1500/day and solve the problem again. (Be sure that the lower bound ≤ upper bound!)

c. What is the minimum-cost menu meeting the nutritional requirements using the foods you indicated? Indicate the solution in the right 2 columns of the table below.

Example solution: Note that only six foods are included in the optimal solution! This is a very economical menu, satisfying nutritional requirements, but probably not very satisfying in other ways!

Quantity (# servings)	Cost	Foods	Quantity (# servings)	Cost
3.09	0.22	1. Carrots,Raw	3.10	0.22
10.00	0.40	2. Celery, Raw	10.00	0.40
1.44	0.03	3. Lettuce,Iceberg,Raw	2.19	0.04
		4. Roasted Chicken		
		5. Spaghetti W/ Sauce		
		6. Wheat Bread		
		7. White Bread		
		8. Chocolate Chip Cookies		
		9. Butter,Regular		
		10. 3.3% Fat,Whole Milk		
		11. 2% Lowfat Milk		
2.13	0.28	12. Skim Milk	1.85	0.24
		13. White Rice		
3.18	0.22	14. Peanut Butter	0.15	0.01
9.95	0.40	15. Popcorn,Air-Popped	8.22	0.33
Total Cost :	$1.54/day		**Total Cost :**	$1.24/day

* **New restrictions on calories : 1,000 ≤ calories ≤ 1,500**

2. Below are several simplex tableaus. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through **G**, according to the descriptions below. *Also answer the question accompanying each classification, if any.*

(**A**) Nonoptimal, nondegenerate tableau with bounded solution. *Circle a pivot element which would improve the objective.*

(**B**) Nonoptimal, degenerate tableau with bounded solution. *Circle an appropriate pivot element. Would the objective improve with this pivot?*

(**C**) Unique nondegenerate optimum.

(**D**) Optimal tableau, with alternate optimum. *State the values of the basic variables. Circle a pivot element which would lead to another optimal basic solution. Which variable will enter the basis, and at what value?*

(**E**) Objective unbounded (below). *Specify a variable which, when going to infinity, will make the objective arbitrarily low.*

(F) Tableau with infeasible primal

Solution:

(i)	-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
	1	-3	0	1	1	0	0	2	3	-45	
	0	0	0	-4	0	0	1	0	0	9	A
	0	-6	0	3	-2	1	0	2	3	5	
	0	4	1	2	-5	0	0	1	1	8	

(ii)	-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
	1	3	0	-1	3	0	0	2	-2	-45	
	0	0	0	-4	0	0	1	3	0	9	B
	0	-4	1	2	-5	0	0	-2	1	0	
	0	-6	0	3	-2	1	0	-4	3	5	

(iii)	-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
	1	3	0	1	1	0	0	3	5	-45	
	0	0	0	-4	0	0	1	3	0	3	C
	0	4	1	2	-5	0	0	2	1	7	
	0	-6	0	3	-2	1	0	-4	3	15	

(iv)	-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
	1	3	0	1	-3	0	0	2	0	-45	
	0	0	0	-1	0	0	1	3	0	9	E
	0	4	1	-4	-5	0	0	2	1	3	
	0	-6	0	3	-2	1	0	-4	3	5	

(v)	-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	
	1	3	0	0	1	0	0	0	12	-45	
	0	0	0	-4	0	0	1	3	0	9	D
	0	4	1	2	-5	0	0	2	1	8	
	0	-6	0	3	-2	1	0	-4	3	5	

56:171 Operations Research
Homework #3 Solution, Fall 2001

1. ***Revised Simplex Algorithm:*** Consider the LP:

$$\text{Minimize } z = 3x_1 + 2x_2 + 6x_3$$

$$\text{subject to } \begin{cases} 4x_1 + 8x_2 - x_3 \le 5 \\ 7x_1 - 2x_2 + 2x_3 \ge 4 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

By introducing slack and surplus variables, the problem is rewritten as Min cx subject to Ax=b, x≥0 where

C = [3, 2, 6, 0, 0], b = [5, 4] and A = $\begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}$.

Note: In the computation that follows, you need not use more than 3 significant digits.

Suppose that "Phase I" has found the initial basis B ={1,2} for the constraints, i.e., basic variables x_1 and x_2.

a. Then using the revised simplex method requires computation of:

$$c_B = [3 \quad 2], A^B = \begin{bmatrix} 4 & 8 \\ 7 & -2 \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix},$$

$$x_B = \left(A^B\right)^{-1} b = [0.65625 \quad 0.296875], \pi = c_B \left(A^B\right)^{-1} = [0.3125 \quad 0.25]$$

b. Use the simplex multiplier vector π to compute the reduced cost of x_3:

$$\overline{c}_3 = c_3 - \pi A^3 = 6 - [0.3125, 0.25]\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5.8125$$

c. Will entering x_3 into the basis improve the solution? _NO_ (since the reduced cost is positive!)

d. Use the simplex multiplier vector π to compute the reduced cost of x_4:

$$\overline{c}_4 = c_4 - \pi A^4 = -0.3125$$

e. Will entering x_4 into the basis improve the solution? YES, since its reduced cost is negative

f. Select either x_3 or x_4 to enter the basis, and compute the substitution rates (where j=3 or 4):

$$\alpha = \left(A^B\right)^{-1} A^j = \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.03125 \\ 0.125 \end{bmatrix}$$

g. Perform the minimum ratio test to determine which variable leaves the basis.

$$\min\left\{\frac{x_B}{\alpha_B} : \alpha_B > 0\right\} = \min\left\{\frac{0.65625}{0.03125}, \frac{0.296875}{0.109375}\right\} = \min\{21, 2.71429\} = 2.71429$$

Since the second ratio is minimum, the second basic variable is replaced by the entering variable, and the new basis is B = {1, 4}.

h. Compute, for this new basis,

$$c_B = [3,0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix},$$

$$x_B = \left(A^B\right)^{-1} b = [0.571429 \quad 2.71429], \pi = c_B \left(A^B\right)^{-1} = [0 \quad 0.428571]$$

i. Find a nonbasic variable, if any, which would improve the solution if entered into the basis, and determine which variable would be replaced in the basis.

The reduced costs of the nonbasic variables are now: $\overline{c}_2 = 2.85714, \overline{c}_3 = 5.14286, \overline{c}_5 = 0.428571$

Since the reduced costs are all nonnegative, the current solution is optimal.

2. ***LP Duality:*** Write the dual of the following LP:

$$Min\ 3x_1 + 2x_2 - 4x_3$$

$$\text{subject to} \begin{cases} 5x_1 - 7x_2 + x_3 \geq 12 \\ x_1 - x_2 + 2x_3 = 18 \\ 2x_1 - x_3 \leq 6 \\ x_2 + 2x_3 \geq 10 \\ x_j \geq 0, \text{ j=1,2,3} \end{cases}$$

Solution:

$$Maximize\ \ 12y_1 + 18y_2 + 6y_3 + 10y_4$$

subject to

$$\begin{cases} 5y_1 + y_2 + y_3 \leq 3 \\ -7y_1 - y_2 + y_4 \leq 2 \\ y_1 + 2y_2 - y_3 + 2y_4 \leq -4 \end{cases}$$

with sign restrictions: $y_1 \geq 0$, $y_3 \leq 0$, $y_4 \geq 0$ (y_2 unrestricted in sign)

3. Consider the following primal LP problem:

$$Max\ x_1 + 2x_2 - 9x_3 + 8x_4 - 36x_5$$

$$\text{subject to} \begin{cases} 2x_2 - x_3 + x_4 - 3x_5 \leq 40 \\ x_1 - x_2 + 2x_4 - 2x_5 \leq 10 \\ x_j \geq 0, \text{ j=1,2,3,4,5} \end{cases}$$

a. Write the dual LP problem

Solution:

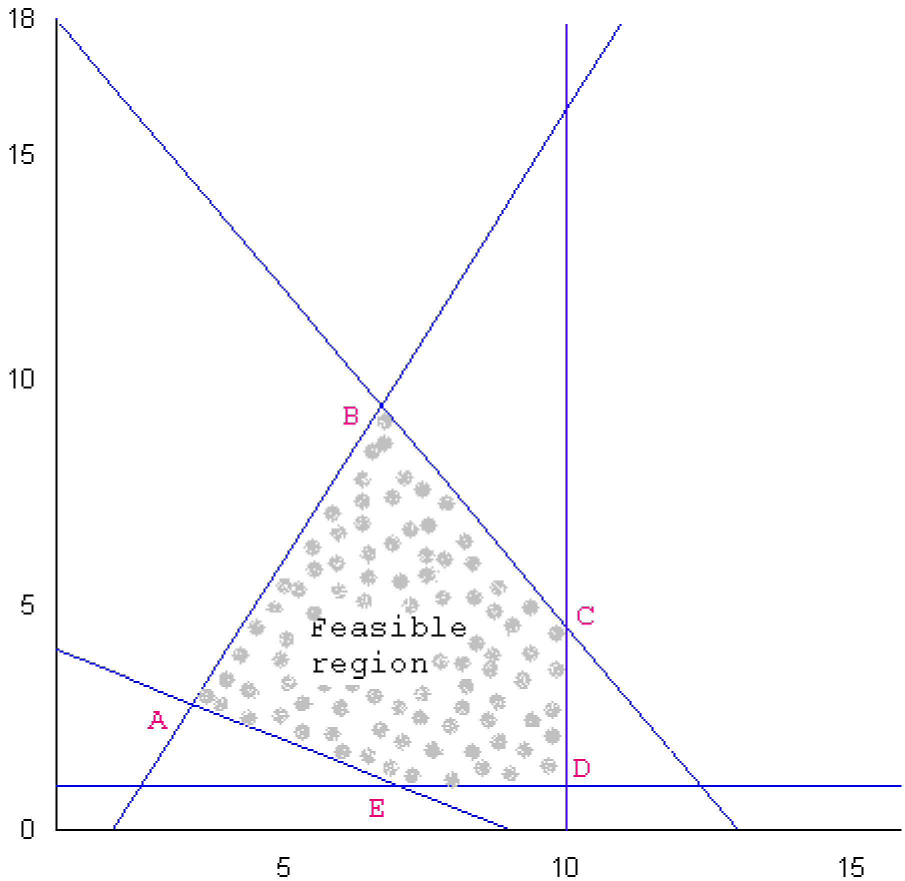
$$Min\ \ 40y_1 + 10y_2$$

subject to:

$$\begin{cases} y_2 \geq 1 \\ 2y_1 - y_2 \geq 2 \\ -y_1 \geq -9 \\ y_1 + 2y_2 \geq 8 \\ -3y_1 - 2y_2 \geq -36 \end{cases}$$

and $y_j \geq 0$, j=1,2

b. Sketch the feasible region of the dual LP in 2 dimensions, and use it to find the optimal solution.



The objective function evaluated at the points A(2.4, 2.8), B(4.667, 7.333), C(9, 4.5), D(9, 1), and E(3.5, 1) are 124, 260, 405, 370, and 150, respectively, so that the minimum value(=124) is achieved at (2.4, 2.8), i.e., y_1=2.4, y_2=2.8.

c. Using complementary slackness conditions,

- ♦ write equations which must be satisfied by the optimal primal solution x*

Solution: Since both y_1 , y_2 are positive, primal constraint (1) and (2) must be tight, i.e.,

$2x_2 - x_3 + x_4 - 3x_5 = 40$, $x_1 - x_2 + 2x_4 - 2x_5 = 10$.

- ♦ which primal variables must be zero?

Solution: since constraints (1), (3), and (5) are slack, the primal variables x_1, x_3, and x_5 must be zero.

d. Using the information in (c.), determine the optimal solution x*.

Solution: x_2 = 14 & x_4 = 12 , while x_j = 0 for j=1, 3, 5.

e. Compare the optimal objective values of the primal and dual solutions.

Solution: at x*, the objective function is 2×14 + 8×12=124, which is identical to the optimal dual objective value.

4. ***LP Sensitivity Analysis***: Cornco produces two products: PS and QT. The sales price for each product and the maximum quantity of each that can be sold during each of the next three months are:

	Month 1		Month 2		Month 3	
Product	Price	Demand	Price	Demand	Price	Demand
PS	\$40	50	\$60	45	\$55	50
QT	\$35	43	\$40	50	\$44	40

Each product must be processed through two assembly lines: 1 & 2. The number of hours required by each product on each assembly line are:

Product	Line 1	Line 2
PS	3 hours	2 hours
QT	2 hours	2 hours

The number of hours available on each assembly line during each month are:

Line	Month 1	Month 2	Month 3
1	200	160	190
2	140	150	110

Each unit of PS requires 4 pounds of raw material while each unit of QT requires 3 pounds. A total of 710 units of raw material can be purchased during the three-month interval at $3 per pound. At the beginning of month 1, 10 units of PS and 5 units of QT are available. It costs $10 to hold a unit of a unit of either product in inventory for a month.

Solution:

Define variables

Pt = # units of product PS produced in month t, t=1,2,3

Qt = # units of product QT produced in month t, t=1,2,3

R = (total) # units of raw material purchased

St = # units of product PS sold in month t, t=1,2,3

Tt = # units of product QT sold in month t, t=1,2,3

It = # units of product PS in inventory at end of month t, t=0,1,2

Jt = # units of product QT in inventory at end of month t, t=0,1,2

Objective: Maximize profit =

40S1 + 60S2 + 55S3	*(revenue from sale of PS)*
+35T1 + 40T2 + 44T3	*(revenue from sale of QT)*
- 3R	*(purchase of raw material)*
- 10I1 - 10I2	*(storage cost of PS)*
- 10J1 - 10 J2	*(storage cost of QT)*

Subject to the constraints:

R ≤ 710	*(limited availability of raw material)*
S1 ≤ 50, S2 ≤ 45, S3 ≤ 50	*(demand constraints for PS)*
T1 ≤ 43, T2 ≤ 50, T3 ≤ 40	*(demand constraints for QT)*
3P1 +2Q1 ≤ 200	*(hours available on line 1, month 1)*
3P2 +2Q2 ≤ 160	*(hours available on line 1, month 2)*
3P3 +2Q3 ≤ 190	*(hours available on line 1, month 3)*
2P1 +2Q1 ≤ 140	*(hours available on line 2, month 1)*
2P2 +2Q2 ≤ 150	*(hours available on line 2, month 2)*
2P3 +2Q3 ≤ 110	*(hours available on line 2, month 3)*
P1 + I0 = 50 + S1+I1	*(material balance of PS, month 1)*
P2 + I1 = 45 + S2+I2	*(material balance of PS, month 2)*
P3 + I2 = 50 + S3	*(material balance of PS, month 3)*
Q1 + J0 = 43 + T1+J1	*(material balance of QT, month 1)*
Q2 + J1 = 50 + T2+J2	*(material balance of QT, month 2)*
Q3 + J2 = 40 + T3	*(material balance of QT, month 3)*
4P1+3Q1+4P2+3Q2+4P3+3Q3 ≤ R	*(consumption of raw material)*

Note: the upper bounds on R, St, Tt, etc. could be imposed either by using the "simple upper bound" (SUB) command or by adding a row to the problem. The former is preferred!

LINDO output:

```
MAX     40 S1 + 60 S2 + 55 S3 + 35 T1 + 40 T2 + 44 T3 - 3 R - 10 I1
       - 10 I2 - 10 J1 - 10 J2
  SUBJECT TO
        2)   3 P1 + 2 Q1 <=   200
        3)   3 P2 + 2 Q2 <=   160
        4)   3 P3 + 2 Q3 <=   190
        5)   2 P1 + 2 Q1 <=   140
        6)   2 P2 + 2 Q2 <=   150
        7)   2 P3 + 2 Q3 <=   110
        8) - S1 - I1 + P1 + I0 =    0
        9) - S2 + I1 - I2 + P2 =    0
       10) - S3 + I2 + P3 =    0
```

```
      11) - T1 - J1 + Q1 + J0 =    0
      12) - T2 + J1 - J2 + Q2 =    0
      13) - T3 + J2 + Q3 =    0
      14) - R + 4 P1 + 3 Q1 + 4 P2 + 3 Q2 + 4 P3 + 3 Q3 <=   0
 END
 SUB       S1       50.00000
 SUB       S2       45.00000
 SUB       S3       50.00000
 SUB       T1       43.00000
 SUB       T2       50.00000
 SUB       T3       40.00000
 SUB        R      710.00000
 SUB       I0       10.00000
 SUB       J0        5.00000

      OBJECTIVE FUNCTION VALUE
      1)      7590.000

 VARIABLE        VALUE          REDUCED COST
       S1        40.000000          0.000000
       S2        45.000000        -10.000000
       S3        50.000000         -6.000000
       T1        20.000000          0.000000
       T2        50.000000         -5.000000
       T3         5.000000          0.000000
        R       710.000000         -2.000000
       I1        25.000000          0.000000
       I2         0.000000         11.000000
       J1         0.000000         10.000000
       J2         0.000000          1.000000
       P1        55.000000          0.000000
       Q1        15.000000          0.000000
       P2        20.000000          0.000000
       Q2        50.000000          0.000000
       P3        50.000000          0.000000
       Q3         5.000000          0.000000
       I0        10.000000        -40.000000
       J0         5.000000        -35.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         5.000000          0.000000
       3)         0.000000         10.000000
       4)        30.000000          0.000000
       5)         0.000000         10.000000
       6)        10.000000          0.000000
       7)         0.000000         14.500000
       8)         0.000000        -40.000000
       9)         0.000000        -50.000000
      10)         0.000000        -49.000000
      11)         0.000000        -35.000000
      12)         0.000000        -35.000000
      13)         0.000000        -44.000000
      14)         0.000000          5.000000

 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      S1       40.000000         5.000000         1.000000
      S2       60.000000         INFINITY        10.000000
      S3       55.000000         INFINITY         6.000000
      T1       35.000000         2.000000         5.000000
      T2       40.000000         INFINITY         5.000000
```

T3	44.000000	1.000000	29.000000
R	-3.000000	INFINITY	2.000000
I1	-10.000000	1.500000	7.500000
I2	-10.000000	11.000000	INFINITY
J1	-10.000000	10.000000	INFINITY
J2	-10.000000	1.000000	INFINITY
P1	0.000000	6.000000	2.000000
Q1	0.000000	2.000000	5.000000
P2	0.000000	7.500000	1.500000
Q2	0.000000	1.000000	5.000000
P3	0.000000	INFINITY	6.000000
Q3	0.000000	6.000000	29.000000
I0	0.000000	INFINITY	40.000000
J0	0.000000	INFINITY	35.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	200.000000	INFINITY	5.000000
3	160.000000	15.000000	3.750000
4	190.000000	INFINITY	30.000000
5	140.000000	11.500000	6.666667
6	150.000000	INFINITY	10.000000
7	110.000000	15.333333	3.333333
8	0.000000	40.000000	10.000000
9	0.000000	40.000000	10.000000
10	0.000000	5.000000	5.000000
11	0.000000	20.000000	23.000000
12	0.000000	15.000000	10.000000
13	0.000000	5.000000	35.000000
14	0.000000	5.000000	23.000000

THE TABLEAU

ROW	(BASIS)	S1	S2	S3	T1	T2	T3
1	ART	0.000	10.000	6.000	0.000	5.000	0.000
2	SLK 2	0.000	0.000	1.000	0.000	0.333	0.000
3	Q2	0.000	0.000	0.000	0.000	1.000	0.000
4	SLK 4	0.000	0.000	-1.000	0.000	0.000	0.000
5	S1	1.000	-1.000	-1.000	0.000	-1.000	0.000
6	SLK 6	0.000	0.000	0.000	0.000	-0.667	0.000
7	Q3	0.000	0.000	-1.000	0.000	0.000	0.000
8	I1	0.000	1.000	0.000	0.000	0.667	0.000
9	T1	0.000	0.000	1.000	1.000	0.333	0.000
10	P3	0.000	0.000	1.000	0.000	0.000	0.000
11	Q1	0.000	0.000	1.000	0.000	0.333	0.000
12	P1	0.000	0.000	-1.000	0.000	-0.333	0.000
13	T3	0.000	0.000	-1.000	0.000	0.000	1.000
14	P2	0.000	0.000	0.000	0.000	-0.667	0.000

ROW	R	I1	I2	J1	J2	P1	Q1
1	2.000	0.000	11.000	10.000	1.000	0.000	0.000
2	-1.000	0.000	1.000	0.333	-0.333	0.000	0.000
3	0.000	0.000	0.000	1.000	-1.000	0.000	0.000
4	0.000	0.000	-1.000	0.000	0.000	0.000	0.000
5	1.000	0.000	0.000	-1.000	1.000	0.000	0.000
6	0.000	0.000	0.000	-0.667	0.667	0.000	0.000
7	0.000	0.000	-1.000	0.000	0.000	0.000	0.000
8	0.000	1.000	-1.000	0.667	-0.667	0.000	0.000
9	-1.000	0.000	1.000	1.333	-0.333	0.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	-1.000	0.000	1.000	0.333	-0.333	0.000	1.000
12	1.000	0.000	-1.000	-0.333	0.333	1.000	0.000
13	0.000	0.000	-1.000	0.000	-1.000	0.000	0.000

14	0.000	0.000	0.000	-0.667	0.667	0.000	0.000

ROW	P2	Q2	P3	Q3	I0	J0	SLK 2
1	0.000	0.000	0.000	0.000	40.000	35.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	1.000
3	0.000	1.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	1.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	1.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	1.000	0.000	0.000	0.000	0.000	0.000	0.000

ROW	SLK 3	SLK 4	SLK 5	SLK 6	SLK 7	SLK 14	
1	10.000	0.000	10.000	0.000	14.500	5.000	7590.000
2	1.333	0.000	0.500	0.000	1.500	-1.000	5.000
3	0.000	0.000	0.000	0.000	0.000	0.000	50.000
4	0.000	1.000	0.000	0.000	-1.000	0.000	30.000
5	-1.000	0.000	-1.500	0.000	-1.500	1.000	40.000
6	-0.667	0.000	0.000	1.000	0.000	0.000	10.000
7	0.000	0.000	0.000	0.000	0.500	0.000	5.000
8	-0.333	0.000	0.000	0.000	0.000	0.000	25.000
9	1.333	0.000	2.000	0.000	1.500	-1.000	20.000
10	0.000	0.000	0.000	0.000	0.000	0.000	50.000
11	1.333	0.000	2.000	0.000	1.500	-1.000	15.000
12	-1.333	0.000	-1.500	0.000	-1.500	1.000	55.000
13	0.000	0.000	0.000	0.000	0.500	0.000	5.000
14	0.333	0.000	0.000	0.000	0.000	0.000	20.000
16	0.500	0.000	0.000	0.000	5.000		

Answer the questions below, using the output above for the original problem, if possible. If not possible, you need not run LINDO again.

a. Find the new optimal solution if it costs \$11 to hold a unit of PS in inventory at the end of month 1.

Solution: The current objective coefficient of I1 (the amount of PS in inventory at the end of month 1) is –10.

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
I1	-10.000000	1.500000	7.500000

According to the above LINDO output, the current basis is optimal for values of this coefficient between –10–7.5=–17.5 and –10+11=+1. If the inventory cost were \$11, the new coefficient would be –11, which is within the range [–17.5, +1], so the current basis remains optimal and the values of the basic variables are unchanged.

b. Find the company's new optimal solution if 210 hours on line 1 are available during month 1.

Solution: Currently 200 hours (the right-hand-side of row 2) are available on line 1 in month 1, of which 195 are used (since the slack in this constraint is 5). The range within which the current basis remains optimal is 200–5 to 200+∞, i.e., the range [195, +∞]. Since 210 is within this range, the current basis remains optimal, although the value of the basic variable SLK2 will increase from 5 to 15.

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	200.000000	INFINITY	5.000000

c. Find the company's new profit level if 109 hours are available on line 2 during month 3.

Solution: The right-hand-side of row 7 would be changed: 7) 2 P3 + 2 Q3 <= 110

```
 ROW   SLACK OR SURPLUS     DUAL PRICES
  7)        0.000000        14.500000
```

Currently, all available hours (110) are used. The "dual price" (+14.5)is in this case the "dual variable", indicating that the profit changes at the rate of $14.5/hour within the range

[110–3.3333, 110+15.3333] = [106.6667, 125.3333].

```
                    RIGHTHAND SIDE RANGES
ROW          CURRENT        ALLOWABLE        ALLOWABLE
               RHS          INCREASE          DECREASE
  7       110.000000        15.333333         3.333333
```

Therefore, a decrease of 1 hour will reduce the profit by $14.50.

d. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 2?

Solution: The "dual variable" for row 3 is +10 $/hour, which is valid for any increase up to 15 hours.

```
 ROW   SLACK OR SURPLUS     DUAL PRICES
  3)        0.000000        10.000000

                    RIGHTHAND SIDE RANGES
ROW          CURRENT        ALLOWABLE        ALLOWABLE
               RHS          INCREASE         DECREASE
  3       160.000000        15.000000         3.750000
```

e. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 3?

Solution: There is currently 30 hours of slack in row 4, which imposes the restriction on use of hours on line 1 during month 3. Therefore, the dual variable is zero, indicating that additional time has no value.

```
 ROW   SLACK OR SURPLUS     DUAL PRICES
  4)       30.000000          0.000000
```

f. Find the new optimal solution if PS sells for $50 during month 2.

Solution: The variable S2, the sales of PS during month 2, has an objective coefficient of +60:

```
                    OBJ COEFFICIENT RANGES
 VARIABLE    CURRENT        ALLOWABLE        ALLOWABLE
               COEF         INCREASE         DECREASE
   S2       60.000000         INFINITY        10.000000
```

A drop of $10 in the selling price $60 to $50, is exactly the "allowable decrease", and so the current basis remains optimal. Since the values of the basic variables do not depend upon the objective coefficients, their values remain unchanged.

g. Find the new optimal solution if QT sells for $50 during month 3.

Solution: The variable T3, the amount of QT sold during month 3, is currently basic (= 5). The "allowable increase" in the objective coefficient is only 1, so an increase of $6 is outside the range within which the current basis is optimal.

```
                    OBJ COEFFICIENT RANGES
VARIABLE     CURRENT        ALLOWABLE        ALLOWABLE
             COEF           INCREASE         DECREASE
   T3       44.000000        1.000000        29.000000
```

It is not possible to determine the new basic solution, given the available output from LINDO.

h. Suppose spending $20 on advertising would increase demand for QT in month 2 by 5 units. Should the advertising be done?

Solution: The demand for QT in month 2 is currently 50, the upper bound on the variable T2.

```
 VARIABLE        VALUE          REDUCED COST
   T2        50.000000          -5.000000
```

The "reduced cost" is defined by LINDO as the rate at which the objective (profit) deteriorates as T2 is increased. So that means that increasing the sales of QT in month 2 will improve the profit at the rate of $5 per unit. The cost of increasing sales by the proposed advertising is $20/5 units = $4/unit, and so it appears that the advertising is cost-effective. However, the LINDO output does not easily allow us to know that this $5/unit is valid for an increase of 5 units of sales. (The increase allowed without changing the basis could be calculated by performing a minimum ratio test using the substitution rates for T2 in the tableau, however:

ROW (BASIS)	T2	RHS	*ratio*
1 ART	5.000	7590.000	
2 SLK 2	0.333	5.000	5/0.333 = 15

3	Q2	1.000	50.000	50/1 = 50
4	SLK 4	0.000	30.000	
5	S1	-1.000	40.000	
6	SLK 6	-0.667	10.000	
7	Q3	0.000	5.000	
8	I1	0.667	25.000	25/0.667 = 37.5
9	T1	0.333	20.000	20/0.333 = 60
10	P3	0.000	50.000	
11	Q1	0.333	15.000	15/0.333 = 45
12	P1	-0.333	55.000	
13	T3	0.000	5.000	
14	P2	-0.667	20.000	

This calculation indicates that an increase of 15 units in T2 is allowed before a basic variable (SLK 2) is reduced to zero, preventing any further increase of T2.

56:171 Operations Research
Homework #4 Solutions -- Fall 2001

1. Linear Programming sensitivity. A paper recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are:

Input type	Cost $/ton	Pulp content
Box board	5	15%
Tissue paper	6	20%
Newsprint	8	30%
Book paper	10	40%

Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs $20 to de-ink a ton of any input. The process of de-inking removes 10% of the input's pulp. It costs $15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process removes 20% of the input's pulp. At most 3000 tons of input can be run through the asphalt dispersion process or the de-inking process. Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper. The LP model below was formulated to minimize the cost of meeting the demands for pulp.

Define the variables

BOX = tons of purchased boxboard
TISS = tons of purchased tissue
NEWS = tons of purchased newsprint
BOOK = tons of purchased book paper
BOX1 = tons of boxboard sent through de-inking
TISS1 = tons of tissue sent through de-inking
NEWS1 = tons of newsprint sent through de-inking
BOOK1 = tons of book paper sent through de-inking
BOX2 = tons of boxboard sent through asphalt dispersion
TISS2 = tons of tissue sent through asphalt dispersion
NEWS2 = tons of newsprint sent through asphalt dispersion
BOOK2 = tons of book paper sent through asphalt dispersion
PBOX = tons of pulp recovered from boxboard
PTISS = tons of pulp recovered from tissue
PNEWS= tons of pulp recovered from newsprint
PBOOK = tons of pulp recovered from book paper
PBOX1 = tons of boxboard pulp used for grade 1 paper,
PBOX2 = tons of boxboard pulp used for grade 2 paper, etc.
...
PBOOK3 = tons of book paper pulp used for grade 3 paper.

The LP model using these variables is:

```
MIN  5 BOX +6 TISS +8 NEWS +10 BOOK +20 BOX1 +20 TISS1 +20 NEWS1
          +20 BOOK1 +15 BOX2 +15 TISS2 +15 NEWS2 +15 BOOK2
 SUBJECT TO
        2) - BOX + BOX1 + BOX2 <=   0
        3) - TISS + TISS1 + TISS2 <=   0
        4) - NEWS + NEWS1 + NEWS2 <=   0
        5) - BOOK + BOOK1 + BOOK2 <=   0
        6)   0.135 BOX1 + 0.12 BOX2 - PBOX =    0
        7)   0.18 TISS1 + 0.16 TISS2 - PTISS =    0
        8)   0.27 NEWS1 + 0.24 NEWS2 - PNEWS =    0
        9)   0.36 BOOK1 + 0.32 BOOK2 - PBOOK =    0
       10) - PBOX + PBOX2 + PBOX3 <=   0
       11) - PTISS + PTISS2 + PTISS3 <=   0
       12) - PNEWS + PNEWS1 + PNEWS3 <=   0
       13) - PBOOK + PBOOK1 + PBOOK2 <=   0
       14)   PNEWS1 + PBOOK1 >=   500
       15)   PBOX2 + PTISS2 + PBOOK2 >=   500
```

```
      16)   PBOX3 + PTISS3 + PNEWS3 >=   600
      17)   BOX1 + TISS1 + NEWS1 + BOOK1 <=   3000
      18)   BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
END
```

- Rows 2-5 state that only the supply of each input material which is purchased can be processed, either by de-inking or asphalt dispersion.
- Row 6 states that the recovered pulp in the boxboard is 90% of that in the boxboard which is processed by de-inking, i.e., (0.90)(0.15)BOX1, since boxboard is 15% pulp, plus 80% of that in the boxboard which is processed by asphalt dispersion, i.e., (0.80)(0.15)BOX2.
- Rows 7-9 are similar to row 6, but for different input materials.
- Rows 10-13 state that no more than the pulp which is recovered from each input may be used in making paper (grades 1, 2, &/or 3). Note that some variables are omitted, e.g., PBOX1, since boxboard cannot be used in Grade 1 paper.
- Rows 14-16 state that the demand for each grade of paper must be satisfied.
- Rows 17-18 state that each process (de-inking & asphalt dispersion) has a maximum throughput of 3000 tons. (The problem statement is unclear here, and might be interpreted as stating the total throughput of both processes cannot exceed 3000 tons, in which case rows 17&18 would be replaced by

```
17)   BOX1 + TISS1 + NEWS1 + BOOK1
    + BOX2 + TISS2 + NEWS2 + BOOK2 <=   3000
```

The solution found by LINDO is as follows:

```
 LP OPTIMUM FOUND AT STEP     25
       OBJECTIVE FUNCTION VALUE

       1)      140000.0

 VARIABLE        VALUE          REDUCED COST
      BOX         0.000000          0.000000
     TISS         0.000000          6.000000
     NEWS      2500.000000          0.000000
     BOOK      2833.333252          0.000000
     BOX1         0.000000         11.124999
    TISS1         0.000000          1.499999
    NEWS1         0.000000          0.249999
    BOOK1      2333.333252          0.000000
     BOX2         0.000000          9.333334
    TISS2         0.000000          0.222223
    NEWS2      2500.000000          0.000000
    BOOK2       500.000000          0.000000
     PBOX         0.000000          0.000000
    PTISS         0.000000          0.000000
    PNEWS       600.000000          0.000000
    PBOOK      1000.000000          0.000000
    PBOX2         0.000000         19.444445
    PBOX3         0.000000          0.000000
   PTISS2         0.000000         19.444445
   PTISS3         0.000000          0.000000
   PNEWS1         0.000000         19.444445
   PNEWS3       600.000000          0.000000
   PBOOK1       500.000000          0.000000
   PBOOK2       500.000000          0.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000          5.000000
       3)         0.000000          0.000000
       4)         0.000000          8.000000
       5)         0.000000         10.000000
       6)         0.000000       -102.777779
       7)         0.000000       -102.777779
       8)         0.000000       -102.777779
       9)         0.000000        -83.333336
      10)         0.000000        102.777779
      11)         0.000000        102.777779
      12)         0.000000        102.777779
      13)         0.000000         83.333336
      14)         0.000000        -83.333336
      15)         0.000000        -83.333336
```

16)	0.000000	-102.777779
17)	666.666687	0.000000
18)	0.000000	1.666667

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
BOX	5.000000	INFINITY	5.000000
TISS	6.000000	INFINITY	6.000000
NEWS	8.000000	0.333334	4.666667
BOOK	10.000000	6.000000	1.999989
BOX1	20.000000	INFINITY	11.124999
TISS1	20.000000	INFINITY	1.499999
NEWS1	20.000000	INFINITY	0.249999
BOOK1	20.000000	0.249999	0.750001
BOX2	15.000000	INFINITY	9.333333
TISS2	15.000000	INFINITY	0.222222
NEWS2	15.000000	0.222221	4.666667
BOOK2	15.000000	0.666667	0.222221
PBOX	0.000000	INFINITY	77.777779
PTISS	0.000000	INFINITY	1.388890
PNEWS	0.000000	1.388890	19.444443
PBOOK	0.000000	19.444443	83.333336
PBOX2	0.000000	INFINITY	19.444443
PBOX3	0.000000	19.444443	77.777779
PTISS2	0.000000	INFINITY	19.444443
PTISS3	0.000000	19.444443	1.388890
PNEWS1	0.000000	INFINITY	19.444443
PNEWS3	0.000000	1.388890	19.444443
PBOOK1	0.000000	19.444443	83.333336
PBOOK2	0.000000	19.444443	83.333336

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	0.000000	0.000000	INFINITY
3	0.000000	INFINITY	0.000000
4	0.000000	2500.000000	INFINITY
5	0.000000	2833.333252	INFINITY
6	0.000000	0.000000	600.000000
7	0.000000	0.000000	600.000000
8	0.000000	120.000008	600.000000
9	0.000000	240.000015	840.000000
10	0.000000	600.000000	0.000000
11	0.000000	600.000000	0.000000
12	0.000000	600.000000	120.000008
13	0.000000	840.000000	240.000015
14	500.000000	240.000015	500.000000
15	500.000000	240.000015	500.000000
16	600.000000	120.000008	600.000000
17	3000.000000	INFINITY	666.666687
18	3000.000000	2625.000000	500.000000

THE TABLEAU

ROW	(BASIS)	BOX	TISS	NEWS	BOOK	BOX1	TISS1
1	ART	0.000	6.000	0.000	0.000	11.125	1.500
2	BOOK	0.000	0.000	0.000	1.000	-0.062	-0.083
3	SLK 3	0.000	-1.000	0.000	0.000	0.000	1.000
4	SLK 17	0.000	0.000	0.000	0.000	0.500	0.333
5	BOOK1	0.000	0.000	0.000	0.000	0.500	0.667
6	PBOX	0.000	0.000	0.000	0.000	-0.135	0.000
7	PTISS	0.000	0.000	0.000	0.000	0.000	-0.180
8	PNEWS	0.000	0.000	0.000	0.000	0.135	0.180
9	PBOOK	0.000	0.000	0.000	0.000	0.000	0.000
10	PBOX3	0.000	0.000	0.000	0.000	-0.135	0.000
11	PTISS3	0.000	0.000	0.000	0.000	0.000	-0.180
12	PNEWS3	0.000	0.000	0.000	0.000	0.135	0.180

13	PBOOK2	0.000	0.000	0.000	0.000	0.000	0.000
14	PBOOK1	0.000	0.000	0.000	0.000	0.000	0.000
15	NEWS2	0.000	0.000	0.000	0.000	0.562	0.750
16	NEWS	0.000	0.000	1.000	0.000	0.562	0.750
17	BOX	1.000	0.000	0.000	0.000	-1.000	0.000
18	BOOK2	0.000	0.000	0.000	0.000	-0.562	-0.750

ROW	NEWS1	BOOK1	BOX2	TISS2	NEWS2	BOOK2	PBOX
1	0.250	0.000	9.333	0.222	0.000	0.000	0.000
2	-0.125	0.000	0.056	0.037	0.000	0.000	0.000
3	0.000	0.000	0.000	1.000	0.000	0.000	0.000
4	0.000	0.000	0.444	0.296	0.000	0.000	0.000
5	1.000	1.000	-0.444	-0.296	0.000	0.000	0.000
6	0.000	0.000	-0.120	0.000	0.000	0.000	1.000
7	0.000	0.000	0.000	-0.160	0.000	0.000	0.000
8	0.000	0.000	0.120	0.160	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	-0.120	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	-0.160	0.000	0.000	0.000
12	0.000	0.000	0.120	0.160	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	1.125	0.000	0.500	0.667	1.000	0.000	0.000
16	0.125	0.000	0.500	0.667	0.000	0.000	0.000
17	0.000	0.000	-1.000	0.000	0.000	0.000	0.000
18	-1.125	0.000	0.500	0.333	0.000	1.000	0.000

ROW	PTISS	PNEWS	PBOOK	PBOX2	PBOX3	PTISS2	PTISS3
1	0.000	0.000	0.000	19.444	0.000	19.444	0.000
2	0.000	0.000	0.000	3.241	0.000	3.241	0.000
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	0.000	0.000	0.926	0.000	0.926	0.000
5	0.000	0.000	0.000	-0.926	0.000	-0.926	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	1.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	1.000	0.000	-1.000	0.000	-1.000	0.000
9	0.000	0.000	1.000	1.000	0.000	1.000	0.000
10	0.000	0.000	0.000	1.000	1.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	1.000	1.000
12	0.000	0.000	0.000	-1.000	0.000	-1.000	0.000
13	0.000	0.000	0.000	1.000	0.000	1.000	0.000
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.000	0.000	0.000	-4.167	0.000	-4.167	0.000
16	0.000	0.000	0.000	-4.167	0.000	-4.167	0.000
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.000	0.000	0.000	4.167	0.000	4.167	0.000

ROW	PNEWS1	PNEWS3	PBOOK1	PBOOK2	SLK 2	SLK 3	SLK 4
1	19.444	0.000	0.000	0.000	5.000	0.000	8.000
2	3.241	0.000	0.000	0.000	0.000	0.000	0.000
3	0.000	0.000	0.000	0.000	0.000	1.000	0.000
4	0.926	0.000	0.000	0.000	0.000	0.000	0.000
5	-0.926	0.000	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	-1.000	0.000	0.000	0.000	0.000	0.000	0.000
9	1.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.000	1.000	0.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	1.000	0.000	0.000	0.000
14	1.000	0.000	1.000	0.000	0.000	0.000	0.000
15	-4.167	0.000	0.000	0.000	0.000	0.000	0.000
16	-4.167	0.000	0.000	0.000	0.000	0.000	-1.000
17	0.000	0.000	0.000	0.000	-1.000	0.000	0.000
18	4.167	0.000	0.000	0.000	0.000	0.000	0.000

ROW	SLK 5	SLK 10	SLK 11	SLK 12	SLK 13	SLK 14	SLK 15
1	10.000	102.778	102.778	102.778	83.333	83.333	83.333

2	-1.000	0.463	0.463	0.463	-2.778	-2.778	-2.778
3	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	3.704	3.704	3.704	2.778	2.778	2.778
5	0.000	-3.704	-3.704	-3.704	-2.778	-2.778	-2.778
6	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.000	-1.000	-1.000	-1.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	-1.000	-1.000	-1.000
10	0.000	1.000	0.000	0.000	0.000	0.000	0.000
11	0.000	0.000	1.000	0.000	0.000	0.000	0.000
12	0.000	-1.000	-1.000	0.000	0.000	0.000	0.000
13	0.000	0.000	0.000	0.000	0.000	0.000	-1.000
14	0.000	0.000	0.000	0.000	0.000	-1.000	0.000
15	0.000	-4.167	-4.167	-4.167	0.000	0.000	0.000
16	0.000	-4.167	-4.167	-4.167	0.000	0.000	0.000
17	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.000	4.167	4.167	4.167	0.000	0.000	0.000

ROW	SLK 16	SLK 17	SLK 18	RHS
1	0.10E+03	0.00E+00	1.7	-0.14E+06
2	0.463	0.000	0.111	2833.333
3	0.000	0.000	0.000	0.000
4	3.704	1.000	0.889	666.667
5	-3.704	0.000	-0.889	2333.333
6	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000
8	-1.000	0.000	0.000	600.000
9	0.000	0.000	0.000	1000.000
10	0.000	0.000	0.000	0.000
11	0.000	0.000	0.000	0.000
12	-1.000	0.000	0.000	600.000
13	0.000	0.000	0.000	500.000
14	0.000	0.000	0.000	500.000
15	-4.167	0.000	0.000	2500.000
16	-4.167	0.000	0.000	2500.000
17	0.000	0.000	0.000	0.000
18	4.167	0.000	1.000	500.000

a. *Complete the following statements:* the optimal solution is to purchase only newsprint and book paper, process 500 tons of the book paper and 2500 tons of the newsprint by asphalt dispersion, and the remaining book paper by de-inking. This yields 600 tons of pulp from the newsprint and 1000 tons of pulp from the book paper. One-half of the pulp from book paper is used in each of grades 1 & 2 paper, and the newsprint is used in grade 3 paper. This plan will use $\frac{3000-666.66}{3000} = 77.78\%$ % of the de-inking capacity and 100% of the asphalt dispersion capacity. Note that BOX is a basic variable, but because it has a value of zero, this solution is categorized as degenerate.

b. How much must newsprint increase in price in order that less would be used? $0.33 /ton

c. In the optimal solution, no newsprint is processed by the de-inking. Suppose that 5 tons of newsprint were to be de-inked. How should the solution best be modified to compensate? In particular, what should be the adjusted values of:

Quantity	Current value	*Subs. rate*	Adjusted value
BOX = tons of purchased boxboard	0	0	
TISS = tons of purchased tissue	0		0
NEWS = tons of purchased newsprint	2500	+0.125	2499.375
BOOK = tons of purchased book paper	2833.33	–1.25	2839.58
TISS1 = tons of tissue sent through de-inking	0		
NEWS1 = tons of newsprint sent through de-inking	0		5
BOOK1 = tons of book paper sent through de-inking	2333.33	+1	2328.33
PNEWS= tons of pulp recovered from newsprint	600	0	600

***Solution**:* The nonbasic variable NEWS1 should be increased by 5 units. The substitution rates of NEWS1 for the basic variables are shown above. Thus we see, for example, that 5×0.125 fewer tons of newsprint and 5×1.25 more tons of book paper should be purchased.

d. Suppose that ten additional tons of pulp for grade 3 paper were required. Is this within the range of requirements for which the current basis is optimal? YES ***Solution***: The requirement (600 tons) for pulp for grade 3 paper is imposed by the constraint in row 16. The ALLOWABLE INCREASE in that right-hand-side is 120 tons, and so the increase of ten is within the range for which the current basis remains optimal.

What would be the effect on the cost? increase by 10 tons×\$102.78/ton = \$1027.78

Solution: The "dual price" of row 16 is –102.78 (\$/ton)—as the right-hand-side increases, the constraint is more restrictive and the cost will increase (i.e., the dual variable is +102.78 \$/ton).

How would the quantities of the four raw materials change?

Raw material	Current value	Subs. rate	Adjusted value
BOX = tons of purchased boxboard	0	0	0
TISS = tons of purchased tissue	0		0
NEWS = tons of purchased newsprint	2500	–4.167	25416.7
BOOK = tons of purchased book paper	2833.33	+0.463	2828.7

Solution: Row 16 in equation form is PBOX3+PTISS3+PNEWS3 – SLK16 = 600. (SLK16 is actually a "surplus" variable, despite the name chosen by LINDO!) If the pulp used for grade 3 paper (PBOX3+PTISS3+PNEWS3) is 610, then SLK16 has increased by 10. The substitution rate (+0.463) indicates that BOOK (tons of purchased book paper) will decrease by 4.63 tons while NEWS (tons of purchased newsprint) will increase by 41.67.

Other nonzero substitution rates are

BOOK1: substitution rate = –3.704 which implies that BOOK1 will increase by 37.04

PNEWS: substitution rate = –1 which implies that PNEWS will increase by 10

PNEWS3: substitution rate = –1 which implies that PNEWS3 will increase by 10

NEWS2: substitution rate = –4.167 which implies that NEWS2 will increase by 41.67

BOOK2: substitution rate = +4.167 which implies that BOOK2 will decrease by 41.67

To summarize, then, we buy 41.67 additional tons of newsprint, which is sent through the asphalt dispersion process. Because the asphalt dispersion process was operating at capacity, we must reduce the tons of book paper sent through that process by 41.67 tons. We buy 4.63 fewer tons of book paper, however, so that the increase in book paper sent to the de-inking process is only 41.67–4.63 = 37.04 tons.

2. (A modification of Exercise 3, page 317, of Operations Research, by W. Winston). You have been assigned to evaluate the efficiency of the Port Charles Police Department. Eight precincts are to be evaluated. The inputs and outputs for each precinct are as follows:

Inputs:

- Number of policemen
- Number of vehicles used

Outputs:

- Number of patrol units responding to service requests (thousands/year)
- Number of convictions obtained each year (hundreds/year)

The following data has been collected:

Precinct	# policemen	# vehicles	# responses	#convictions
A	200	60	6	8
B	250	65	5.5	9
C	300	90	8	9.5
D	400	120	10	11
E	350	100	9.5	9
F	300	80	5	7.5
G	275	85	9	8
H	325	75	4.5	10

The city wishes to use this information to determine which precincts, if any, are inefficient.

a. Write the LP model which can be used to compute the efficiency of precinct C.

$$\text{Maximize } 8v_1 + 9.5v_2$$

subject to $300u_1 + 90u_2 = 1$

$$6v_1 + 8v_2 - 200u_1 - 60u_2 \le 0$$
$$5.5v_1 + 9v_2 - 250u_1 - 65u_2 \le 0$$
$$8v_1 + 9.5v_2 - 300u_1 - 90u_2 \le 0$$
$$10v_1 + 11v_2 - 400u_1 - 120u_2 \le 0$$
$$4.5v_1 + 10v_2 - 325u_1 - 75u_2 \le 0$$

$$9.5v_1 + 9v_2 - 350u_1 - 100u_2 \le 0$$
$$5v_1 + 7.5v_2 - 300u_1 - 80u_2 \le 0$$
$$9v_1 + 8v_2 - 275u_1 - 85u_2 \le 0$$
$$u_1 \ge 0, u_2 \ge 0, v_1 \ge 0, v_2 \ge 0,$$

b. What is the total number of LP problems which need to be solved in order to compute the efficiencies of the eight precincts? _8 (one LP per precinct)__

One might use LINDO to do the computation, or any of several other software packages for data envelopment analysis—see, for example, the website
http://www.wiso.uni-dortmund.de/lsfg/or/scheel/doordea.htm)
The output below was computed by the APL workspace "DEA" which can be downloaded from the website at URL:
http://asrl.ecn.uiowa.edu/dbricker/APL_software.html

i	ID	Efficiency	Freq	R
1	A	1	4	1
2	B	1	2	3
3	C	0.8727	0	6
4	D	0.809	0	7
5	E	0.9042	0	5
6	F	0.6875	0	8
7	G	1	3	2
8	H	0.963	0	4

```
Freq = frequency of appearance in reference sets of inefficient DMUs
R = rank based upon (Efficiency + Freq)
```

Slack inputs/outputs

i/o	C	D	E	F	H
responses	0	0	0	0	1.61111
convictions	0	0	0	0	0
policemen	2.43056	4.86111	22.7083	6.25	35.1852
vehicles	0	0	0	0	0

Prices

i	ID	responses	convictions	policemen	vehicles
1	A	0.125	0.03125	0.005	0
2	B	0	0.111111	0.00111111	0.0111111
3	C	0.0925926	0.0138889	0	0.0111111
4	D	0.0694444	0.0104167	0	0.00833333
5	E	0.0833333	0.0125	0	0.01
6	F	0.025	0.075	0	0.0125
7	G	0.0909091	0.0227273	0.00363636	0
8	H	0	0.0962963	0	0.0133333

Reference Sets

For each DMU, the DMUs in its reference set are listed:

3 C	4 D	5 E	6 F	8 H
1 A	1 A	7 G	2 B	2 B
7 G	7 G	1 A	1 A	

c. In order to make itself look as "efficient" as possible, what "prices" would be assigned by precinct C to the outputs (# responses & # convictions) and to the inputs (# policemen & # vehicles)?

Variable	Price
# responses	__0.0925926/thousand responses = 92.5926/response___
# convictions	__0.0138889/hundred convictions = 13.8889/conviction__
# policemen	__0/policeman_____
# vehicles	__0.0111111/vehicle_________

d. Using these prices for precinct C, compute the ratio of the total value of the output variables *responses* and *convictions* to the total value of input variables *policemen* and *vehicles*.

Solution: $\frac{0.092596 \times 8 + 0.0138889 \times 9.5}{0 \times 300 + 0.011111 \times 90} = \frac{0.87271255}{1} \approx 87.3\%$

which is in agreement with the efficiency computed for precinct C.

e. Using these same prices which would be assigned by precinct C, which precincts would be judged to be 100% efficient? ***Solution***: Precincts _A__ and __G__.

f. By how much should precinct C cut its number of policemen in order to become "efficient" (assuming that they could maintain their current output levels)? ***Solution***: _2.43056__

3. The ZapCon Company is considering investing in three projects. If it fully invests in a project, the realized cash flows (in millions of dollars) will be as listed in the table below.

Time (years)	*Cash flow project 1*	*Cash flow project 2*	*Cash flow project 3*
0	–3	–2	–2.0
0.5	–1	–0.5	–2.0
1	–1.8	1.5	–1.8
1.5	0.4	1.5	1
2	1.8	1.5	1
2.5	1.8	0.2	1
3	5.5	–1.0	6

For example, project 1 requires an initial cash outflow of \$3 million, smaller outlays six months and one year from now, begins paying a small return 1.5 years from now, and a final payback of \$5.5 million 3 years from now. Today ZapCon has \$2 million in cash. At each time point (0, 0.5, 1, 1.5 2, and 2.5 years from today) the com[any can, if desired, borrow up to \$2 million at 3.5% (per 6 months) interest. Leftover cash earns 3% (per six months) interest. For example, if after borrowing and investing at time 0, ZapCon has \$1 million, it would receive \$30,000 in interest at time 0.5 year. the company's goal is to maximize cash on hand after cash flows 3 years from now are accounted for. What investment and borrowing strategy should it use? Assume that the company can invest in a fraction of a project. For example, if it invests in 0.5 of project 3, it has, for example, cash outflows of -\$1 million at times 0 and 0.5. No more than 100% investment in a project is possible, however.

a. Formulate a linear programming model to optimize the investment plan.

Solution: Define variables

F30 = incoming cash flow at time 3.0 years

Pj = investment level in project j, j=1,2,3

Bt = amount borrowed (\$millions) at time t=00, 05, 10, 15, 20, 25, 30

Lt = amount loaned (\$millions) at time t=00, 05, 10, 15, 20, 25, 30

For each of the 7 time periods, there is a cash flow balance equation: flow out = flow in. In addition, there are upper bounds of 1 on the variables P1, P2, and P3. (These are best handled as "simple upper bound" (SUB) constraints:

```
MAX     F30
SUBJECT TO
       2)   B00 - 3 P1 - 2 P2 - 2 P3 - L00 =  - 2
       3) - 1.035 B00 - P1 - 0.5 P2 - 2 P3 + 1.03 L00 + B05 - L05 =    0
       4) - 1.8 P1 - 0.5 P2 - 2 P3 - 1.035 B05 + 1.03 L05 + B10 - L10 =    0
       5)   0.4 P1 + 1.5 P2 + P3 - 1.035 B10 + 1.03 L10 + B15 - L15 =    0
       6)   1.8 P1 + 1.5 P2 + P3 - 1.035 B15 + 1.03 L15 + B20 - L20 =    0
       7)   1.8 P1 + 0.2 P2 + P3 - 1.035 B20 + 1.03 L20 + B25 - L25 =    0
       8) - F30 + 5.5 P1 - P2 + 6 P3 - 1.035 B25 + 1.03 L25 =    0
END
SUB       P1        1.00000
SUB       P2        1.00000
SUB       P3        1.00000
```

PICTURE command output:

```
F B       L B L B L B L B L B L
3 0 P P P 0 0 0 1 1 1 1 2 2 2 2
0 0 1 2 3 0 5 5 0 0 5 5 0 0 5 5
```

```
1: 1      '     '     '     '     ' MAX
2:   1-3-2-2-1 '     '     '     ' =-2
3: '-A-1-T-2'A 1-1'  '  '  '  '  ' =
4:    -A-T-2  -A A 1-1     '     ' =
5:     T A 1   '  -A A 1-1 '     ' =
6: '  'A A 1'  '  '  '-A'A 1-1'  ' =
7:     A T 1   '     '    -A A 1-1 =
8:-1   A-1 6   '     '     '  -A A =
```

b. Use LINDO (or other LP solver) to find the optimal solution.
Solution:

```
        OBJECTIVE FUNCTION VALUE
        1)      7.338224

  VARIABLE        VALUE          REDUCED COST
       F30         7.338224          0.000000
       B00         3.000000          0.000000
        P1         1.000000         -2.793699
        P3         1.000000         -2.086015
       B05         6.105000          0.000000
       B10        10.118674          0.000000
       B15         9.072828          0.000000
       B20         6.590377          0.000000
       B25         4.021039          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         -1.229255
        3)         0.000000         -1.187686
        4)         0.000000         -1.147523
        5)         0.000000         -1.108718
        6)         0.000000         -1.071225
        7)         0.000000         -1.035000
        8)         0.000000         -1.000000
```

The optimal solution is to invest in projects 1 and 3 at their full level. This requires borrowing 3 million dollars initially. Additional amounts are borrowed at later points in time, e.g., 10.118 million \$ at t = 1 year. The net cash on hand after 3 years is \$7,338,224.

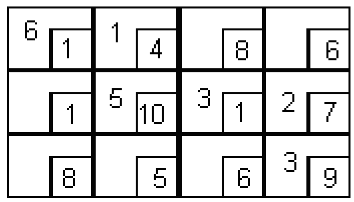
Homework #5 Solutions -- Fall 2001

1. ***Transportation Problem*** Consider the following “balanced” transportation problem with three sources and four destinations, where the transportation cost/unit shipped, supplies available, and amounts required are shown in the table:

Plant \ Warehouse	1	2	3	4	**Supply**
A	1	4	8	6	**7**
B	1	10	1	7	**10**
C	8	5	6	9	**3**
Demand	**6**	**6**	**3**	**5**	

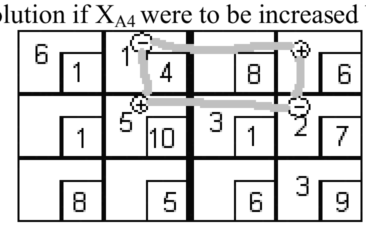
a. A linear programming model of this problem will have _7_ equality constraints (not counting the objective) and _6_ basic variables.

b. Find an initial feasible basic solution, using the “Northwest Corner Rule”:
Solution



c. The shipping cost of this solution is _____
Solution : $6\times1+1\times4+5\times10+3\times1+2\times7+3\times9=$ 104 .

d. Compute the reduced cost of the variable X_{A4} by identifying the “cycle” of adjustments that would be required in the NW-corner solution if X_{A4} were to be increased by one unit.



Solution: Reduced cost the is $+6 -7 +10 -4 = +5$.

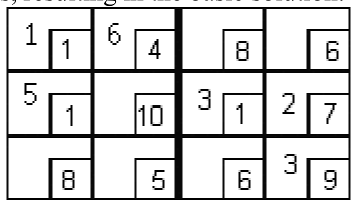
e. Entering X_{A4} into basis will increase the objective function by _5__ per unit shipped from A to 4.

f. Compute a set of “dual variables” corresponding to the initial NW-corner solution, and use them to compute the reduced cost of X_{A4} :

Solution: There are infinitely-many correct answers possible, depending upon the choice of the dual variable to be given an initial assignment, and the value of that assignment. Here, I have chosen to initially assign $U_A=0$:
$X_{A1}>0 \Rightarrow U_A+V_1=1 \Rightarrow V_1=1$
$X_{A2}>0 \Rightarrow U_A+V_2=4 \Rightarrow V_2=4$
$X_{B2}>0 \Rightarrow U_B+V_2=10 \Rightarrow U_B=6$
$X_{B3}>0 \Rightarrow U_B+V_3=1 \Rightarrow V_3=-5$
Etc.
Corresponding to supply constraints: U_A=__0__, U_B=__6_, U_C=___8___
Corresponding to demand constraints: V_1=__1___, V_2=__4___, V_3=__–5___, V_4=__1__

Reduced cost of X_{A4} is $C_{A4}-(U_A+V_4)$ = __6__ − _(0+1)__ = _+5__

g. The reduced cost of X_{B1} is is $C_{B1}-(U_B+V_1)$ = __–6__. Entering X_{B1} into the basis would cause the variable _ X_{B2}_ to leave the basis, resulting in the basic solution:



The increase in X_{A4} (__5_ units) times the reduced cost (_-6_) is _–30__, so that the cost of the new solution is 104–30= 74 .

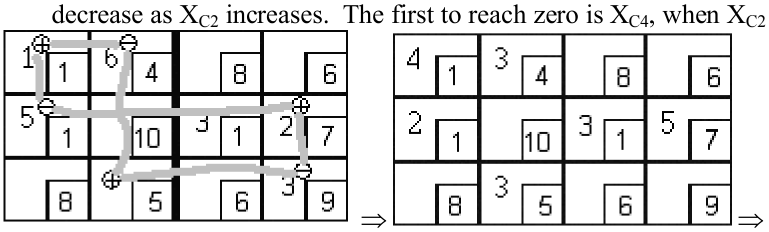
h. Continue changing the basis until you have found the optimal solution:

Solution: Recomputing the dual variables:

Corresponding to supply constraints: U_A=__0__, U_B=__0_, U_C=___2___

Corresponding to demand constraints: V_1=__1___, V_2=__4___, V_3=__1___, V_4=__7__

Using these dual variables, we find that the reduced cost of X_{C2} is 5–(2+4)=–1<0, and so we enter X_{C2} into the solution: The cycle is more complex than the previous iteration, and three basic variables decrease as X_{C2} increases. The first to reach zero is X_{C4}, when X_{C2} = 3.

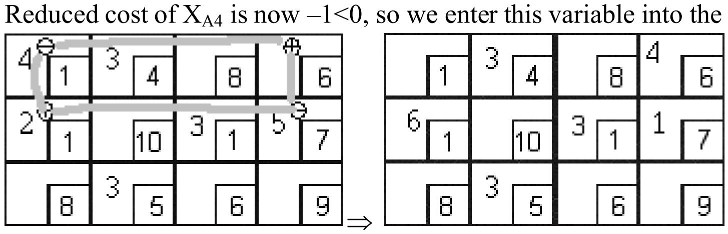


Recomputing the dual variables:

Corresponding to supply constraints: U_A=__0__, U_B=__0_, U_C=___1___

Corresponding to demand constraints: V_1=__1___, V_2=__4___, V_3=__1___, V_4=__7__

Reduced cost of X_{A4} is now –1<0, so we enter this variable into the basis:



Recomputing the dual variables:

Corresponding to supply constraints: U_A=__0__, U_B=__1_, U_C=___1___

Corresponding to demand constraints: V_1=__0___, V_2=__4___, V_3=__0___, V_4=__6__

The reduced costs are now all nonnegative!

i. The optimal cost is _69_ .

2. *Powerhouse* produces capacitors at three locations: Los Angeles, Chicago, and New York. Capacitors are shipped from these locations to public utilities in five regions of the country: northeast (NE), northwest (NW), midwest (MW), southeast (SE), and southwest (SW). The cost of producing and shipping a capacitor from each plant to each region of the country is given in the table below.

From \ to	**NE**	**NW**	**MW**	**SE**	**SW**
LA	$27.86	$4.00	$20.54	$21.52	$13.87
Chicago	$8.02	$20.54	$2.00	$6.74	$10.67
NY	$2.00	$27.86	$8.02	$8.41	$15.20

Each plant has an annual production capacity of 100,000 capacitors. Each year, each region of the country must receive the following number of capacitors: NE: 55,000; NW: 50,000; MW: 60,000; SE: 60,000; SW: 45,000.

Powerhouse feels shipping costs are too high, and the company is therefore considering building one or two more production plants. Possible sites are Atlanta and Houston. The costs of producing a capacitor and shipping it to each region of the country are:

From \ to	**NE**	**NW**	**MW**	**SE**	**SW**
Atlanta	$8.41	$21.52	$6.74	$3.00	$7.89
Houston	$15.20	$13.87	$10.67	$7.89	$3.00

It costs $3 million (in current dollars) to build a new plant, and operating each plant incurs a fixed cost (in addition to variable shipping and production costs) of $50,000 per year. A plant at Atlanta or Houston will have the capacity to produce 100,000 capacitors per year.

Assume that future demand patterns and production costs will remain unchanged. If costs are discounted at a rate of $11\ ^1/_9$ % per year, how can *Powerhouse* minimize the present value of all costs associated with meeting current and future demands?

Solution: We may either convert the construction costs of the proposed plants into equivalent annual costs, or convert the annual costs over an infinite time period into present values. I have arbitrarily selected the latter. With the given discount rate 0.1111111, an infinite sequence of annual costs of $1/year is equivalent to a present value of ($1/0.111111) = $9.

There are four options to consider. For each option, we solve a transportation problem to compute the annual production & shipping cost (exclusive of the fixed operating cost and construction costs).

I. No added plants:

```
                Shipments
     f|            to
     r|
     o|  NE    NW    MW    SE    SW  dummy
     m|
     1|     0 50000     0     0 20000 30000
     2|     0     0 60000 15000 25000     0
     3|55000     0     0 45000     0     0
Cost = 1453700
```
($ per year)

Present value	
production & shipping costs : 9×1453700 =	$13,083,300
operating costs of plants: 9×3×$50,000 =	$1,350,000
construction costs:	0
Total present value:	$14,433,300

II. Add plant at Atlanta only

```
                Shipments
     f|            to
     r|
     o|  NE    NW    MW    SE    SW  dummy
     m|
     1|     0 50000     0     0     0 50000
     2|     0     0 60000     0  5000 35000
     3|55000     0     0     0     0 45000
     4|     0     0     0 60000 40000     0
Cost = 978950
```
($ per year)

Present value	
production & shipping costs : 9×978950=	$8,810,550
operating costs of plants: 9×4×$50,000 =	$1,800,000
construction cost of Atlanta plant:	$3,000,000
Total present value:	$13,610,550

III. Add plant at Houston only

```
                Shipments
```

```
         f|              to
         r|
         o|  NE    NW    MW    SE    SW  dummy
         m|
         1|     0 50000     0     0     0 50000
         2|     0     0 60000 40000     0     0
         3|55000     0     0     0     0 45000
         4|     0     0     0 20000 45000 35000
Cost = 992400
```

Present value	
production & shipping costs : 9×992400=	$8,931,600
operating costs of plants: 9×4×$50,000 =	$1,800,000
construction cost of Houston plant:	$3,000,000
Total present value:	$13,731,600

IV. Add plants at both Atlanta & Houston

```
                     Shipments
         f|              to
         r|
         o|  NE    NW    MW    SE    SW  dummy
         m|
         1|     0 50000     0     0     0 50000
         2|     0     0 60000     0     0 40000
         3|55000     0     0     0     0 45000
         4|     0     0     0 60000     0 40000
         5|     0     0     0     0 45000 55000
Cost = 745000
```

Present value	
production & shipping costs : 9×745,000=	$6,705,000
operating costs of plants: 9×5×$50,000 =	$2,250,000
construction cost of Atlanta & Houston plants:	$6,000,000
Total present value:	$14,955,000

The minimum-cost decision is to build the plant at Atlanta. The L.A. plant will then ship 50,000 annually to the NW region. The Chicago plant will ship 60,000 annually the the MW region and 5000 to the SW region. The NY plant will ship 55,000 to the NE region. The Atlanta plant will ship 60,000 to the SE region and 40,000 to the SW region.

3. The coach of a swim team needs to assign four swimmers to a 400-meter medley relay team. The "best times" (in seconds for 100 meters) achieved by his seven swimmers in each of the strokes are given below. Which swimmer should the coach assign to each of the four strokes? Which swimmers will *not* be assigned to the relay team? Are there more than one optimal solution?

Stroke	Alan	Ben	Carl	Don	Ed	Fred	George
Backstroke	66	67	66	64	70	68	64
Breaststroke	71	72	70	69	72	72	73
Butterfly	65	67	71	74	65	64	64
Freestyle	59	59	55	59	54	54	56

Solution: This is an assignment problem. Although it isn't necessary, the matrix has been transposed below, so that the "agents" correspond to the 7 swimmers and the "tasks" to the four strokes:

```
Cost matrix:
          f|    to
          r|
          o| 1  2  3  4
          m|
          1|66 71 65 59
          2|67 72 67 59
          3|66 70 71 55
          4|64 69 74 59
          5|70 72 65 54
```

```
                                    6|68 72 64 54
                                    7|64 73 64 56
m = #agents = 7
n = #jobs = 4
3 "Dummy" jobs were defined
```

Since each row already contains a zero, no row reduction is possible/necessary.
After column reduction (subtracting 64 from column #1, 69 from column 2, etc.):

```
2 2  1 5 0 0 0
3 3  3 5 0 0 0
2 1  7 1 0 0 0
0 [0] 10 5 0 0 0
6 3  1 [0] 0 0 0
4 3  [0] 0 0 0 0
[0] 4  0 2 0 0 0
```

Seven lines are required to cover all of the zeroes, and so a zero-cost assignment (shown by boxed elements) is possible and therefore optimal.

```
  i      ->      j
GEORGE  ->  BACKSTROKE
DON     ->  BREASTSTROKE
FRED    ->  BUTTERFLY
ED      ->  FREESTYLE
ALAN    ->  dummy 5
BEN     ->  dummy 6
CARL    ->  dummy 7

Minimum Cost = 251
```

Alan, Ben, and Carl are not given positions on the relay team. If all swimmers were to perform at their best level, the total time would be 251 seconds. There are no meaningful alternate optimal solutions (except that idle swimmers could be assigned other “dummy” tasks, e.g., ALAN -> dummy 6, etc.)

56:171 Operations Research
Homework #6 Solutions -- Fall 2001

1. ***Integer LP Model*** A court decision has stated that the enrollment of each high school in Metropolis be at least 20% black. The numbers of black and white high school students in each of the city's five school districts are:

District	Whites	Blacks
1	80	30
2	70	5
3	90	10
4	50	40
5	60	30

The distance (in miles) that a student in each district must travel to each high school is:

District	HS#1	HS#2
1	1.0	2.0
2	0.5	1.7
3	0.8	0.8
4	1.3	0.4
5	1.5	0.6

School board policy requires that all students in a given district must attend the same school, and that each school must have an enrollment of at least 150 students. Formulate an integer LP to determine how to minimize the total distance that Metropolis students must travel to high school, and use LINDO (or other ILP solver) to compute the optimal solution.

Decision Variables :

$$Xij = \begin{cases} 1, & \text{if students from district i are sent to school j} \\ 0, & \text{otherwise} \end{cases}$$

Integer Programming Formulation :

The objective is to minimize the total distance students travel (which would be equivalent to minimizing the average distance traveled), so the coefficient of Xij is the population of district i times the distance from district i to school j.

$$\begin{aligned} \text{Min} \quad & \{(80+30)*1.0\}\ X11 + \{(70+5)*0.5\}\ X21 + \{(90+10)*0.8\}\ X31 \\ & + \{(50+40)*1.3\}\ X41 + \{(60+30)*1.5\}\ X51 \\ & + \{(80+30)*2.0\}\ X12 + \{(70+5)*1.7\}\ X22 + \{(90+10)*0.8\}\ X32 \\ & + \{(50+40)*0.4\}\ X42 + \{(60+30)*0.6\}\ X52 \end{aligned}$$

s.t.

Minimum enrollment at schools:

$$(80+30)\ X11 + (70+5)\ X21 + (90+10)\ X31 + (50+40)\ X41 + (60+30)\ X51 \geq 150$$
$$(80+30)\ X12 + (70+5)\ X22 + (90+10)\ X32 + (50+40)\ X42 + (60+30)\ X52 \geq 150$$

Minimum proportion of black students in each school:

$$\frac{30\ X11 + 5\ X21 + 10\ X31 + 40\ X41 + 30\ X51}{(80+30)\ X11 + (70+5)\ X21 + (90+10)\ X31 + (50+40)\ X41 + (60+30)\ X51} \geq 0.2$$

$$\frac{30\ X12 + 5\ X22 + 10\ X32 + 40\ X42 + 30\ X52}{(80+30)\ X12 + (70+5)\ X22 + (90+10)\ X32 + (50+40)\ X42 + (60+30)\ X52} \geq 0.2$$

"Multiple choice" constraints: Each district is to be assigned to one of the two schools:

$X11 + X12 = 1$, $X21 + X22 = 1$, $X31 + X32 = 1$, $X41 + X42 = 1$, $X51 + X52 = 1$

LINDO input

```
Min    110 X11 +  37.5 X21 + 80 X31 + 117 X41 + 135 X51
     + 220 X12 + 127.5 X22 + 80 X32 +  36 X42 +  54 X52
s.t.
110 X11 + 75 X21 + 100 X31 + 90 X41 + 90 X51   >= 150
110 X12 + 75 X22 + 100 X32 + 90 X42 + 90 X52   >= 150
8X11 - 10X21 - 10X31 + 22X41 + 12X51 >= 0
8X12 - 10X22 - 10X32 + 22X42 + 12X52 >= 0
X11 + X12 = 1
X21 + X22 = 1
X31 + X32 = 1
X41 + X42 = 1
X51 + X52 = 1
END
INTE 10
```

(Here, zero/one variable (binary) restrictions are imposed by the command INTE)

LINDO output

```
LP OPTIMUM FOUND AT STEP      6
OBJECTIVE VALUE =   324.863647


NEW INTEGER SOLUTION OF    398.500000     AT BRANCH      0 PIVOT       6
RE-INSTALLING BEST SOLUTION...

       OBJECTIVE FUNCTION VALUE

       1)      398.5000

 VARIABLE        VALUE          REDUCED COST
      X11         1.000000        110.000000
      X21         1.000000         37.500000
      X31         0.000000         80.000000
      X41         1.000000        117.000000
      X51         0.000000        135.000000
      X12         0.000000        220.000000
      X22         0.000000        127.500000
      X32         1.000000         80.000000
      X42         0.000000         36.000000
      X52         1.000000         54.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)       125.000000         0.000000
       3)        40.000000         0.000000
       4)        20.000000         0.000000
       5)         2.000000         0.000000
       6)         0.000000         0.000000
       7)         0.000000         0.000000
       8)         0.000000         0.000000
       9)         0.000000         0.000000
      10)         0.000000         0.000000

NO. ITERATIONS=       7
BRANCHES=    0 DETERM.=  1.000E    0
```

Optimal decision :

Students from district 1 are sent to school 1,
Students from district 2 are sent to school 1,

Students from district 3 are sent to school 2,
Students from district 4 are sent to school 1,
Students from district 5 are sent to school 2.

Corresponding total distance traveled by students is 398.5 miles(which is an average of 0.857 miles for each of the 465 students, ranging from 0.5 mile to 1.3 mile.)

✲ ✲ ✲ ✲ ✲ ✲ ✲ ✲ ✲

2. ***Integer LP Model*** A company sells seven types of boxes, ranging in volume from 17 to 33 cubic feet. The demand and size of each box are given below.

Product#:	1	2	3	4	5	6	7
Size	33	30	26	24	19	18	17
Demand	400	300	500	700	200	400	200

The variable cost (in dollars) of producing each box is equal to the box's volume. A fixed cost of $1000 is incurred to produce any of a particular box. If the company desires, demand for a box may be satisfied by a box of larger size. Formulate an integer LP model to minimize the cost of meeting the demand for boxes, and solve, using LINDO (or another ILP solver).

Decision Variables :

Xi = the number of type i boxes produced.

$$Yi = \begin{cases} 1, & \text{if company produces type i box} \\ 0, & \text{otherwise} \end{cases}$$

LINDO input

```
Min  33 X1 + 30 X2 + 26 X3 + 24 X4 + 19 X5 + 18 X6 + 17 X7
   + 1000 Y1 + 1000 Y2 + 1000 Y3 + 1000 Y4 + 1000 Y5 + 1000 Y6 + 1000 Y7
Subject to
X1 >= 400
X1 +   X2 >= 700
X1 +   X2 +  X3 >= 1200
X1 +   X2 +  X3 +  X4 >= 1900
X1 +   X2 +  X3 +  X4 +  X5 >= 2100
X1 +   X2 +  X3 +  X4 +  X5 +  X6 >= 2500
X1 +   X2 +  X3 +  X4 +  X5 +  X6 +  X7 >= 2700
X1 - 2700 Y1 <= 0
X2 - 2300 Y2 <= 0
X3 - 2000 Y3 <= 0
X4 - 1500 Y4 <= 0
X5 - 800 Y5 <= 0
X6 - 600 Y6 <= 0
X7 - 200 Y7 <= 0
end
inte Y1
inte Y2
inte Y3
inte Y4
inte Y5
inte Y6
inte Y7
```

LINDO output

```
 LP OPTIMUM FOUND AT STEP     60
 OBJECTIVE VALUE =   68845.2500

 FIX ALL VARS.(     2)  WITH RC >  0.000000E+00
 SET       Y2 TO <=      0 AT    1, BND= -0.7047E+05 TWIN=-0.7057E+05     70
 SET       Y3 TO >=      1 AT    2, BND= -0.7122E+05 TWIN=-0.7372E+05     73
```

```
SET       Y4 TO >=     1 AT    3, BND= -0.7175E+05 TWIN=-0.7215E+05     76
SET       Y5 TO <=     0 AT    4, BND= -0.7250E+05 TWIN=-0.7250E+05     79
DELETE       Y5 AT LEVEL     4
DELETE       Y4 AT LEVEL     3
DELETE       Y3 AT LEVEL     2
FLIP       Y2 TO >=         1 AT    1 WITH BND=   -70566.664
SET       Y3 TO >=     1 AT    2, BND= -0.7132E+05 TWIN=-0.7232E+05     82
SET       Y4 TO >=     1 AT    3, BND= -0.7185E+05 TWIN=-0.7225E+05     85
SET       Y5 TO <=     0 AT    4, BND= -0.7260E+05 TWIN=-0.7260E+05     88
DELETE       Y5 AT LEVEL     4
DELETE       Y4 AT LEVEL     3
DELETE       Y3 AT LEVEL     2
DELETE       Y2 AT LEVEL     1
RELEASE FIXED VARIABLES
SET       Y2 TO <=     0 AT    1, BND= -0.7082E+05 TWIN=-0.7092E+05    101
SET       Y3 TO >=     1 AT    2, BND= -0.7157E+05 TWIN=-0.7407E+05    104
SET       Y4 TO >=     1 AT    3, BND= -0.7210E+05 TWIN=-0.7250E+05    107
SET       Y5 TO >=     1 AT    4, BND= -0.7210E+05 TWIN=-0.7285E+05    113

NEW INTEGER SOLUTION OF    72100.0000     AT BRANCH     18 PIVOT     113
BOUND ON OPTIMUM:  69697.10
DELETE       Y5 AT LEVEL     4
DELETE       Y4 AT LEVEL     3
DELETE       Y3 AT LEVEL     2
FLIP       Y2 TO >=         1 AT    1 WITH BND=   -70916.664
SET       Y3 TO >=     1 AT    2, BND= -0.7167E+05 TWIN=-0.7267E+05    116
SET       Y4 TO <=     0 AT    3, BND= -0.7260E+05 TWIN=-0.7220E+05    119
DELETE       Y4 AT LEVEL     3
DELETE       Y3 AT LEVEL     2
DELETE       Y2 AT LEVEL     1
ENUMERATION COMPLETE. BRANCHES=    20 PIVOTS=     119

LAST INTEGER SOLUTION IS THE BEST FOUND
RE-INSTALLING BEST SOLUTION...

       OBJECTIVE FUNCTION VALUE
       1)      72100.00

 VARIABLE        VALUE          REDUCED COST
       Y1         1.000000       1000.000000
       Y2         0.000000      -5900.000000
       Y3         1.000000       1000.000000
       Y4         1.000000       1000.000000
       Y5         1.000000       1000.000000
       Y6         0.000000        400.000000
       Y7         0.000000        600.000000
       X1       700.000000          0.000000
       X2         0.000000          0.000000
       X3       500.000000          0.000000
       X4       700.000000          0.000000
       X5       800.000000          0.000000
       X6         0.000000          0.000000
       X7         0.000000          0.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)       300.000000          0.000000
       3)         0.000000         -7.000000
       4)         0.000000         -2.000000
       5)         0.000000         -5.000000
       6)       600.000000          0.000000
       7)       200.000000          0.000000
       8)         0.000000        -19.000000
       9)      2000.000000          0.000000
```

```
10)         0.000000           3.000000
11)      1500.000000           0.000000
12)       800.000000           0.000000
13)         0.000000           0.000000
14)         0.000000           1.000000
15)         0.000000           2.000000
```

Optimal decision :
700 type **1**, 500 type **3**, 700 type **4**, and 800 type **5** boxes are required to produce to meet the demand. The corresponding total cost is $72,100

Note: another formulation might define Y as before, but Z instead of X:

Z_{ij} = fraction of type i boxes used to satisfy need for type j boxes

Ci = cost of box of type i

Dj = demand for box of type j

Fi = setup cost for box type i ($1000 in this instance)

$$Minimize\ \ F_i \sum_{i=1}^{7} Y_i + \sum_{i=1}^{7} C_i \sum_{j=i}^{7} D_j Z_{ij}$$

$$s.t. \quad \sum_{j=i}^{7} Z_{ij} \le 7Y_i, \quad i = 1,...7 \text{ or (better) } Z_{ij} \le Y_i \quad \forall i \,\&\, j$$

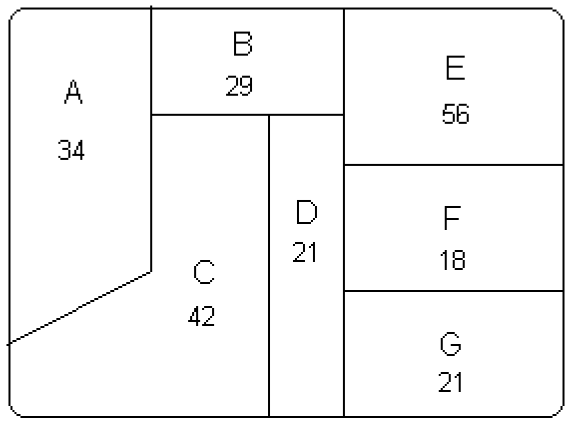
$$\sum_{i=1}^{j} Z_{ij} = 1, \quad j = 1, 2, ...7$$

$$Y_i \in \{0,1\}, Z_{ij} \ge 0 \quad \forall \text{i,j}$$

This model is essentially the same as that of the ***uncapacitated*** (or "*simple*") ***plant location problem***.

✲ ✲ ✲ ✲ ✲ ✲ ✲ ✲ ✲

3. ***Integer LP Model*** WSP Publishing sells textbooks to college students. WSP has two sales representatives available to assign to the seven-state area (states A through G):



The number of college students (in thousands) in each area is indicated in the figure above. Each sales representative must be assigned to two adjacent states. For example, a sales rep could be assigned to A & B, but not A&D. WSP's goal is to maximize the number of total students in the states assigned to the sales reps. Formulate an integer LP whose solution will tell WSP where to assign the sales reps. Use LINDO (or another ILP solver) to compute the optimal assignment.

Decision Variables :

$$Xi = \begin{cases} 1, & \text{if state i is served by a sales representative} \\ 0, & \text{otherwise} \end{cases}$$

$$Yij = \begin{cases} 1, & \text{if an sales representative is assigned to i \& j} \\ 0, & \text{otherwise} \end{cases}$$

Integer Programming Formulation :

The objective is to maximize the number of total students in the states assigned to the sales representatives.

```
Max  34 Xa + 29 Xb + 42 Xc + 21 Xd + 56 Xe + 18 Xf + 21 Xg
```

s.t.

Xj must be zero unless at least one representative is assigned to state i & j.

```
Xa  <= Yab + Yac
Xb  <= Yab + Ybc + Ybd + Ybe
Xc  <= Yac + Ybc + Ycd
Xd  <= Ybd + Ycd + Yde + Ydf + Ydg
Xe  <= Ybe + Yde + Yef
Xf  <= Ydf + Yef + Yfg
Xg  <= Ydg + Yfg
```

Two sales representatives are to be assigned:

```
Yab + Yac + Ybc + Ybd + Ybe + Ycd + Yde + Ydf + Ydg + Yef + Yfg = 2
```

LINDO input

```
Max  34 Xa + 29 Xb + 42 Xc + 21 Xd + 56 Xe + 18 Xf + 21 Xg
Subject to
Xa  - Yab - Yac <= 0
Xb  - Yab - Ybc - Ybd - Ybe <= 0
Xc  - Yac - Ybc - Ycd <= 0
Xd  - Ybd - Ycd - Yde - Ydf - Ydg <= 0
Xe  - Ybe - Yde - Yef <= 0
Xf  - Ydf - Yef - Yfg <= 0
Xg  - Ydg - Yfg <= 0
Yab + Yac + Ybc + Ybd + Ybe + Ycd + Yde + Ydf + Ydg + Yef + Yfg = 2
```

```
end
inte 18
```

(Here, zero/one variable (binary) restrictions are imposed by the command INTE)

LINDO output

```
LP OPTIMUM FOUND AT STEP     30
OBJECTIVE VALUE =   161.000000


NEW INTEGER SOLUTION OF    161.000000     AT BRANCH      0 PIVOT      30
RE-INSTALLING BEST SOLUTION...

       OBJECTIVE FUNCTION VALUE

       1)      161.0000

 VARIABLE        VALUE          REDUCED COST
       XA         1.000000        -34.000000
       XB         1.000000        -29.000000
       XC         1.000000        -42.000000
       XD         0.000000        -21.000000
       XE         1.000000        -56.000000
       XF         0.000000        -18.000000
       XG         0.000000        -21.000000
      YAB         0.000000          0.000000
      YAC         1.000000          0.000000
      YBC         0.000000          0.000000
      YBD         0.000000          0.000000
      YBE         1.000000          0.000000
      YCD         0.000000          0.000000
      YDE         0.000000          0.000000
      YDF         0.000000          0.000000
      YDG         0.000000          0.000000
      YEF         0.000000          0.000000
      YFG         0.000000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         0.000000          0.000000
       3)         0.000000          0.000000
       4)         0.000000          0.000000
       5)         0.000000          0.000000
       6)         0.000000          0.000000
       7)         0.000000          0.000000
       8)         0.000000          0.000000
       9)         0.000000          0.000000

NO. ITERATIONS=      30
BRANCHES=    0 DETERM.=  1.000E    0
```

Optimal decision :

One sales representative is assigned to **A and C**, and another sales representative is assigned to **B and E**. This plan maximizes the number of total students in the states assigned to the sales reps, namely **161** thousand.

56:171 Operations Research

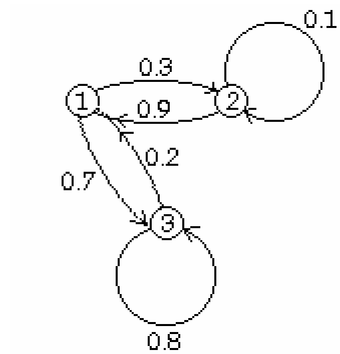
Homework #7 Solutions -- Fall 2001

1. A Markov chain has the transition probability matrix

$$P = \begin{bmatrix} 0 & 0.3 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

a. Draw the transition diagram, with probabilities indicated.

Solution:



b. Find the probability distributions of the state for the first five steps, given that it begins in state 3.

Solution:

$$P^2 = \begin{bmatrix} 0.41 & 0.03 & 0.56 \\ 0.09 & 0.28 & 0.63 \\ 0.16 & 0.06 & 0.78 \end{bmatrix}, P^3 = \begin{bmatrix} 0.139 & 0.126 & 0.735 \\ 0.378 & 0.055 & 0.567 \\ 0.21 & 0.054 & 0.736 \end{bmatrix}, P^4 = \begin{bmatrix} 0.2604 & 0.0543 & 0.6853 \\ 0.1629 & 0.1189 & 0.7182 \\ 0.1958 & 0.0684 & 0.7858 \end{bmatrix}, P^5 = \begin{bmatrix} 0.1859 & 0.08355 & 0.7305 \\ 0.2507 & 0.06076 & 0.6886 \\ 0.2087 & 0.06558 & 0.7257 \end{bmatrix}$$

The probability distributions are given by the 3rd row of the matrices P, P^2, …P^5.

c. Find the expected first passage time from state 3 to state 1.

Solution:

$$M = \begin{bmatrix} 4.833 & 15 & 1.905 \\ 1.111 & 14.5 & 3.016 \\ 5 & 20 & 1.381 \end{bmatrix}$$

So the expected number of stages required for the system to reach state 1, given that it begins in state 3, is $m_{31}=5$.

d. What property does this Markov chain have that guarantees the existence of a steady state probability distribution?

Solution: This is a regular Markov chain, indicated by the fact that the elements of P^2 are strictly postive.

e. Write the equations which must be solved in order to compute the steady state distribution.

Solution:

$$\pi = \pi P, i.e., \begin{cases} \pi_1 = 0.9\pi_2 + 0.2\pi_2 \\ \pi_2 = 0.3\pi_1 + 0.1\pi_2 \\ \pi_3 = 0.7\pi_1 + 0.8\pi_3 \end{cases}$$

(or any two of the preceding equations), and the "normalizing" equation

$$\pi_1 + \pi_2 + \pi_3 = 1$$

f. What is the steady state probability distribution?

Solution: The solution of the system of equations in (e) is

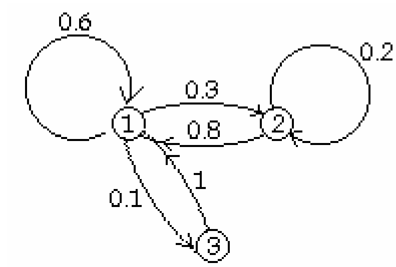
$$\begin{cases} \pi_1 = 0.2069 \\ \pi_2 = 0.06897 \\ \pi_3 = 0.7241 \end{cases}$$

2. An office has two printers, which are very unreliable. It has been observed that when both are working in the morning, there is a 30% chance that one will fail by evening, and a 10% chance that both will fail. If it happens that only one printer is working in the morning, there is a 20% chance that it will fail by evening. . Any printers that fail during the day are picked up by a repairman the next morning, and returned the following morning. (Assume that he can work on more than one printer at a time.)

Model this situation as a Markov chain with the state being the number of failed printers observed in the morning after the repairman has returned any printers but before any failures have occurred. The states then, are 0, 1, & 2.

a. Draw the transition diagram, with probabilities indicated.

Solution:



b. Write the transition probability matrix.

Solution:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.8 & 0.2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c. What is the probability distribution of the number of failed printers on Wednesday evening if both printers are working on Monday morning?

Solution:

$$P^3 = \begin{bmatrix} 0.672 & 0.258 & 0.07 \\ 0.688 & 0.248 & 0.064 \\ 0.7 & 0.24 & 0.06 \end{bmatrix}$$

and so, if both printers are working Monday morning (state 0), there is 67.2% probability that 0 printers are failed, 25.8% probability that 1 printer is failed, and 7% probability that 2 printers are in the failed condition on Wednesday evening (after 3 days).

d. What property does this Markov chain have that guarantees the existence of a steady state probability distribution?

Solution: This Markov chain is regular, as evidenced by the fact that P^3 has strictly positive elements.

e. Write the equations which must be solved in order to compute the steady state distribution.

Solution:

$$\pi = \pi P \Rightarrow \begin{cases} \pi_1 = 0.6\pi_1 + 0.8\pi_2 + \pi_3 \\ \pi_2 = 0.3\pi_1 + 0.2\pi_2 \\ \pi_3 = 0.1\pi_1 \end{cases}$$

and $\pi_1 + \pi_2 + \pi_3 = 1$

f. What is the steady state probability distribution?

Solution:

$$\begin{cases} \pi_1 = 0.678 \\ \pi_2 = 0.2542 \\ \pi_3 = 0.0678 \end{cases}$$

3. ***(s,S) Model of Inventory System*** A periodic inventory replenishment system with reorder point **s**=2 and order-up to level **S**=7 is modeled. At the end of each period of demand (day), the inventory is tallied, and if the level is less than or equal to the reorder point (s), enough is ordered (& immediately received) so as to bring the inventory level up to S. The probability distribution is discrete and Poisson, with expected demand 2/day.

The state of the system is the inventory position: if no backorders are permitted, as in this case, this is the stock-on-hand. (Otherwise it is the stock-on-hand if nonnegative, and the number of unfilled orders if negative.)

The following output was obtained using the MARKOV workspace (APL code) which is available from the URL: http://asrl.ecn.uiowa.edu/dbricker/APL_software.html

a. Over a long period of time, what is the percent of the days in which you would expect there to be a stockout (zero inventory)? ***Solution:*** $\pi_0 = 0.09024$

b. What will be the average end-of-day inventory level? ***Solution:*** $\sum_{i=0}^{7} i\pi_i = 3.443$

c. How often (i.e. once every how many days?) will the inventory be full at the end of the day?

Solution: average interval between visits to state 7 is $m_{77} = \frac{1}{\pi_7} = 19.17$ days

d. How often will the inventory be restocked? ***Solution***: The probability that the inventory is re-stocked is $\sum_{i=0}^{2} \pi_i = 0.09024 + 0.09892 + 0.1442 = 0.3333$, which implies that the inventory is restocked, on average, once every three days.

e. If the shelf is full Monday morning, what is the probability that a replenishment occurs Friday evening? ***Solution***: The probability that the system is in states 0, 1, or 2 after 5 stages, given that it begins in state 7, is $\sum_{j=0}^{2} p_{7,j}^{(5)} = 0.08798 + 0.09716 + 0.1428 = 0.3279$. *Note that this is very nearly the same as the answer to (d)!*

f. If the shelf is full Monday morning, what is the probability that the *first* stockout occurs Friday evening?

Solution: The first-passage probability $f_{7,0}^{(5)} = 0.07153$

g. What is the expected number of days, starting with a full inventory, until a stockout occurs?

Solution: $m_{7,0} = 11.08$

h. Starting with a full inventory, what is the expected number of stockouts during the first 30 days? What is the expected number of times that the inventory is restocked? ***Solution***: $\sum_{n=1}^{30} p_{7,0}^{(n)} = 2.619$

i. What is the average daily cost of this inventory system--including holding cost of \$0.50/unit, replenishment cost of \$10 per replenishment, and shortage penalties of \$5 per stockout (regardless of the unsatisfied demand)?

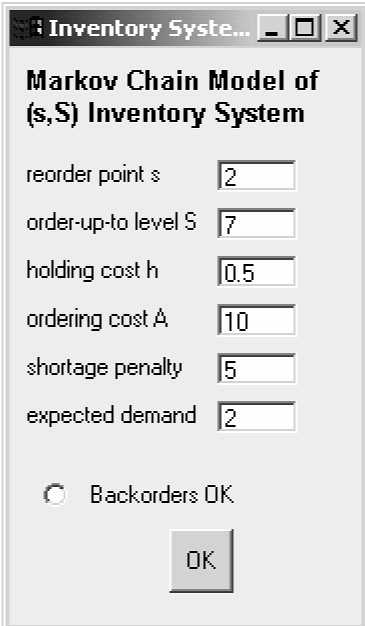
Solution: holding cost $0.50 \times \sum_{i=0}^{7} i\pi_i = 1.721$ \$/day

replenishment cost: $10 \times \sum_{i=0}^{2} \pi_i = 3.3333$ \$/day

shortage penalties: $5 \times \pi_0 = 0.4512$ \$/day

The sum is 5.505 \$/day.

If the shortage penalty depended upon the magnitude of the unsatisfied demand, the computation would be somewhat more complicated!



Transition Probability Matrix

	1	2	3	4	5	6	7	8
1	0.004534	0.01203	0.03609	0.09022	0.1804	0.2707	0.2707	0.1353
2	0.004534	0.01203	0.03609	0.09022	0.1804	0.2707	0.2707	0.1353
3	0.004534	0.01203	0.03609	0.09022	0.1804	0.2707	0.2707	0.1353
4	0.3233	0.2707	0.2707	0.1353	0	0	0	0
5	0.1429	0.1804	0.2707	0.2707	0.1353	0	0	0
6	0.05265	0.09022	0.1804	0.2707	0.2707	0.1353	0	0
7	0.01656	0.03609	0.09022	0.1804	0.2707	0.2707	0.1353	0
8	0.004534	0.01203	0.03609	0.09022	0.1804	0.2707	0.2707	0.1353

Cost Vector

i	State	Cost
1	SOH=zero	10.0
2	SOH=one	10.5
3	SOH=two	11.0
4	SOH=three	1.5
5	SOH=four	2.0
6	SOH=five	2.5
7	SOH=six	3.0
8	SOH=seven	3.5

5-th Power

	1	2	3	4	5	6	7	8
1	0.08798	0.09716	0.1428	0.1681	0.1703	0.1606	0.1212	0.05186
2	0.08798	0.09716	0.1428	0.1681	0.1703	0.1606	0.1212	0.05186
3	0.08798	0.09716	0.1428	0.1681	0.1703	0.1606	0.1212	0.05186
4	0.09102	0.09849	0.1418	0.163	0.164	0.1598	0.126	0.05585
5	0.09283	0.1009	0.1456	0.1667	0.1649	0.156	0.1203	0.0527
6	0.09241	0.1013	0.1473	0.1699	0.1674	0.155	0.1166	0.05016
7	0.08993	0.09931	0.1457	0.1702	0.1699	0.1575	0.1174	0.04999
8	0.08798	0.09716	0.1428	0.1681	0.1703	0.1606	0.1212	0.05186

Steady State Distribution

i	state	P{i}
1	SOH=zero	0.09024
2	SOH=one	0.09892
3	SOH=two	0.1442
4	SOH=three	0.1675
5	SOH=four	0.1678
6	SOH=five	0.1585
7	SOH=six	0.1207
8	SOH=seven	0.05217

Expected no. of visits during first 30 stages

	1	2	3	4	5	6	7	8
1	2.619	2.886	4.236	4.982	5.073	4.854	3.73	1.621
2	2.619	2.886	4.236	4.982	5.073	4.854	3.73	1.621
3	2.619	2.886	4.236	4.982	5.073	4.854	3.73	1.621
4	2.888	3.085	4.382	4.945	4.879	4.671	3.59	1.56
5	2.764	3.043	4.428	5.089	4.974	4.613	3.547	1.542
6	2.705	2.987	4.384	5.123	5.1	4.695	3.489	1.517
7	2.662	2.936	4.31	5.067	5.129	4.82	3.585	1.49
8	2.619	2.886	4.236	4.982	5.073	4.854	3.73	1.621

First Visit Probabilities to State 1 from State 8

n	P
1	0.004534
2	0.07452
3	0.1048
4	0.08298
5	0.07153
6	0.06623
7	0.05974
8	0.05352
9	0.04818
10	0.0434
11	0.03905
12	0.03514
13	0.03163
14	0.02847
15	0.02562

Mean First Passage Time Matrix

	1	2	3	4	5	6	7	8
1	11.08	10.11	6.936	5.746	5.373	5.305	7.086	19.17
2	11.08	10.11	6.936	5.746	5.373	5.305	7.086	19.17
3	11.08	10.11	6.936	5.746	5.373	5.305	7.086	19.17
4	8.094	8.101	5.922	5.969	6.529	6.462	8.243	20.32
5	9.472	8.527	5.604	5.104	5.959	6.824	8.605	20.69
6	10.12	9.087	5.911	4.903	5.209	6.311	9.08	21.16
7	10.6	9.609	6.419	5.239	5.038	5.518	8.287	21.66
8	11.08	10.11	6.936	5.746	5.373	5.305	7.086	19.17

56:171 Operations Research
Homework #8 Solutions -- Fall 2001

1. A factory has a buffer with a capacity of 4 m^3 for temporarily storing waste produced by the factory. Each week the factory produces k m^3 waste with a probability of p_k, where $p_0 = {}^1/_8$, $p_1 = ½$, $p_2 = ¼$, and $p_3 = {}^1/_8$. If the amount of waste produced in one week exceeds the remaining capacity of the buffer, the excess is specially removed at a cost of $100 per m^3. At the end of each week, there is a regular opportunity to remove waste from the storage buffer at a fixed cost of $50 and a variable cost of $10 per m^3. The following policy is used. If at the end of the week the storage buffer contains more than 2 m^3 the buffer is emptied; otherwise no waste is removed. Determine
   a) the frequency of overflows
   b) the frequency that the buffer is emptied
   c) the long-run average cost per week

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

Solution: Define six states: $X_n \in \{0, 1, 2, 3, 4, 5\}$
where the state is the volume of waste at the end of the week *before* removing any excess. Note that the description of the system implies that this volume will never exceed 5 m^3, since the week will begin with no more than 2 m^3, and the maximum amount of waste generated is 3 m^3!
The transition probability matrix is

$$P = \begin{bmatrix} 0.125 & 0.5 & 0.25 & 0.125 & 0 & 0 \\ 0 & 0.125 & 0.5 & 0.25 & 0.125 & 0 \\ 0 & 0 & 0.125 & 0.5 & 0.25 & 0.125 \\ 0.125 & 0.5 & 0.25 & 0.125 & 0 & 0 \\ 0.125 & 0.5 & 0.25 & 0.125 & 0 & 0 \\ 0.125 & 0.5 & 0.25 & 0.125 & 0 & 0 \end{bmatrix}$$

Note that if the system is in states 3, 4, or 5, the storage buffer is empty at the beginning of the next week, and so the transition probabilities are identical to those of state 0.
The results of the computation of the steadystate distribution and mean first passage times are shown below:

Steady State Distribution

i	state	P{i}
1	ZERO	0.05724
2	ONE	0.2617
3	TWO	0.2804
4	THREE	0.2629
5	FOUR	0.1028
6	FIVE	0.03505

Mean First Passage Time Matrix

	0	1	2	3	4	5
0	17.47	2.571	2.933	3.804	9.727	28.53
1	19.27	3.821	2.4	3.271	8.545	28
2	18.61	3.714	3.567	2.773	8.091	25.6
3	17.47	2.571	2.933	3.804	9.727	28.53
4	17.47	2.571	2.933	3.804	9.727	28.53
5	17.47	2.571	2.933	3.804	9.727	**28.53**

a. The frequency of overflows is the mean recurrence time m_{55} (= $1/\pi_5$ = 28.53) for state 5 (which is the only state in which an overflow has occurred.) That is, an overflow occurs, on average, once every 28.53 weeks.

b. The frequency with which the buffer is emptied will be the reciprocal of the steadystate probability of states in which the buffer is emptied, namely states 3, 4, & 5. Thus,

$$\sum_{i=3}^{5} \pi_i = 0.2629 + 0.1028 + 0.03505 = 0.4007,$$

so that the buffer is emptied, on average, once every 2.496 weeks.

c. The long-run average cost per week is

$$80\pi_3 + 90\pi_4 + 190\pi_5 = (80\times 0.2629) + (90\times 0.1028) + (190\times 0.03505)$$
$$= 36.94$$

❋❋❋❋❋❋❋❋❋❋❋❋❋

2. For simplicity, suppose that fresh blood obtained by a hospital will spoil if it is not transfused within five days. The hospital receives 100 pints of fresh blood daily from a local blood bank. Two policies are possible for determining the order in which blood is transfused. The following table gives the probabilities of transfusion for blood of various ages under each policy:

	0 day old	1 day old	2 days old	3 days old	4 days old
Policy 1	10%	20%	30%	40%	50%
Policy 2	50%	40%	30%	20%	10%

For example, under policy 1, blood has a 10% chance of being transfused during its first day at the hospital. Under policy 2, four-day-old blood has a 10% chance of being transfused.

a. A FIFO (first in, first out) blood-issuing policy issues "old" blood first, whereas a LIFO (last in, first out) policy issues "young" blood first. Which policy above represents a LIFO policy, and which represents a FIFO policy?

Solution: Policy #1 is FIFO and Policy #2 is LIFO.

Define Markov chain models with seven states, where the state is determined when first obtained and every 24 hours thereafter.

State	Description
1	0 days old
2	1 day old
3	2 days old
4	3 days old
5	4 days old
6	Transfused
7	Spoiled

States 6 & 7 are absorbing states, and states 1 – 5 are transient.

FIFO policy (#1). The transition probability matrix is P=

```
  | 1  2    3    4    5    6    7
1 | 0  0.9  0    0    0    0.1  0
2 | 0  0    0.8  0    0    0.2  0
3 | 0  0    0    0.7  0    0.3  0
4 | 0  0    0    0    0.6  0.4  0
5 | 0  0    0    0    0    0.5  0.5
```

```
6 | 0 0   0   0   0   1   0
7 | 0 0   0   0   0   0   1
```

The submatrices Q and R are:

$$Q = \begin{bmatrix} 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \& R = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0 \\ 0.3 & 0 \\ 0.4 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Then

$$E = (I-Q)^{-1} = \begin{bmatrix} 1 & -0.9 & 0 & 0 & 0 \\ 0 & 1 & -0.8 & 0 & 0 \\ 0 & 0 & 1 & -0.7 & 0 \\ 0 & 0 & 0 & 1 & -0.6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0.9 & 0.72 & 0.504 & 0.3024 \\ 0 & 1 & 0.8 & 0.56 & 0.336 \\ 0 & 0 & 1 & 0.7 & 0.42 \\ 0 & 0 & 0 & 1 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A = ER = \begin{bmatrix} 0.8488 & 0.1512 \\ 0.832 & 0.168 \\ 0.79 & 0.21 \\ 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$$

The first-passage probabilities $f_{1,6}^{(n)}, n = 1,2,3,4,5,6$ are

n	$f_{1,6}^{(n)}$
1	0.1
2	0.18
3	0.216
4	0.2016
5	0.1512
6	0

b. For FIFO policy (#1), the probability that a new pint of blood (state 0) will eventually spoil (reach state 7) is a_{17}=15.12%

c. The average number of pints of blood in inventory may be found from the matrix E: on any typical day, 100% of the 100 new pints are in inventory, 90% of the one-day-old pints, 72% of the two-day-old pints, 50.4% of the three-day-old pints, and 30.24% of the four-day-old pints. Thus, the average inventory will be

100+90+72+50.4+30.24=342.6.

d. The average age of transfused blood is the expected first passage time, i.e., $\sum_{n=1}^{5} n \times f_{1,6}^{(n)} = 2.67$ days.

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

LIFO policy (#2). The transition probability matrix is P=

```
  | 1  2   3   4   5   6   7
1 | 0 0.5 0   0   0   0.5 0
2 | 0 0   0.6 0   0   0.4 0
3 | 0 0   0   0.7 0   0.3 0
4 | 0 0   0   0   0.8 0.2 0
5 | 0 0   0   0   0   0.1 0.9
6 | 0 0   0   0   0   1   0
7 | 0 0   0   0   0   0   1
```

The submatrices Q and R are:

$$Q = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \& R = \begin{bmatrix} 0.5 & 0 \\ 0.4 & 0 \\ 0.3 & 0 \\ 0.2 & 0 \\ 0.1 & 0.9 \end{bmatrix}$$

Then

$$E = (I-Q)^{-1} = \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 \\ 0 & 1 & -0.6 & 0 & 0 \\ 0 & 0 & 1 & -0.7 & 0 \\ 0 & 0 & 0 & 1 & -0.8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0.5 & 0.3 & 0.21 & 0.168 \\ 0 & 1 & 0.6 & 0.42 & 0.336 \\ 0 & 0 & 1 & 0.7 & 0.56 \\ 0 & 0 & 0 & 1 & 0.8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A = ER = \begin{bmatrix} 0.8488 & 0.1512 \\ 0.6976 & 0.3024 \\ 0.496 & 0.504 \\ 0.28 & 0.72 \\ 0.1 & 0.9 \end{bmatrix}$$

The first-passage probabilities $f_{1,6}^{(n)}, n = 1,2,3,4,5,6$ are

n	$f_{1,6}^{(n)}$
1	0.5
2	0.2
3	0.09
4	0.042
5	0.0168
6	0

The analysis of this policy is similar to the FIFO policy:

b. The probability that a new pint of blood will eventually spoil is a_{17} = 15.12%, *which is the same as for the FIFO policy!*

c. The average number of pints of blood in inventory is 100×(1.0 + 0.5+ 0.3 +0.21+ 0.168) = 217.8

d. The average age of transfused blood is 1×0.5 +2×0.2 +3×0.09+4× 0.042 +5×0.0168= 1.422 days

Summary

Policy	Probability of Spoilage	Average inventory	Average age of transfused blood
FIFO	15.12%	342.6	2.67 days
LIFO	15.12%	217.8	1.422 days

Surprisingly, the LIFO (last-in, first-out) policy performs just as well as the FIFO policy with respect to the probability of spoilage, and out-performs the FIFO policy with respect to the other two criteria!

56:171 Operations Research

Homework #9 Solutions -- Fall 2001

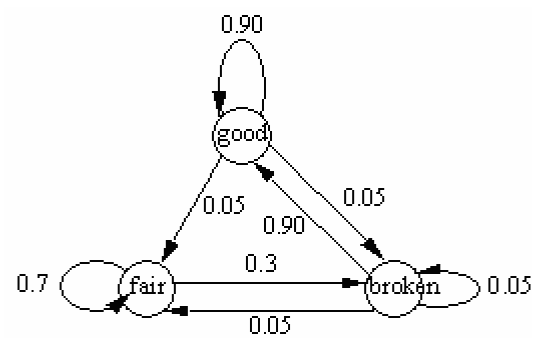
1. Discrete-time Markov Chain. *(Based upon Exercise 4, §19.5, page 982 of text by W. Winston)*

At the beginning of each year, my car is in good, fair, or broken-down condition. A good car will be good at the beginning of next year with probability 90%, fair with probability 5%, or broken-down with probability 5%. A fair car will be fair at the beginning of the next year with probability 70%, or broken-down with probability 30%. It costs $12000 to purchase a good car; a fair car can be traded in for $5000; and a broken-down car has no trade-in value and must immediately be replaced by a good car. It costs $1000 per year to operate a good car and $2000 to operate a fair car. Should I replace my car as soon as it becomes a fair car, or should I drive my car until it breaks down? Assume that the cost of operating a car during a year depends on the type of car on hand at the *beginning* of the year (after a new car, if any, arrives).

Define a Markov chain model with three states (Good, Fair, & Broken-down). Assume, as implied by the problem statement, that break-down occurs only at the *end* of a year, and then (at the beginning of the next year) the broken-down car "must immediately be replaced".
Note: assume that state 1= Good, state 2= Fair, and state 3= Broken-down.

Policy A: Replace when car has broken down:

a. Draw a diagram of the Markov chain and write down the transition probability matrix.
Solution:



b. Write down the equations which could be solved to obtain the steadystate probabilities.
Solution:

$$\pi = \pi P = \pi \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0 & 0.7 & 0.3 \\ 0.9 & 0.05 & 0.05 \end{bmatrix} \Rightarrow \begin{cases} \pi_1 = 0.9\pi_1 + 0.9\pi_3 \\ \pi_2 = 0.05\pi_1 + 0.7\pi_2 + 0.05\pi_3 \\ \pi_3 = 0.05\pi_1 + 0.3\pi_2 + 0.05\pi_3 \end{cases}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

c. Solve the equations, either manually or using appropriate computer software.
Solution: $\pi = \left[0.7714,\ \ 0.1429,\ \ 0.08571\right]$

d. Compute the average cost per year for the replacement policy.
Solution:

i	State	π_i	C_i	$\pi_i \times C_i$
1	GOOD	0.7714	1000	771.4
2	FAIR	0.1429	2000	285.7
3	BROKEN	0.08571	13000	1114

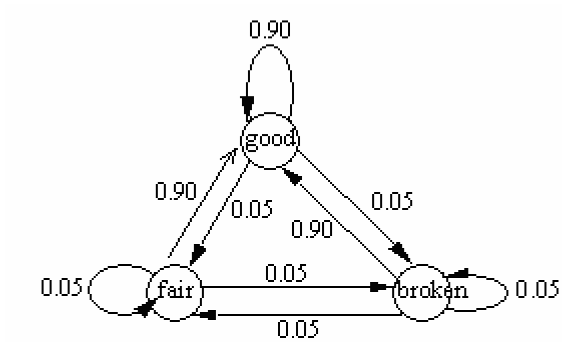
The average cost/period in steady state is $2171/year

e. What is the expected time between break-downs?
Solution: $m_{33} = 1/\pi_3 = 1/0.08571 = 11.67$ *, i.e, 11.67 years*

Policy B: Replace when car is in FAIR condition

a. *Draw a diagram of the Markov chain and write down the transition probability matrix.*

Solution:



b. *Write down the equations which could be solved to obtain the steadystate probabilities.*

Solution:

$$\pi = \pi P \Rightarrow \begin{cases} \pi_1 = 0.9\pi_1 + 0.9\pi_2 + 0.9\pi_3 \\ \pi_2 = 0.05\pi_1 + 0.05\pi_2 + 0.05\pi_3 \\ \pi_3 = 0.05\pi_1 + 0.05\pi_2 + 0.05\pi_3 \end{cases}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

c. *Solve the equations, either manually or using appropriate computer software.*

Solution: Since each row of P is identical, the solution is obviously $\pi = \left[0.9,\quad 0.05,\quad 0.05\right]$

d. *Compute the average cost per year for the replacement policy.*

Solution:

i	State	π_i	C_i	$\pi_i \times C_i$
1	GOOD	0.9	1000	900
2	FAIR	0.05	8000	400
3	BROKEN	0.05	13000	650

The average cost/period in steady state is $1950/year

Note that the cost for state 2 FAIR includes the replacement cost (12000) minus trade-in value (5000) plus the operating cost for the replacement car (1000)!

e. *What is the expected time between break-downs?*

Solution: The probability that the system leaves state 1 is 10%, so it will occur every 10 years

f. What replacement policy do you recommend?

Solution:

Policy	*Average Cost/Year*
A: Replace when Broken-down	$2171
B: Replace when in Fair condition	$1950

The policy "Replace in FAIR condition" is lower in cost by $221/year.

2. Continuous-time Markov Chain. In exercise 1, the Markov chain model assumes that break-down occurs only at the *end* of a year, and then (at the beginning of the next year) the broken-down car "must immediately be replaced" with a car in good condition. In fact, of course, the change in condition can occur at any time during the year, and a continuous-time Markov chain model would be a closer representation of reality. Let's assume that when my car breaks down, it takes me an average of 0.02 years (about 1 week) to find and purchase a replacement car (and that this delay has exponential distribution.) Again define a Markov chain model with three states (Good, Fair, & Brokendown).

a. What should be the transition rates, so that the probability of a change of condition during a one-year period is in agreement with the probabilities given in exercise 1?

Hint: The cdf of the exponential distribution is

$$F(t) = P\{\text{time to next event } \leq t\} = 1 - e^{-\lambda t}$$

If in state 1 (good car) there is a 10% probability that the system has changed states during the next year, the transition rate λ_1 *should therefore satisfy*

$$F(1) = 1 - e^{-\lambda} = 0.10$$

The value of λ_{12} *should be equal to the value of* λ_{13} *(since the transition probabilities* p_{12} *and* p_{13} *were each 5%), and* $\lambda_1 = \lambda_{12} + \lambda_{13}$*, so* $\lambda_{12} = \lambda_{13} = 0.5\,\lambda_1$*. To get the transition rate* λ_{31}*, observe that the expected value of the length of time required to replace my broken-down car is* $1/\lambda_{31} = 0.02$ *years.*

Solution: Reasoning as above, we obtain:

$$F(1) = 1 - e^{-\lambda_1} = 0.10 \Rightarrow e^{-\lambda_1} = 0.9 \Rightarrow \lambda_1 = 0.1053 \Rightarrow \lambda_{12} = \lambda_{13} = 0.0526$$

$$F(1) = 1 - e^{-\lambda_2} = 0.3 \Rightarrow e^{-\lambda_2} = 0.7 \Rightarrow \lambda_2 = 0.3567$$

$$1/\lambda_{31} = 0.02\text{yr} \Rightarrow \lambda_{31} = 50/\text{yr}$$

Policy A: Replace only when broken.

b. Write the matrix of transition rates.

Solution: the transition rate matrix is

$$\Lambda = \begin{bmatrix} -0.1053 & 0.0526 & 0.0526 \\ 0 & -0.3567 & 0.3567 \\ 50 & 0 & -50 \end{bmatrix}$$

Note that the diagonal element in each row is chosen to be the negative of the sum of off-diagonal elements.

c. Write the set of equations that must be solved for a steadystate distribution.

Solution:

$$\pi\Lambda = 0 \Rightarrow \begin{cases} -0.1053\pi_1 + 50\pi_3 = 0 \\ 0.0526\pi_1 - 0.3567\pi_2 = 0 \\ 0.0526\pi_1 + 0.3567\pi_2 - 50\pi_3 = 0 \end{cases}$$

and $\pi_1 + \pi_2 + \pi_3 = 1$

d. Find the steadystate distribution.

Solution: $\pi = [0.8699,\ 0.1283,\ 0.0018]$

e. What does this model predict will be my average operating cost/year (not including replacement costs)?

Solution: $(\$1000/\text{yr})\pi_1 + (\$2000/\text{yr})\pi_2 = \$1126.45/\text{yr}$

To compute the average replacement costs per year is not quite so simple. (We must multiply the replacement costs by the expected number of replacements/year, not by π_3 *(the fraction of the year spent in state 3). Let T = average time between replacements. Then*

$$\pi_3 = \frac{\text{average time from breakdown to replacement}}{\text{average length of time between replacements}} = \frac{0.02\,yr}{T}$$

What then is T? The number of replacements per year should then be 1/T.

***Solution**:* $\text{T} = \frac{0.02\text{yr}}{\pi_3} = \frac{0.02\text{yr}}{0.0018} = 11.11 \text{ yr} \Rightarrow \frac{1}{\text{T}} = 0.09/\text{yr}$

f. What average replacement cost per year is predicted by this model?

Solution: \$12000×0.09/yr = \$1080/yr

Total cost per year with policy A: \$1126/yr + \$1080/yr = \$2206/yr

Policy B: Replace when in FAIR condition.

b. The transition rate matrix will be

$$\Lambda = \begin{bmatrix} -0.1053 & 0.0526 & 0.0526 \\ 50 & -50 & 0 \\ 50 & 0 & -50 \end{bmatrix}$$

c. The equations defining the steadystate distribution will be

$$\pi\Lambda = \pi \begin{bmatrix} -0.1053 & 0.0526 & 0.0526 \\ 50 & -50 & 0 \\ 50 & 0 & -50 \end{bmatrix} = 0 \Rightarrow \begin{cases} -0.1053\pi_1 + 50\pi_2 + 50\pi_3 = 0 \\ 0.0526\pi_1 - 50\pi_2 = 0 \\ 0.0526\pi_1 - 50\pi_3 = 0 \end{cases}$$

and $\pi_1 + \pi_2 + \pi_3 = 1$

d. Find the steadystate distribution. ***Solution***: $\pi = [0.9979,\ 0.00105,\ 0.00105]$

e. What does this model predict will be my average operating cost/year (not including replacement costs)?

Solution: (\$1000/yr)π_1 = \$997.90/yr

f. What average replacement cost per year is predicted by this model?

Solution: As before, the fraction of the time spent replacing the automobile is equal to the ratio 0.02 year to the cycle length T: $\pi_2 + \pi_3 = \frac{0.02\,yr}{T} \Rightarrow \frac{1}{T} = \frac{\pi_2 + \pi_3}{0.02} = 0.1050/\text{yr}$,

i.e., average # replacements/yr is 0.1050

The replacement cost when the system reaches state (2) FAIR differs from that in state (3) BROKEN, since a trade-in value is received. Since $\pi_2 = \pi_3$, it appears that half of the replacements result in a trade-in allowance. And so average replacement cost/year will be

(0.5× 0.1050/yr × \$12000) + (0.5× 0.1050/yr × [12000 - \$5000]) = \$997.50/yr.

The total expected operating and replacement costs would therefore be \$997.90+\$997.40 = \$1995.40.

✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷ ✷

Comparison of Policies:

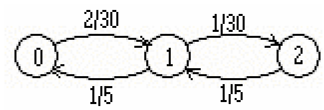
Policy	*Average Cost/Year*
A: Replace when Broken-down	\$2206
B: Replace when in Fair condition	\$1995

As in exercise 1, Policy B (Replace when in FAIR or BROKEN state) is less costly (by about \$211/yr) than Policy A (Replace only when in BROKEN state), and the costs predicted by the continuous-time Markov chain model are only slightly higher than those predicted by the discrete-time Markov chain model.

56:171 Operations Research

Homework #10 solution-- Fall 2001

1. Birth-death process (exercise 2, section 22.3 of text of Winston. *Numerical values modified*)
My home uses two light bulbs. On average, a light bulb lasts for 30 days (exponentially distributed). When a light bulb burns out, it takes me an average of 5 days (exponentially distributed) before I replace the bulb (one at a time!)



(rates are in failures/day)

a. Formulate a three-state birth-death model of this situation.
Solution: we may define the state of the system to be either the number of bulbs functioning or the number burned out. (In the diagram above, the state, i.e., "population", is the number of burned-out bulbs.) The failure rate of *each* bulb is the reciprocal of the average lifetime, i.e., 1/30 per day.

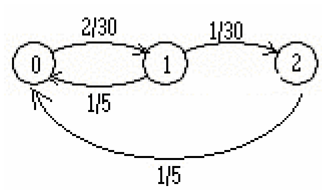
b. Determine the fraction of the time that both light bulbs are working.
Solution: To compute the steadystate distribution for this birth-death process, we compute

$$\frac{1}{\pi_0} = 1 + \frac{2/30}{1/5} + \frac{2/30}{1/5} \times \frac{1/30}{1/5} = 1 + \frac{1}{3} + \frac{1}{6} = \frac{25}{18} \Rightarrow \pi_0 = \frac{18}{25} = 72\%$$

Then $\pi_1 = \frac{1}{3}\pi_0 = \frac{6}{25} = 24\%$ & $\pi_2 = \frac{1}{6}\pi_0 = \frac{1}{25} = 4\%$

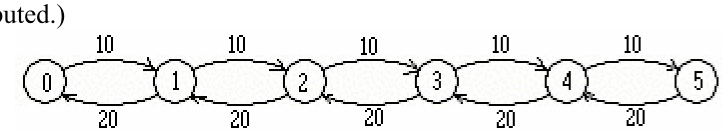
c. Determine the fraction of the time that neither light bulb is working.
Solution: $\pi_2 = 4\%$

d. Suppose that, when both bulbs are burned out and I replace a bulb, I replace both bulbs simultaneously. Why is this no longer a birth-death process?
Solution: The continuous-time Markov chain becomes



which is not a birth-death process, because "deaths" must be one-at-a-time in a birth-death process!

2. **Birth-death process** A local takeout Chinese restaurant has space to accommodate at most five customers. During the frigid Iowa winter, it is noticed that when customers arrive and the restaurant is full, virtually no one waits outside in the subfreezing weather, but instead goes next door to Luigi's Pizza Palace. Customers arrive at the restaurant at the average rate of 10 per hour, according to a Poisson process. The restaurant serves customers one at a time, first-come, first-served, in an average of 3 minutes each (the actual time being exponentially distributed.)



(a.) What is the steady-state distribution of the number of customers in the restaurant?

Solution: First compute π_0:

$$\frac{1}{\pi_0} = 1 + \frac{10}{20} + \left(\frac{10}{20}\right)^2 + \left(\frac{10}{20}\right)^3 + \left(\frac{10}{20}\right)^4 + \left(\frac{10}{20}\right)^5 = \frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = 1.96875 \Rightarrow \pi_0 = 0.5079365 \approx 50.8\%$$

Population in restaurant	Probability
0	0.507937
1	0.253968
2	0.126984
3	0.0634921
4	0.031746
5	0.015873

(b.) What is the average number of customers in the Chinese restaurant at any time?

Solution: $L = \sum_{i=0}^{5} i\pi_i = 0.904762$

(c.) What is the *average* arrival rate, considering that when there are 5 customers in the restaurant, the arrival rate is zero?

Solution: $\bar{\lambda} = \sum_{i=0}^{5} \lambda_i \pi_i = 10\pi_0 + 10\pi_1 + \cdots + 10\pi_4 + 0\pi_5 = 9.68254$, i.e., 9.68254 customers per hour

(d.) According to Little's Law, what is the expected amount of time that a customer spends in the restaurant?

Solution: $L = \bar{\lambda} W \Rightarrow W = L/\bar{\lambda} = \frac{0.904762}{9.68254/hr}$ = 0.0934426 hr = 5.60656 min

(e.) What is the fraction of potential customers who are lost to the pizza establishment? ***Solution***: $\pi_5 \approx 1.6\%$

What is the number of customers per day lost to the pizza establishment?

Solution: $(10/hour) \times \pi_5 \approx 0.16/hour$

3. **Deterministic Dynamic Programming** (A variation of the example presented in class) A utility company must plan expansion of its generating capacity over the next eight years. A forecast has been prepared, specifying the number of additional power plants R_t required at the end of each year t. Each year, at most three plants may be added. The cost of adding a power plant in year t is C_t per plant, plus a fixed cost of F_t (unless no plants are added).

Year t	*Reqd* R_t	*Fixed cost* F_t	*Marginal cost* C_t
1	1	2.4	3.4
2	2	2.4	3.5
3	3	2.5	3.5
4	5	2.5	3.5
5	7	2.6	3.4
6	8	2.6	3.4

A dynamic programming model with forward recursion is developed, so that stage 1 = first year (now), stage 6 = final year of planning period. Time value of money is to be considered, with a discount factor = 0.83333. The computations at each stage are shown below in order to minimize the present value of the cost of adding the generating capacity. *Note*: A value "9999.9999" in the table indicates an *infeasible* combination of state & decision.

---Stage 6 (*final year of planning period*) ---

s \ x:	0	1	Minimum
7	9999.9999	6.0000	6.0000
8	0.0000	9999.9999	0.0000

---Stage 5---

s \ x:	0	1	2	3	Minimum
5	9999.9999	9999.9999	14.4000	12.8000	12.8000
6	9999.9999	11.0000	9.4000	9999.9999	9.4000
7	5.0000	6.0000	9999.9999	9999.9999	5.0000
8	0.0000	9999.9999	9999.9999	9999.9999	0.0000

---Stage 4---

s \ x:	0	1	2	3	Minimum
4	9999.9999	16.6667	17.3333	17.1667	16.6667
5	10.6667	13.8333	13.6667	13.0000	10.6667
6	7.8333	10.1667	9.5000	9999.9999	7.8333
7	4.1667	6.0000	9999.9999	9999.9999	4.1667
8	0.0000	9999.9999	9999.9999	9999.9999	0.0000

---Stage 3---

s \ x:	0	1	2	3	Minimum
2	9999.9999	9999.9999	23.3889	21.8889	21.8889
3	9999.9999		18.3889	19.5278	18.3889
4	13.8889	14.8889	16.0278	16.4722	13.8889
5	8.8889	12.5278	12.9722	13.0000	8.8889
6	6.5278	9.4722	9.5000	9999.9999	6.5278
7	3.4722	6.0000	9999.9999	9999.9999	3.4722
8	0.0000	9999.9999	9999.9999	9999.9999	0.0000

---Stage 2---

s \ x:	0	1	2	3	Minimum
1	9999.9999	24.1407	24.7241	24.4741	24.1407
2	18.2407	21.2241	20.9741	20.3074	18.2407
3	15.3241	17.4741	16.8074	18.3398	15.3241
4	11.5741	13.3074	14.8398	15.7935	11.5741
5	7.4074	11.3398	12.2935	12.9000	7.4074
6	5.4398	8.7935	9.4000	9999.9999	5.4398
7	2.8935	5.9000	9999.9999	9999.9999	2.8935
8	0.0000	9999.9999	9999.9999	9999.9999	0.0000

---Stage 1 (*first year of planning period*)---

s \ x:	0	1	2	3	Minimum
0	9999.9999	25.9173	24.4006	25.3701	24.4006

a. What annual percent return on investment is implied by the discount factor $\beta = 0.833333$?

Solution: $\beta = \frac{1}{1+r} \Rightarrow r = \frac{1}{\beta} - 1 = 20\%$

b. One value is missing in the table, i.e., the total cost of years 3, 4, 5, and 6 if, at the beginning of year 3, the company has already added 3 plants and decides to add 1 additional plant. What is this value?

Construction cost in year 3	$2.5 + 3.5 = 6$
Discounted minimum cost of years 4, 5, & 6	$\beta \times f_4(4) = 0.83333 \times 16.6667 = 13.8889$
Total	19.8889

b. What is the minimum present value of the total construction cost to meet the requirements?

Solution: $f1(0) = 24.4006$

c. What is the optimal schedule for adding plants?

Solution: The "minimum" column in each table above displays the value

$F_n(s)$ = minimum cost of stages n, n+1, …8 if s plants are already built at the beginning of stage n

and the column in which this minimum was found indicates the optimal decision at that stage.

One could, therefore, determine tables for all the optimal values and decisions:

Stage 1

Current State	Optimal Decision	Optimal Value	Next State
0 added	Build 2	24.4006	2 added

Stage 2

Current State	Optimal Decision	Optimal Value	Next State
1 added	Build 1	24.1407	2 added
2 added	Idle	18.2407	2 added
3 added	Idle	15.3241	3 added
4 added	Idle	11.5741	4 added
5 added	Idle	7.4074	5 added
6 added	Idle	5.4398	6 added
7 added	Idle	2.8935	7 added
8 added	Idle	0.0000	8 added

Stage 3

Current State	Optimal Decision	Optimal Value	Next State
2 added	Build 3	21.8889	5 added
3 added	Build 2	18.3889	5 added
4 added	Idle	13.8889	4 added
5 added	Idle	8.8889	5 added
6 added	Idle	6.5278	6 added
7 added	Idle	3.4722	7 added
8 added	Idle	0.0000	8 added

```
Stage 4
 Current   Optimal   Optimal     Next
 State     Decision  Value       State
  4 added  Build 1     16.6667    5 added
  5 added  Idle        10.6667    5 added
  6 added  Idle         7.8333    6 added
  7 added  Idle         4.1667    7 added
  8 added  Idle         0.0000    8 added
```

```
Stage 5
 Current   Optimal   Optimal     Next
 State     Decision  Value       State
  5 added  Build 3     12.8000    8 added
  6 added  Build 2      9.4000    8 added
  7 added  Idle         5.0000    7 added
  8 added  Idle         0.0000    8 added
```

```
Stage 6
 Current   Optimal   Optimal     Next
 State     Decision  Value       State
  7 added  Build 1      6.0000    8 added
  8 added  Idle         0.0000    8 added
```

We can then trace through the table to find the optimal schedule for adding capacity:

Year t	*# plants to add*	*Cumulative # plants added*	*# plants required*
1	2	2	1
2	0	2	2
3	3	5	3
4	0	5	5
5	3	8	7
6	0	8	8

d. Suppose that (for unspecified reasons) the number of plants added during the first year is one (not optimal!). What is the best schedule for adding capacity during the remaining five years?

Solution: If the system begins stage 2 in state 1, we can trace through the tables beginning with stage 2 in order to obtain the following schedule for adding capacity:

Year t	*# plants to add*	*Cumulative # plants added*	*# plants required*
1	1	1	1
2	1	2	2
3	3	5	3
4	0	5	5
5	3	8	7
6	0	8	8

56:171 Operations Research

Homework #11 Solution -- Fall 2001

1. ***Redistricting Problem*** A state is to be allocated twenty representatives (Reps) to be sent to the national legislature. There are nine districts within the state, whose boundaries are fixed. Every district should be assigned at least one representative. The allocation should be done according to the population (Pop) of the districts:

District	1	2	3	4	5	6	7	8	9
Population	50	60	70	50	70	100	20	70	40

The "target allocation" of district i is Reps times Pop[i] divided by the population of the state, but this target is generally non-integer. The objective is the assign the representatives to the districts in such a way that the maximum absolute deviation from the targets is as small as possible.

In the DP model, there is a stage for each district, and the state of the system is the number of representatives not yet assigned when we decide upon the allocation to district i. The optimal value function is defined by a forward recursion:

$$\begin{cases} f_n(s) = \underset{x\in\{1,2,3,4\}}{\text{minimum}} \ \max\left\{\left|\alpha_n - x\right|, f_{n+1}(s-x)\right\} \\ f_0(0) = 0 \ \& \ f_0(s) = +\infty \text{ for } s > 0 \end{cases}$$

That is, the optimal value function $f_n(s)$ at stage n with state s is the smallest possible value of the maximum absolute deviations from the targets α of the allocation to districts n, n+1, …. 9 if the total number of representatives available to those districts is given by the state s.

a. What is the "target" allocation α_i of each district?

District	1	2	3	4	5	6	7	8	9
Population	50	60	70	50	70	100	20	70	40
Target	1.88	2.26	2.64	1.88	2.64	3.77	0.75	2.64	1.51
Rounded:	2	2	3	2	3	4	1	3	2

Note that by rounding the "target" to the nearest integer, the number of representatives allocated is 22 (>20!)

b. Compute the missing values in the table below for stage 3.

Solution: The target allocation α_3 is 2.64. Therefore:

(s=12, x=1): $\max\left\{\left|\alpha_n - x\right|, f_{n+1}(s-x)\right\} = \max\left\{\left|2.64-1\right|, f_4(11)\right\} = \max\{1.64, 0.77\} = 1.64$

(s=12, x=2): $\max\left\{\left|2.64-2\right|, f_4(10)\right\} = \max\{0.64, 0.89\} = 0.89$

(s=12, x=3): $\max\left\{\left|2.64-3\right|, f_4(9)\right\} = \max\{0.36, 1.64\} = 1.64$

(s=12, x=1): $\max\left\{\left|2.64-1\right|, f_4(11)\right\} = \max\{1.64, 0.77\} = 1.64$

(s=13, x=1): $\max\left\{\left|\alpha_n - x\right|, f_{n+1}(s-x)\right\} = \max\left\{\left|2.64-1\right|, f_4(12)\right\} = \max\{1.64, 0.64\} = 1.64$, ***etc.***

c. There are three optimal solutions to this problem. For each solution, what are the optimal allocations of representatives to districts? (Enter in tables below.)

Solution #1:

District	1	2	3	4	5	6	7	8	9
Allocation	2	2	2	2	3	4	1	3	1
Deviation	+0.12	–0.26	**–0.64**	+0.12	+0.36	+0.23	+0.25	+0.36	–0.51

Solution #2:

District	1	2	3	4	5	6	7	8	9
Allocation	2	2	3	2	2	4	1	3	1
Deviation	+0.12	–0.26	+0.36	+0.12	**–0.64**	+0.23	+0.25	+0.36	–0.51

Solution #3:

District	1	2	3	4	5	6	7	8	9
Allocation	2	2	3	2	3	4	1	2	1
Deviation	+0.12	–0.26	+0.36	+0.12	+0.36	+0.23	+0.25	**–0.64**	–0.51

d. Does one of the three solutions seem “better” than the others with respect to some other considerations?
???? They seem very comparable to me in every way!

e. Which district has the largest positive deviation from its target allocation?
Solution #1: district 3; **Solution** #2: District 5; **Solution** #3: District 8

f. Which district has the largest negative deviation from its target allocation? Same as (e)—in every case, the deviation with the maximumabsolute value is negative, i.e., the district is *underrepresented*.

---Stage 9---

s \ x:	1	2	3	4	Min
1	0.51	999.99	999.99	999.99	0.51
2	999.99	0.49	999.99	999.99	0.49
3	999.99	999.99	1.49	999.99	1.49
4	999.99	999.99	999.99	2.49	2.49

---Stage 8---

s \ x:	1	2	3	4	Min
2	1.64	999.99	999.99	999.99	1.64
3	1.64	0.64	999.99	999.99	0.64
4	1.64	0.64	0.51	999.99	0.51
5	2.49	1.49	0.49	1.36	0.49
6	999.99	2.49	1.49	1.36	1.36
7	999.99	999.99	2.49	1.49	1.49
8	999.99	999.99	999.99	2.49	2.49

---Stage 7---

s \ x:	1	2	3	4	Min
3	1.64	999.99	999.99	999.99	1.64
4	0.64	1.64	999.99	999.99	0.64
5	0.51	1.25	2.25	999.99	0.51
6	0.49	1.25	2.25	3.25	0.49
7	1.36	1.25	2.25	3.25	1.25
8	1.49	1.36	2.25	3.25	1.36
9	2.49	1.49	2.25	3.25	1.49
10	999.99	2.49	2.25	3.25	2.25
11	999.99	999.99	2.49	3.25	2.49
12	999.99	999.99	999.99	3.25	3.25

---Stage 6---

s \ x:	1	2	3	4	Min
4	2.77	999.99	999.99	999.99	2.77
5	2.77	1.77	999.99	999.99	1.77
6	2.77	1.77	1.64	999.99	1.64
7	2.77	1.77	0.77	1.64	0.77
8	2.77	1.77	0.77	0.64	0.64
9	2.77	1.77	0.77	0.51	0.51
10	2.77	1.77	1.25	0.49	0.49
11	2.77	1.77	1.36	1.25	1.25
12	2.77	2.25	1.49	1.36	1.36
13	3.25	2.49	2.25	1.49	1.49
14	999.99	3.25	2.49	2.25	2.25
15	999.99	999.99	3.25	2.49	2.49

---Stage 5---

s \ x:	1	2	3	4	Min
5	2.77	999.99	999.99	999.99	2.77
6	1.77	2.77	999.99	999.99	1.77
7	1.64	1.77	2.77	999.99	1.64
8	1.64	1.64	1.77	2.77	1.64
9	1.64	0.77	1.64	1.77	0.77
10	1.64	0.64	0.77	1.64	0.64
11	1.64	0.64	0.64	1.36	0.64
12	1.64	0.64	0.51	1.36	0.51
13	1.64	1.25	0.49	1.36	0.49
14	1.64	1.36	1.25	1.36	1.25
15	2.25	1.49	1.36	1.36	1.36
16	2.49	2.25	1.49	1.36	1.36

---Stage 4---

s \ x:	1	2	3	4	Min
8	1.64	1.77	2.77	999.99	1.64
9	1.64	1.64	1.77	2.77	1.64
10	0.89	1.64	1.64	2.11	0.89
11	0.89	0.77	1.64	2.11	0.77
12	0.89	0.64	1.11	2.11	0.64
13	0.89	0.64	1.11	2.11	0.64
14	0.89	0.51	1.11	2.11	0.51
15	1.25	0.49	1.11	2.11	0.49
16	1.36	1.25	1.11	2.11	1.11
17	1.36	1.36	1.25	2.11	1.25

1.64 0.89 1.64 1.64 1.64

---Stage 3---

s \ x:	1	2	3	4	Min
12	1.64	0.89	1.64	1.64	0.89
13	1.64	0.77	0.89	1.64	0.77
14	1.64	0.64	0.77	1.36	0.64
15	1.64	0.64	0.64	1.36	0.64
16	1.64	0.64	0.64	1.36	0.64
17	1.64	0.64	0.51	1.36	0.51
18	1.64	1.11	0.49	1.36	0.49

---Stage 2---

s \ x:	1	2	3	4	Min
16	1.26	0.64	0.77	1.74	0.64
17	1.26	0.64	0.74	1.74	0.64
18	1.26	0.64	0.74	1.74	0.64
19	1.26	0.51	0.74	1.74	0.51

---Stage 1---

s \ x:	1	2	3	4	Min
20	0.89	0.64	1.11	2.11	0.64

2. ***(Deterministic) Equipment Replacement.*** The optimal policy for replacement of a machine over the next ten years is required. The cost of a new machine is $10,000. The table below indicates the annual operating cost of the machine, and the trade-in value, according to its age. The policy is to keep a machine for no more than six years. A new machine has just been purchased (whose cost should not be considered), and at the end of the ten-year planning period, a new machine is required.

Age of machine (yrs)	Operating cost/year ($)	Trade-in value ($)
0	1400	7500
1	1800	6000
2	2400	5000
3	3000	4200
4	3500	3500
5	4000	2500
6	4500	0

a. What is the total cost of the policy which replaces the machine at the end of year 5 and year 10?

Define the functions

g(n) = minimum total operating & replacement cost (including trade-in value) if, with n years remaining in the planning period, you have a new machine

y(n) = optimal age at which the machine is to be replaced if, with n years remaining in the planning period, you have a new machine.

Yrs to go (n)	*g(n)*	*y(n)*
10	25600	2, 3, or 4
9	21800	3
8	18400	2 or 3
7	15000	2, 3, or 4
6	11200	3
5	7800	2 or 3
4	4400	2 or 4
3	600	3
2	–2800	2
1	–6100	1
0	0	

For example, if n=2 years remain with a new machine, which is replaced at the end of two years, i.e., y(2)=2, the total cost is: operating cost: 1400+1800 = 3400

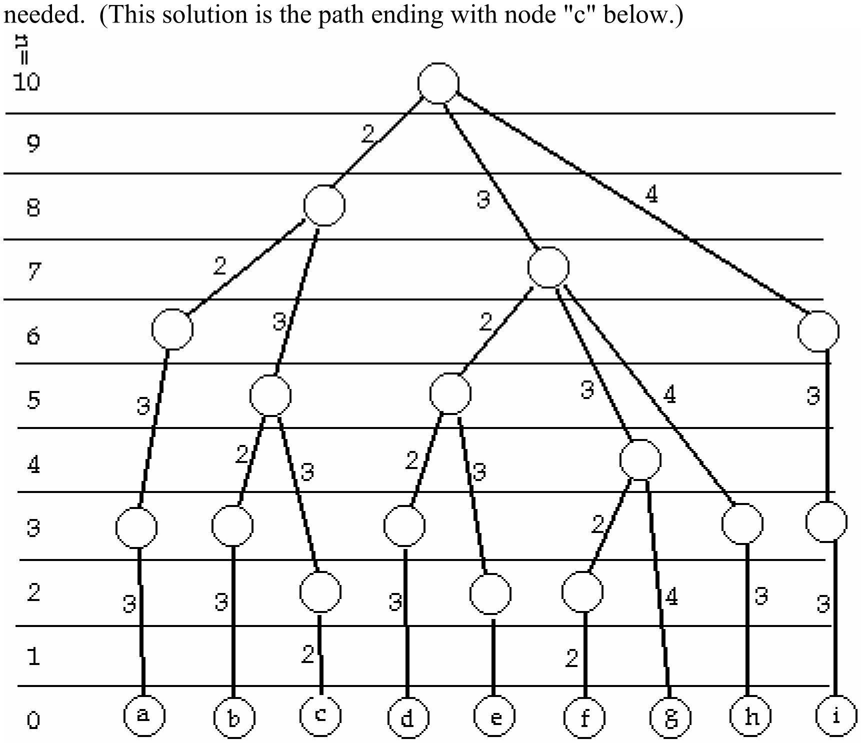
cost of new machine:	10000
trade-in value:	–2500
Total:	7200

b. Complete the computations in the table above. ***Solution***: see above.
For example, when n=8:

If y=	1	2	3	4	5	6
then OC=	1400	3200	5600	8600	12100	16100
and RC=	2500	4000	5000	5800	6500	7500
and g(8-y) =	15000	11200	7800	4400	600	–2800
So total=	18900	18400	18400	18800	19200	20800

c. If "now" is January 1, 2002, what are the optimal dates at which the machine should be replaced?
Solution: There are several (nine in all!) optimal policies:

For example, if we choose y(10)=2, the first replacement should be Jan. 1 of 2004, with 8 years remaining.
If we then choose y(8)=3, the next replacement will be Jan. 1, 2007, with 5 years remaining.
If we then choose y(5)=3, the next replacement will be Jan. 1, 2010, with 2 years remaining.
The only optimal choice then is y(2)=2, so that with 0 years remaining on Jan. 1, 2012, there is no replacement needed. (This solution is the path ending with node "c" below.)



❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖❖

3. ***Stochastic Production Planning.*** The state of the system is the "inventory position", which if positive is the stock on hand, and if negative is the number of backorders which have accumulated, which is observed at the end of the business day. The decision is the number of units to be produced, which is no more than 3. There is a setup cost of $10 if any units are produced, plus $4 per unit. We assume that production is completed in time to meet any demand that occurs the next day.
The demand is a discrete random variable with stationary distribution

D	0	1	2	3	4
P{D}	0.1	0.2	0.3	0.2	0.2

In addition, there is a storage cost of $1 per unit, based upon the end-of-day inventory, and a shortage cost of $15 per unit, based upon any backorders. Finally, at the end of the planning period (5 days), a salvage value of $2 per unit is received for any remaining inventory. If any shortage exists at the end of the planning period, enough must be produced to satisfy that shortage.
A backward recursion is used, where F(N) is a vector (one element per state) containing the minimum expected cost of the final N days of the planning period if the initial inventory position corresponds to that state.
Initially, with 5 days remaining (i.e., stage 5) there are 2 units in inventory.

Below are the tables used to compute the optimal production policy.

a. What is the missing value in the table for stage 1? _22.80_

Production cost (x=2): 10+2×4 = 18
Storage cost (s=1): 1
If demand = 0, final state is 3, so salvage value is -6
If demand = 1, final state is 2, so salvage value is -4
If demand = 2, final state is 1, so salvage value is -2
If demand = 3, final state is 0, so cost = 0
If demand = 4, final state is -1, i.e., shortage of 1 unit, so cost is 15 shortage cost + 14 production cost=29
Total expected cost: $19 + (0.1\times(-6)) + (0.2\times(-4)) + (0.3\times(-2)) + (0.2\times0) + (0.2\times29) = 19+3.8 = 22.8$

b. What is the missing value in the table for stage 5? _93.32_

Production cost (x=2): 10+2×4 = 18
Storage cost (s=2): 2
If demand = 0, final state is 4, and $f_4(4) = 58.01$
If demand = 1, final state is 3, and $f_4(3) = 65.84$
If demand = 2, final state is 2, and $f_4(2) = 71.10$
If demand = 3, final state is 1, and $f_4(1) = 77.01$
If demand = 4, final state is 0, and $f_4(0) = 88.09$
Total expected cost:
$20 + (0.1\times58.01) + (0.2\times65.84) + (0.3\times71.10) + (0.2\times77.01) + (0.2\times88.09)$
$= 20 + 73.32 = 93.32$

c. What is the optimal production decision at the initial stage (stage 5)? _produce 3_

d. What is the minimum expected cost (total of production, storage, and shortage costs) for the five-day period? __90.18__

e. Suppose that the demand in the first day (i.e., stage 5) is 1. What is the optimal production decision for day 2 (i.e., stage 4)? _0_ *The quantity available to satisfy the first day's demand is 2+3=5. If the demand is 1, then the second day is begun with stock-on-hand = 4, and according to the table for stage 4, the optimal production quantity is 0*
.

---Stage 1---

s \ x:	0	1	2	3	Minimum
¯3	9999.99	9999.99	9999.99	114.00	114.00
¯2	9999.99	9999.99	95.00	83.50	83.50
¯1	9999.99	76.00	64.50	51.60	51.60
0	47.00	45.50	32.60	25.80	25.80
1	32.50	29.60	22.80	19.40	19.40
2	16.60	19.80	16.40	18.40	16.40
3	6.80	13.40	15.40	17.40	6.80
4	0.40	12.40	14.40	16.60	0.40
5	¯0.60	11.40	13.60	16.20	¯0.60
6	¯1.60	10.60	13.20	16.40	¯1.60

---Stage 2---

s \ x:	0	1	2	3	Minimum
¯3	9999.99	9999.99	9999.99	150.55	150.55
¯2	9999.99	9999.99	131.55	114.08	114.08
¯1	9999.99	112.55	95.08	77.28	77.28
0	83.55	76.08	58.28	47.26	47.26
1	63.08	55.28	44.26	38.36	38.36
2	42.28	41.26	35.36	33.22	33.22
3	28.26	32.36	30.22	29.48	28.26
4	19.36	27.22	26.48	26.78	19.36
5	14.22	23.48	23.78	26.00	14.22
6	10.48	20.78	23.00	26.60	10.48

---Stage 3---

s \ x:	0	1	2	3	Minimum
¯3	9999.99	9999.99	9999.99	181.63	181.63
¯2	9999.99	9999.99	162.63	141.40	141.40
¯1	9999.99	143.63	122.40	100.44	100.44
0	114.63	103.40	81.44	67.89	67.89
1	90.40	78.44	64.89	57.68	57.68
2	65.44	61.89	54.68	52.09	52.09
3	48.89	51.68	49.09	47.00	47.00
4	38.68	46.09	44.00	42.93	38.68
5	33.09	41.00	39.93	40.00	33.09
6	28.00	36.93	37.00	39.23	28.00

---Stage 4---

s \ x:	0	1	2	3	Minimum
¯3	9999.99	9999.99	9999.99	208.95	208.95
¯2	9999.99	9999.99	189.95	166.08	166.08
¯1	9999.99	170.95	147.08	122.48	122.48
0	141.95	128.08	103.48	88.09	88.09
1	115.08	100.48	85.09	77.01	77.01
2	87.48	82.09	74.01	71.10	71.10
3	69.09	71.01	68.10	65.84	65.84
4	58.01	65.10	62.84	61.46	58.01
5	52.10	59.84	58.46	58.15	52.10
6	46.84	55.46	55.15	57.01	46.84

---Stage 5---

s \ x:	0	1	2	3	Minimum
2	108.65	102.02	93.32	90.18	90.18

56:171 Operations Research

Homework #12 Solutions – Fall 2001

1. **Quiz Show** A person has been invited to be a contestant on a TV quiz show, in which there are seven stages (1,...7). At any stage i, the contestant may choose to quit and receive her accumulated winnings.

If the person chooses to continue, she is presented with a question which, if correctly answered, allows her to advance to the next stage (i+1), but if not correctly answered, forces her to quit with no payoff, i.e., she loses everything.

The questions become progressively more difficult at each stage, of course, and she estimates that the probability that she can answer the question at stage i to be P[i] where P[i+1] < P[i].

If she correctly answers the sixth question, she receives her total winnings ($1+2+4+8+…+64= $127).

Question	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Prize	1	2	4	8	16	32	64
P{correct}	0.9	0.8	0.7	0.6	0.5	0.4	0.3

Having recently taken an O.R. course, she does an analysis using dynamic programming to determine her optimal strategy prior to appearing on the quiz show. She defines the optimal value function $f_n(s)$ = maximum expected future payoff if, at stage (question) n, she is in state s (s=1 for "active", 0 for "inactive"):

$$f_n(s) = \begin{cases} \sum_{i=1}^{n-1} R_i & \text{if s=1 "active" \& x=0 "quit"} \\ p_n f_{n+1}(1) + (1-p_n) f_{n+1}(0) & \text{if x=1 "continue"} \end{cases}$$

$$f_8(s) = 0$$

```
Stage 7
   s \ x: 1        0   | Maximum
   1  |  18.90  63.00|  63.00
   0  |   0.00   0.00|   0.00

Stage 6
   s \ x: 1        0   | Maximum
   1  |  25.20  31.00|  31.00
   0  |   0.00   0.00|   0.00

Stage 5
   s \ x: 1        0   | Maximum
   1  |  15.50  15.00|  15.50
   0  |   0.00   0.00|   0.00

Stage 4
   s \ x: 1        0   | Maximum
   1  |   9.30   7.00|   9.30
   0  |   0.00   0.00|   0.00

Stage 3
   s \ x: 1        0   | Maximum
   1  |   6.51   3.00|   6.51
   0  |   0.00   0.00|   0.00

Stage 2
   s \ x: 1        0   | Maximum
   1  |   5.21   1.00|   5.21
   0  |   0.00   0.00|   0.00

Stage 1
   s \ x: 1        0   | Maximum
   1  |   4.69   0.00|   4.69
   0  |   0.00   0.00|   0.00
```

a. Explain the contestant's optimal strategy: at what stage should she quit and keep her earnings?

Solution: She should decide to continue unless she reaches stage 6, i.e., the $32 question. At this point, she should take her accumulated earnings ($31) and go home, since her expected payoff would only be $25.20 if she were to continue.

b. Assume that she is motivated by economic values alone. A bus ticket to the studio of the TV station will cost her $5. Should she accept the invitation? (Explain.)

Solution: Her expected payoff at stage 1 is only $4.69, and so the optimal decision would be to decline the invitation to compete.

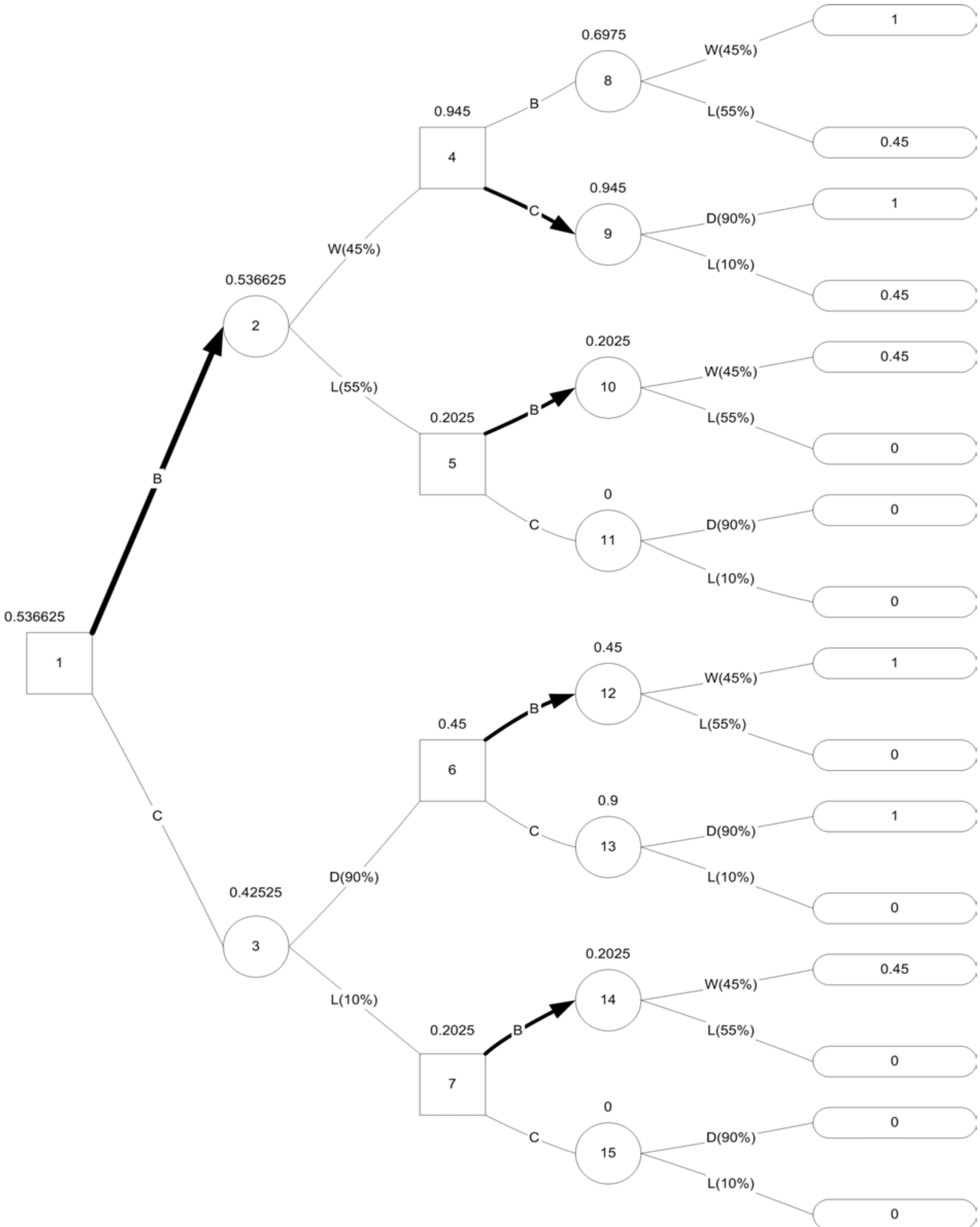
2. Vladimir Ulanowsky is playing Keith Smithson in a two-game **chess match**. Winning a game scores 1 match point, and drawing a game scores $^1/_2$ match point. After the two games are played, the player with more match points is declared the champion. If the two players are tied after two games, there is a "sudden death" playoff, i.e., they continue playing until someone wins a game (the winner of that game will then be the champion).

During each game, Ulanowsky can play one of two ways: **boldly** or **conservatively**. If he plays **boldly**, he has a 45% chance of winning the game and a 55% chance of losing the game. If he plays **conservatively**, he has a 90% chance of drawing the game and a 10% chance of losing the game. *Note that if the match enters a "sudden death" playoff, his obvious strategy is to play boldly at that time, since he has no chance to win otherwise.*

Ulanowsky's goal is to maximize his probability of winning the match. Use dynamic programming to help him accomplish this goal. (If this problem is solved correctly, even though Ulanowsky is the inferior player, his chance of winning the match is over $^1/_2$.)

Solution: A decision tree might be used to do this computation, and even though the amount of computation is slightly more than is required by DP, it is easier to understand. See the decision tree on the next page.

The result is that Ulanowsky should play boldly the first game. If he wins this first game, he should play conservatively the second game, but he should play boldly if he loses the first game. Using this strategy, he has a 53.6625% probability that he will win the match!



0.536625
1
B
C
0.536625
2
0.42525
3
W(45%)
L(55%)
D(90%)
L(10%)
0.945
4
0.2025
5
0.45
6
0.2025
7
B
C
B
C
B
C
B
C
0.6975
8
0.945
9
0.2025
10
0
11
0.45
12
0.9
13
0.2025
14
0
15
W(45%)
L(55%)
D(90%)
L(10%)
W(45%)
L(55%)
D(90%)
L(10%)
W(45%)
L(55%)
D(90%)
L(10%)
W(45%)
L(55%)
D(90%)
L(10%)
1
0.45
1
0.45
0.45
0
0
0
1
0
1
0
0.45
0
0
0

3. Casino Problem Consider the "Casino Problem" presented in the lectures, but with six plays of the game, and the goal being to accumulate at least six chips, beginning with 3 chips, where the probability of winning at each play of the game is 55%.

In the DP model with results presented below, the recursion is "forward", i.e., the stages range from n=1 (first play of the game) to n=6 (final play of the game). The state is the number of chips accumulated, and the decision is the number of chips to bet at the current play of the game.

a. Compute the missing number in the table for stage 1. __0.567__

Solution: Given that he has 3 chips and bets one of them, his maximum probability of accumulating 6 chips is

$$0.55 f_2(4) + 0.45 f_2(2) = 0.55 \times 0.72 + 0.45 \times 0.38 = 0.567$$

b. What is the probability that six chips can be accumulated at the end of six plays of the game? __57__%

c. How many chips should be bet at the first play of the game? __2__ (If more than one value is optimal, choose an answer arbitrarily.) ***Note:*** The print format would make it appear that the optimal decision is 2, but actually, 1 is equally optimal.

d. If one bets the amount you selected in (c) and the first play of the game is won, what should be the bet at the second play of the game?
Solution: If two chips are bet and he wins, he then has five chips, and the optimal decision at stage 2 for state 5 is 1 (or 0) .
If the first play of the game is lost, what should be the bet at the second play of the game?
Solution: If two chips are bet and he loses, he then has one chip, and the optimal decision at stage 2 for state 1 is __1__.

```
                ---Stage 6---
  s \ x:    1    2    3    4    5    6  | Max
 0  | 0.00 XXXXXXXXXXXXXXXXXXXXXXXXXXXXX| 0.00
 1  | 0.00 0.00 XXXXXXXXXXXXXXXXXXXXXXXX| 0.00
 2  | 0.00 0.00 0.00 XXXXXXXXXXXXXXXXXXX| 0.00
 3  | 0.00 0.00 0.00 0.55 XXXXXXXXXXXXXX| 0.55
 4  | 0.00 0.00 0.55 0.55 0.55 XXXXXXXXX| 0.55
 5  | 0.00 0.55 0.55 0.55 0.55 0.55 XXXX| 0.55
 6  | 1.00 0.55 0.55 0.55 0.55 0.55 0.55| 1.00

                ---Stage 5---
  s \ x:    1    2    3    4    5    6  | Max
 0  | 0.00 XXXXXXXXXXXXXXXXXXXXXXXXXXXXX| 0.00
 1  | 0.00 0.00 XXXXXXXXXXXXXXXXXXXXXXXX| 0.00
 2  | 0.00 0.30 0.30 XXXXXXXXXXXXXXXXXXX| 0.30
 3  | 0.55 0.30 0.30 0.55 XXXXXXXXXXXXXX| 0.55
 4  | 0.55 0.55 0.55 0.55 0.55 XXXXXXXXX| 0.55
 5  | 0.55 0.80 0.80 0.55 0.55 0.55 XXXX| 0.80
 6  | 1.00 0.80 0.80 0.80 0.55 0.55 0.55| 1.00

                ---Stage 4---
  s \ x:    1    2    3    4    5    6  | Max
 0  | 0.00 XXXXXXXXXXXXXXXXXXXXXXXXXXXXX| 0.00
 1  | 0.00 0.17 XXXXXXXXXXXXXXXXXXXXXXXX| 0.17
 2  | 0.30 0.30 0.30 XXXXXXXXXXXXXXXXXXX| 0.30
 3  | 0.55 0.44 0.44 0.55 XXXXXXXXXXXXXX| 0.55
 4  | 0.55 0.69 0.69 0.55 0.55 XXXXXXXXX| 0.69
 5  | 0.80 0.80 0.80 0.69 0.55 0.55 XXXX| 0.80
 6  | 1.00 0.91 0.80 0.80 0.69 0.55 0.55| 1.00

                ---Stage 3---
  s \ x:    1    2    3    4    5    6  | Max
 0  | 0.00 XXXXXXXXXXXXXXXXXXXXXXXXXXXXX| 0.00
```

1	0.17	0.17	XXXXXXXXXXXXXXXXXXXXXXXX					0.17
2	0.30	0.38	0.38	XXXXXXXXXXXXXXXXXXX				0.38
3	0.55	0.51	0.51	0.55	XXXXXXXXXXXXXX			0.55
4	0.69	0.69	0.69	0.62	0.55	XXXXXXXXX		0.69
5	0.80	0.86	0.80	0.69	0.62	0.55	XXXX	0.86
6	1.00	0.91	0.86	0.80	0.69	0.62	0.55	1.00

---Stage 2---

s \ x:	0	1	2	3	4	5	6	Max
0	0.00	XXXXXXXXXXXXXXXXXXXXXXXXXXXXX						0.00
1	0.17	0.21	XXXXXXXXXXXXXXXXXXXXXXXX					0.21
2	0.38	0.38	0.38	XXXXXXXXXXXXXXXXXXX				0.38
3	0.55	0.55	0.55	0.55	XXXXXXXXXXXXXX			0.55
4	0.69	0.72	0.72	0.62	0.55	XXXXXXXXX		0.72
5	0.86	0.86	0.80	0.72	0.62	0.55	XXXX	0.86
6	1.00	0.94	0.86	0.80	0.72	0.62	0.55	1.00

---Stage 1---

s \ x:	0	1	2	3	Max
3	0.55		0.57	0.55	0.57