56:171
Operations Research
Homework Fall 2000

Dennis L. Bricker
Dept. of Industrial Engineering
The University of Iowa

56:171 Operations Research
Homework #1 - Due Wednesday, August 30, 2000

In each case below, you must formulate a linear programming model that will solve the problem. Be sure to define the meaning of your variables! Then use LINDO (or other appropriate software) to find the optimal solution. State the optimal objective value, and describe in "layman's terms" the optimal decisions.

1. Walnut Orchard has two farms that grow wheat and corn. Because of differing soil conditions, there are differences in the yields and costs of growing crops on the two farms. The yields and costs are shown in the table below. Farm #1 has 100 acres available for cultivation, while Farm #2 has 150 acres. The farm has contracted to grow 11,000 bushels of corn and 6000 bushels of wheat. Determine a planting plan that will minimize the cost of meeting these contracts.

	Farm #1	Farm #2
Corn yield/acre	100 bushels	120 bushels
Cost/acre of corn	$90	$115
Wheat yield/acre	40 bushels	35 bushels
Cost/acre of wheat	$90	$80

Note: We are assuming that the costs and yields are known with certainty, which is not the case in the "real world"!

Solution:

Decision variables:

C1 = # of acres of Farm 1 planted in corn
W1 = # of acres of Farm 1 planted in wheat
C2 = # of acres of Farm 2 planted in corn
W2 = # of acres of Farm 2 planted in wheat

Constraints:

- Restrictions of the number of acres of each farm which are planted in crops.
  $C1 + W1 \leq 100$
  $C2 + W2 \leq 150$
- Restrictions of the minimum quantity of each crop.
  $100C1 + 120C2 \geq 11000$
  $40W1 + 35W2 \geq 6000$
- Nonnegativity constraint on each of the four variables.
  $C1 \geq 0$, $C2 \geq 0$, $W1 \geq 0$, $W2 \geq 0$

Objective:

```
Min 90 C1 + 115 C2 + 90 W1 + 80 W2
```

Complete LP formulation with solution :

```
MIN     90 C1 + 115 C2 + 90 W1 + 80 W2
SUBJECT TO
       2)   C1 + W1 <=   100
       3)   C2 + W2 <=   150
       4)   100 C1 + 120 C2 >=   11000
       5)   40 W1 + 35 W2 >=   6000
END
      OBJECTIVE FUNCTION VALUE
      1)      24096.15
```

```
  VARIABLE        VALUE          REDUCED COST
        C1         3.846154          0.000000
        C2        88.461540          0.000000
        W1        96.153847          0.000000
        W2        61.538460          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000         17.692308
        3)         0.000000         14.230769
        4)         0.000000         -1.076923
        5)         0.000000         -2.692308
```

That is, the optimal plan is to plant

- 3.85 acres of corn on farm #1 ,
- 88.46 acres of corn on farm #2 ,
- 96.15 acres of wheat on farm #1 and
- 61.54 acres of wheat on farm #2.

The total cost will be $24,096.15.

* * *

2. A firm manufactures chicken feed by mixing three different ingredients. Each ingredient contains four key nutrients: protein, fat, vitamin A, and vitamin B. The amount of each nutrient contained in 1 kilogram of the three basic ingredients is summarized in the table below:

Ingredient	Protein (grams)	Fat (grams)	Vitamin A (units)	Vitamin B (units)
1	25	11	235	12
2	45	10	160	6
3	32	7	190	10

The costs per kg of Ingredients 1, 2, and 3 are $0.55, $0.42, and $0.38, respectively. Each kg of the feed must contain at least 35 grams of protein, a minimum of 8 grams (and a maximum of 10 grams) of fat, at least 200 units of vitamin A and at least 10 units of vitamin B. Formulate an LP model for finding the feed mix that has the minimum cost per kg.
--*revised 8/28/00*

Solution:

Decision variables:
X1 = kg. of Ingredient 1 included in mixture
X2 = kg. of Ingredient 2 included in mixture
X3 = kg. of Ingredient 3 included in mixture

Complete LP Formulation with solution :

```
  MIN     Z = 0.55 X1 + 0.42 X2 + 0.38 X3
  s.t.
         25 X1 +  45 X2 +  32 X3 >=    35  (Protein constraint)
         11 X1 +  10 X2 +   7 X3 >=     8  (Fat constraint)
         11 X1 +  10 X2 +   7 X3 <=    10  (Fat constraint)
        235 X1 + 160 X2 + 190 X3 >=   200  (Vitamin A constraint)
```

```
         12 X1 +   6 X2 +  10 X3 >=    10  (Vitamin B constraint)
            X1 +     X2 +     X3  =     1  (total weight of mixture)
  END
```

According to LINDO, this LP is infeasible! If we modify the problem statement so that we require the mixture contain 35 grams of protein, etc., and discard the last constraint above, we obtain the solution below, in which the total weight of the mixture is approximately 1.077 kg.

```
        OBJECTIVE FUNCTION VALUE
        1)     0.4153846

  VARIABLE        VALUE          REDUCED COST
        X1         0.000000          0.022949
        X2         0.153846          0.000000
        X3         0.923077          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         1.461538          0.000000
        3)         0.000000         -0.024359
        4)         2.000000          0.000000
        5)         0.000000         -0.001103
        6)         0.153846          0.000000
```

The optimal solution to this LP is X1 = 0, X2 = 0.154, X3 = 0.923, Z = 0.415. Thus, the minimum cost mixture costs $0.415 and includes 0.154 kg of Ingredient 2 and 0.923 kg of Ingredient 3.

* * *

3. "Mama's Kitchen" serves from 5:30 a.m. each morning until 1:30 p.m. in the afternoon. Tables are set and cleared by busers working 4-hour shifts beginning on the hour from 5:00 a.m. through 10:00 a.m. Most are college students who hate to get up in the morning, so Mama's pays $9 per hour for the 5:00, 6:00, and 7:00 a.m. shifts, and $7.50 per hour for the others. (That is, a person works a shift consisting of 4 consecutive hours, with the wages equal to 4x$9 for the three early shifts, and 4x$7.50 for the 3 later shifts.) The manager seeks a minimum cost staffing plan that will have at least the number of busers on duty each hour as specified below:

	5 am	6 am	7 am	8 am	9 am	10am	11am	Noon	1 pm
#reqd	2	3	5	5	3	2	4	6	3

Solution:

Decision variables:

Xi = the # of employees who start to work on i^{th} shift. (i = 1, 2, ... , 6)

Complete LP Formulation with solution :

```
  MIN 36 X1 + 36 X2 + 36 X3 + 30 X4 + 30 X5 + 30 X6
  SUBJECT TO
   X1                          >=    2  (Restriction of  # of busers on duty at  5am)
   X1 + X2                     >=    3  (Restriction of  # of busers on duty at  6am)
   X1 + X2 + X3                >=    5  (Restriction of  # of busers on duty at  7am)
   X1 + X2 + X3 + X4           >=    5  (Restriction of  # of busers on duty at  8am)
        X2 + X3 + X4 + X5      >=    3  (Restriction of  # of busers on duty at  9am)
             X3 + X4 + X5 + X6 >=    2  (Restriction of  # of busers on duty at 10am)
```

```
                  X4 + X5 + X6 >=    4 (Restriction of  # of busers on duty at 11am)
                       X5 + X6 >=    6 (Restriction of  # of busers on duty at 12pm)
                            X6 >=    3 (Restriction of  # of busers on duty at   1pm)
   Xi >= 0  (for i = 1,2,3,4,5,6)      (Sign restrictions)

        OBJECTIVE FUNCTION VALUE
        1)      360.0000

  VARIABLE        VALUE          REDUCED COST
        X1         3.000000           0.000000
        X2         0.000000           0.000000
        X3         2.000000           0.000000
        X4         0.000000           0.000000
        X5         3.000000           0.000000
        X6         3.000000           0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         1.000000           0.000000
        3)         0.000000           0.000000
        4)         0.000000          -6.000000
        5)         0.000000         -30.000000
        6)         2.000000           0.000000
        7)         6.000000           0.000000
        8)         2.000000           0.000000
        9)         0.000000         -30.000000
       10)         0.000000           0.000000
```

That is, the optimal staffing plan is to employ
3 busers for the 1st shift(4-hour shift which begins at 5:00a.m.),
2 busers for the 3rd shift(4-hour shift which begins at 7:00a.m.),
3 busers for the 5th shift(4-hour shift which begins at 9:00a.m.), and
3 busers for the 6th shift(4-hour shift which begins at 10:00a.m.).
The total labor cost will be $360/day.

56:171 Operations Research
Homework #3 Solutions -- Fall 2000

1. Simplex Algorithm: Use the simplex algorithm to find the optimal solution to the following LP:

$$\text{Maximize } z = 4x_1 + x_2$$

$$\text{subject to } \begin{cases} 2x_1 + x_2 \le 9 \\ x_2 \le 5 \\ x_1 - x_2 \le 4 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

Show the initial tableau, each intermediate tableau, and the final tableau. Explain how you have decided on the location of each pivot and how you have decided to stop at the final tableau.

Solution:

After adding slack variables X3, X4, and X5 to the three constraints, we obtain the initial tableau as follows :

	-Z	X1	X2	X3	X4	X5	RHS
	1	4	1	0	0	0	0
X3	0	2	1	1	0	0	9
X4	0	0	1	0	1	0	5
X5	0	***1***	-1	0	0	1	4

Entering variable : X1 ; Leaving variable : X5

Either X1 or X2 might be selected to enter the basis-- both have positive relative profits in row 0. Because it has the larger relative profit, we here enter X1 into the basis. The minimum ratio test $\left(i.e., \quad Min\left\{ \frac{9}{2} \quad , \quad \frac{4}{1} \right\} = 4 \right)$ indicates that the pivot should be in the bottom row, i.e., X5 should leave the basis. The resulting tableau is shown below :

	-Z	X1	X2	X3	X4	X5	X6
	1	0	5	0	0	-4	-16
X3	0	0	***3***	1	0	-2	1
X4	0	0	1	0	1	0	5
X1	0	1	-1	0	0	1	4

Entering variable : X2; Leaving variable : X3

Since X2 is the only variable with a positive relative profit in row0, we enter X2 into the basis. The minimum ratio test $\left(i.e., \quad Min\left\{ \frac{1}{3} \quad , \quad \frac{5}{1} \right\} = \frac{1}{3} \right)$ indicates that X3 should leave the basis, i.e., the pivot should be in row 1. The resulting tableau is shown below :

	-Z	X1	X2	X3	X4	X5	X6
	1	0	0	-1.67	0	-0.67	-17.7
X2	0	0	1	0.333	0	-0.67	0.333
X4	0	0	0	-0.33	1	0.667	4.667
X1	0	1	0	0.333	0	0.333	4.333

Since each variable has a nonpositive relative profit in row 0, this is an optimal tableau. Thus, the optimal solution to LP is

Z = 17.7 , X2 = 0.333 , X4 = 4.667 , X1 = 4.333 , X3 = X5 = 0.

2. Below are several ***simplex tableaus***. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through G, according to the descriptions below. *Also answer the question accompanying each classification, if any.*

(A) Nonoptimal, nondegenerate tableau with bounded solution. *Circle a pivot element which would improve the objective.*

(B) Nonoptimal, degenerate tableau with bounded solution. *Circle an appropriate pivot element. Would the objective improve with this pivot?*

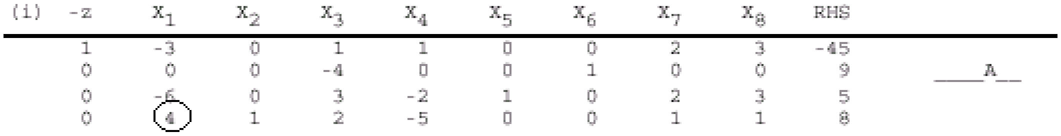
(C) Unique nondegenerate optimum.

(D) Optimal tableau, with alternate optimum. State the values of the basic variables. Circle a pivot element which would lead to another optimal basic solution. Which variable will enter the basis, and at what value?

(E) Objective unbounded (below). *Specify a variable which, when going to infinity, will make the objective arbitrarily low.*

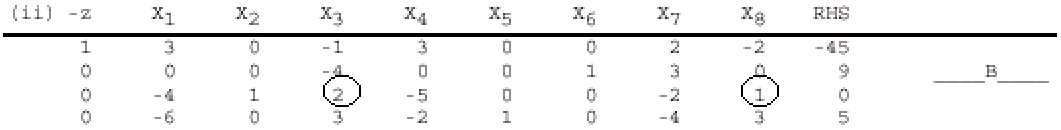
(F) Tableau with infeasible basic solution.

Warning: Some of these classifications might be used for more than one tableau, while others might not be used at all!



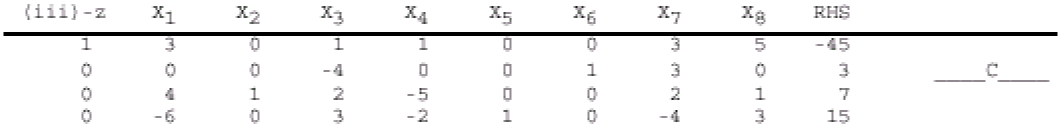
(i) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	-3	0	1	1	0	0	2	3	-45
0	0	0	-4	0	0	1	0	0	9
0	-6	0	3	-2	1	0	2	3	5
0	4	1	2	-5	0	0	1	1	8

____A__



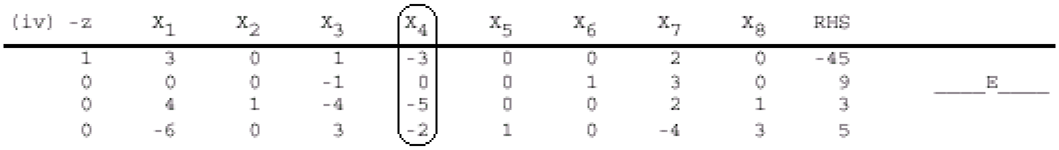
(ii) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	-1	3	0	0	2	-2	-45
0	0	0	-4	0	0	1	3	0	9
0	-4	1	2	-5	0	0	-2	1	0
0	-6	0	3	-2	1	0	-4	3	5

____B____



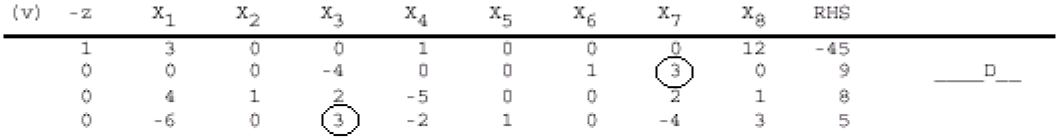
(iii)-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	1	1	0	0	3	5	-45
0	0	0	-4	0	0	1	3	0	3
0	4	1	2	-5	0	0	2	1	7
0	-6	0	3	-2	1	0	-4	3	15

____C____



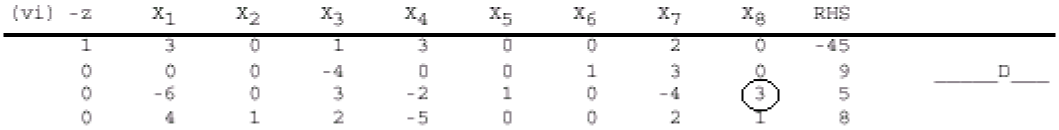
(iv) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	1	-3	0	0	2	0	-45
0	0	0	-1	0	0	1	3	0	9
0	4	1	-4	-5	0	0	2	1	3
0	-6	0	3	-2	1	0	-4	3	5

____E____



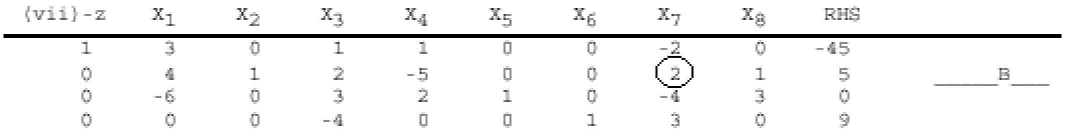
(v) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	0	1	0	0	0	12	-45
0	0	0	-4	0	0	1	3	0	9
0	4	1	2	-5	0	0	2	1	8
0	-6	0	3	-2	1	0	-4	3	5

____D__



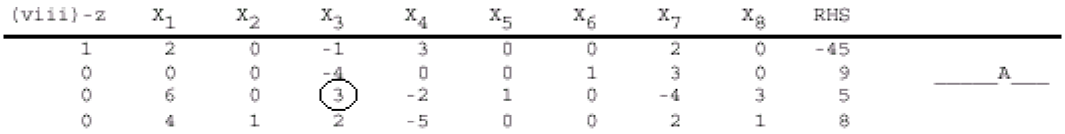
(vi) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	1	3	0	0	2	0	-45
0	0	0	-4	0	0	1	3	0	9
0	-6	0	3	-2	1	0	-4	3	5
0	4	1	2	-5	0	0	2	1	8

_____D___



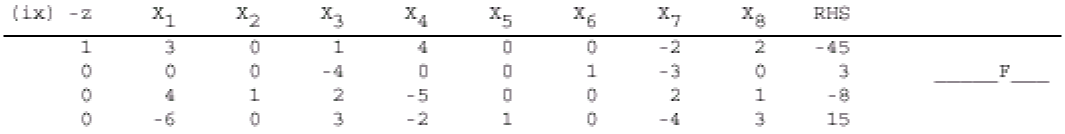
(vii)-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	1	1	0	0	-2	0	-45
0	4	1	2	-5	0	0	2	1	5
0	-6	0	3	2	1	0	-4	3	0
0	0	0	-4	0	0	1	3	0	9

_____B___



(viii)-z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	2	0	-1	3	0	0	2	0	-45
0	0	0	-4	0	0	1	3	0	9
0	6	0	3	-2	1	0	-4	3	5
0	4	1	2	-5	0	0	2	1	8

_____A___



(ix) -z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	3	0	1	4	0	0	-2	2	-45
0	0	0	-4	0	0	1	-3	0	3
0	4	1	2	-5	0	0	2	1	-8
0	-6	0	3	-2	1	0	-4	3	15

_____F___

Note: in (ii) and (v), one of the two pivots indicated might be selected.

3. ***LP Model Formulation*** (from *Operations Research*, by W. Winston (3 rd edition), page 191): Carco uses robots to manufacture cars. The following demands for cars must be met (not necessarily on time, but all demands must be met by end of quarter 4):

At the beginning of the first quarter, Carco has two robots. Robots can be purchased at the beginning of each quarter, but a maximum of two per quarter can be purchased. Each robot can build up to 200 cars per quarter. It costs $5000 to purchase a robot. Each quarter, a robot incurs $500 in maintenance costs (even if it is not being used to build any cars). Robots can also be sold at the beginning of each quarter for $3000. At the end of each quarter, a holding cost of $200 for each car in inventory is incurred. If any demand is backlogged, a cost of $300 per car is incurred for each quarter the customer must wait. At the end of quarter 4, Carco must have at least two robots.

Quarter #	1	2	3	4
Demand	600	800	500	400

a. Formulate an LP to minimize the total cost incurred in meeting the next four quarters' demands for cars. Be sure to define your variables (including units) clearly! (Ignore any integer restrictions.)

Solution:

Decision Variables :

Rt : robots available during quarter t (after robots are bought or sold for the quarter)
Bt : robots bought during quarter t
St : robots sold during quarter t
It : cars in inventory at end of quarter t
Ct : cars produced during quarter t
Dt : backlogged demand for cars at end of quarter t

LP formulation :

```
MIN    500( R1 + R2 + R3 + R4 ) +  200( I1 + I2 + I3 + I4 )
     + 5000( B1 + B2 + B3 + B4 ) – 3000( S1 + S2 + S3 + S4 )
     +  300( D1 + D2 + D3 + D4 )

s.t.
           R1 =  2 + B1 - S1
           R2 = R1 + B2 - S2
           R3 = R2 + B3 - S3
           R4 = R3 + B4 - S4
           I1 - D1 = C1 - 600
           I2 – D2 = I1 - D1 + C2 - 800
           I3 – D3 = I2 - D2 + C3 - 500
           I4 – D4 = I3 - D3 + C4 - 400
           R4 >=   2
           C1 <= 200 R1
           C2 <= 200 R2
           C3 <= 200 R3
           C4 <= 200 R4
           D4 =    0
           B1 <=   2
           B2 <=   2
           B3 <=   2
           B4 <=   2
         All variables >= 0
```

b. Use *LINDO* (or other LP solver) to find the optimal solution and describe it briefly in "plain English". Are integer numbers of robots bought & sold?

Solution:

The formulation above is not in a form to be entered directly into LINDO, which requires that all variables appear on the left of equations or inequalities, and which doesn't recognize the parentheses in the objective function. Thus:

```
MIN    500 R1 + 500 R2 + 500 R3 + 500 R4 + 200 I1 + 200 I2 + 200 I3
     + 200 I4 + 5000 B1 + 5000 B2 + 5000 B3 + 5000 B4 - 3000 S1 - 3000 S2
     - 3000 S3 - 3000 S4 + 300 D1 + 300 D2 + 300 D3 + 300 D4
SUBJECTTO
       2)   R1 - B1 + S1 =    2
       3) - R1 + R2 - B2 + S2 =    0
       4) - R2 + R3 - B3 + S3 =    0
       5) - R3 + R4 - B4 + S4 =    0
```

```
         6)   I1 - D1 - C1 =  - 600
         7) - I1 + I2 + D1 - D2 - C2 =  - 800
         8) - I2 + I3 + D2 - D3 - C3 =  - 500
         9) - I3 + I4 + D3 - D4 - C4 =  - 400
        10)   R4 >=   2
        11) - 200 R1 + C1 <=   0
        12) - 200 R2 + C2 <=   0
        13) - 200 R3 + C3 <=   0
        14) - 200 R4 + C4 <=   0
        15)   D4 =    0
        16)   B1 <=   2
        17)   B2 <=   2
        18)   B3 <=   2
        19)   B4 <=   2
  END

        OBJECTIVEFUNCTIONVALUE
        1)     9750.000

  VARIABLE       VALUE        REDUCED COST
        R1        3.000000        0.000000
        R2        4.000000        0.000000
        R3        2.500000        0.000000
        R4        2.000000        0.000000
        I1        0.000000      190.000000
        I2        0.000000      210.000000
        I3        0.000000      202.500000
        I4        0.000000      200.000000
        B1        1.000000        0.000000
        B2        1.000000        0.000000
        B3        0.000000     2000.000000
        B4        0.000000     2000.000000
        S1        0.000000     2000.000000
        S2        0.000000     2000.000000
        S3        1.500000        0.000000
        S4        0.500000        0.000000
        D1        0.000000      310.000000
        D2        0.000000      290.000000
        D3        0.000000      297.500000
        D4        0.000000        0.000000
        C1      600.000000        0.000000
        C2      800.000000        0.000000
        C3      500.000000        0.000000
        C4      400.000000        0.000000

       ROW   SLACK OR SURPLUS    DUAL PRICES
        2)        0.000000     5000.000000
        3)        0.000000     5000.000000
        4)        0.000000     3000.000000
        5)        0.000000     3000.000000
        6)        0.000000        2.500000
        7)        0.000000       12.500000
        8)        0.000000        2.500000
        9)        0.000000        0.000000
       10)        0.000000    -3500.000000
       11)        0.000000        2.500000
       12)        0.000000       12.500000
       13)        0.000000        2.500000
       14)        0.000000        0.000000
       15)        0.000000     -300.000000
       16)        1.000000        0.000000
       17)        1.000000        0.000000
       18)        2.000000        0.000000
       19)        2.000000        0.000000
```

Note that rows 16-19 could have been omitted, and the upper bounds instead could have been imposed by the commands

```
SUB  B1 2.0
SUB  B2 2.0
SUB  B3 2.0
SUB  B4 2.0
```

This would reduce the size of the basis and therefore save in computation by LINDO, while yielding the same solution. (Furthermore, the output of RANGE (sensitivity analysis) to be studied next would be more meaningful.)

56:171 Operations Research
Homework #4 Solution -- Fall 2000

1. LP Duality: Write the dual of the following LP:

$$Min \quad 3x_1 + 2x_2 - 4x_3$$

$$subject \quad to \quad \begin{cases} 5x_1 - 7x_2 + x_3 \geq 12 \\ x_1 - x_2 + 2x_3 = 18 \\ 2x_1 - x_3 \leq 6 \\ x_2 + 2x_3 \geq 10 \\ x_j \geq 0, \; j = 1,2,3 \end{cases}$$

Solution: Consult the following table (from the class notes):

Maximize	Minimize
Type of constraint i: ≤ = ≥	Sign of variable i: nonnegative unrestricted in sign nonpositive
Sign of variable j: nonnegative unrestricted in sign nonpositive	Type of constraint i: ≥ = ≤

According to the relationships in this table, the dual problem is

$$\text{Max } 12y_1 + 18y_2 + 6y_3 + 10y_4$$

$$\text{subject to} \begin{cases} 5y_1 + y_2 + 2y_3 \leq 3 \\ -7y_1 - y_2 + y_4 \leq 2 \\ y_1 + 2y_2 - y_3 + 2y_4 \leq -4 \\ y_1 \geq 0, \; y_2 \text{ urs}, \; y_3 \leq 0, \; y_4 \geq 0 \end{cases}$$

2. Consider the following primal LP problem:

$$Max \quad x_1 + 2x_2 - 9x_3 + 8x_4 - 36x_5$$

$$subject \quad to \quad \begin{cases} 2x_2 - x_3 + x_4 - 3x_5 \leq 40 \\ x_1 - x_2 + 2x_4 - 2x_5 \leq 10 \\ x_j \geq 0, \; j = 1,2,3,4,5 \end{cases}$$

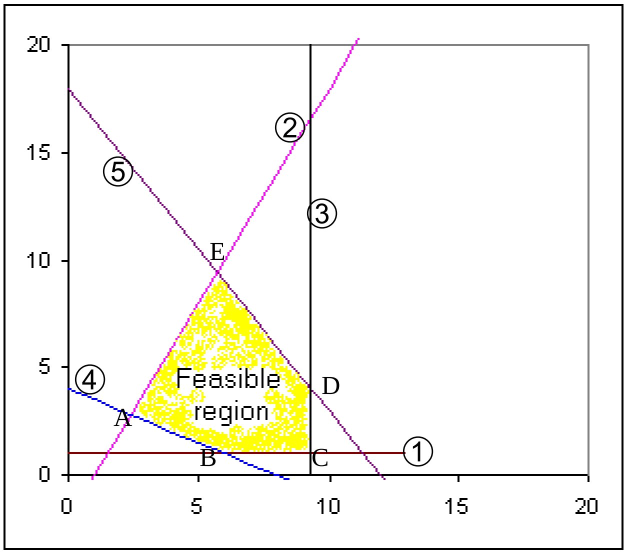
a. Write the dual LP problem

$$Min \quad 40Y_1 + 10Y_2$$

$$subject \quad to \quad \begin{cases} Y_2 \geq 1 & \cdots\cdots\cdot \quad (1) \\ 2Y_1 - Y_2 \geq 2 & \cdots\cdots\cdot \quad (2) \\ -Y_1 \geq -9 & \cdots\cdots\cdot \quad (3) \\ Y_1 + 2Y_2 \geq 8 & \cdots\cdots\cdot \quad (4) \\ -3Y_1 - 2Y_2 \geq -36 & \cdots\cdots\cdot \quad (5) \\ Y_j \geq 0, \; j = 1,2 \end{cases}$$

b. Sketch the feasible region of the dual LP in 2 dimensions, and use it to find the optimal solution.

Solution:



Corner point	Y_1	Y_2	Cost
A	2.4	2.8	124
B	6	0	240
C	9	1	370
D	9	4.5	405
E	40/7	66/7	2260/7 ≈ 322.86

The optimal solution is therefore at point A = (2.4, 2.8).

c. Using complementary slackness conditions,

- write equations which must be satisfied by the optimal primal solution x*
- which primal variables must be zero?

Solution:

At extreme Point A

$Y_1 > 0 \Rightarrow$ first primal constraint is tight, i.e., $2x_2 - x_3 + x_4 - 3x_5 = 40$

$Y_2 > 0 \Rightarrow$ second primal constraint is tight, i.e., $x_1 - x_2 + 2x_4 - 2x_5 = 10$

Dual constraints #1, #3, and #5 are slack

$\Rightarrow$ corresponding variables of the primal problem, X_1, X_3, and X_5 are zero.

d. Using the information in (c.), determine the optimal solution x*.

Solution:

Substituting zero for x_1, x_3, and x_5 in the 2 equations above yields 2 equations with 2 variables:

$$\begin{cases} 2x_2 + x_4 = 40 \\ -x_2 + 2x_4 = 10 \end{cases}$$

which has the solution $x_2 = 14$, $x_4 = 12$. Thus the optimal primal solution is:

$$x_1 = x_3 = x_5 = 0, x_2 = 14, x_4 = 12$$

3. ***Sensitivity Analysis*** (based on LP model Homework #3 from *Operations Research*, by W. Winston (3rd edition), page 191): Carco uses robots to manufacture cars. The following demands for cars must be met (not necessarily on time, but all demands must be met by end of quarter 4):

Quarter #	1	2	3	4
Demand	600	800	500	400

At the beginning of the first quarter, Carco has two robots. Robots can be purchased at the beginning of each quarter, but a maximum of two per quarter can be purchased. Each robot can build up to 200 cars per quarter. It costs $5000 to purchase a robot. Each quarter, a robot incurs $500 in maintenance costs (even if it is not being used to build any cars). Robots can also be sold at the beginning of each quarter for $3000. At the end of each quarter, a holding cost of $200 for each car in inventory is incurred. If any demand is backlogged, a cost of $300 per car is incurred for each quarter the customer must wait. At the end of quarter 4, Carco must have at least two robots.

Decision Variables :

Rt : robots available during quarter t (after robots are bought or sold for the quarter)
Bt : robots bought during quarter t
St : robots sold during quarter t
It : cars in inventory at end of quarter t
Ct : cars produced during quarter t
Dt : backlogged demand for cars at end of quarter t

Using the LINDO output below, answer the following questions:

a. During the first quarter, a one-time offer of 20% discount on robots is offered. Will this change the optimal solution shown below?

Solution:

A 20% discount would mean a $1000 decrease in the cost of variable R1. This exceeds the ALLOWABLE DECREASE ($500), and so the optimal basis will change.

b. In the optimal solution, is any demand backlogged?

Solution:

No, there is no demand backlogged in the current optimal solution, i.e., D1=D2=D3=D4=0.

c. Suppose that the penalty for backlogging demand is $250 per month instead of $300. Will this change the optimal solution? Note: this change applies to all quarters simultaneously!

Solution:

Decreases of $50 is within the ALLOWABLE DECREASE in each of the objective coefficients of D1, D2, D3, and D4. However, since four costs are changed simultaneously, we must apply the "100% Rule":

Variable	ALLOWABLE DECREASE	% of ALLOWABLE DECREASE
D1	310	50/310 = 16.13%
D2	290	50/290 = 17.24%
D3	297	50/297 = 16.84%
D4	300	50/300 = 16.67%

The sum of the changes as percents of ALLOWABLE DECREASE is 66.87% < 100%, indicating that the basis will not be changed. Also, the objective function value will remain the same, since the variables whose costs are changing are all zero.

d. If the demand in quarter #3 were to increase by 100 cars, what would be the change in the objective function?

Solution:

- Dual Price for row (8) is 2.5 ($/car).
- The increase of 100 cars would change the right-hand-side of row 8 from –800 to –900, i.e., a ***decrease*** of 100.
- This is less than the ALLOWABLE DECREASE in the range of RHS of row (8), which is 300, so the dual price ($2.5/car) remains valid for the entire increase.

- According to LINDO's definition, the *dual price* of a constraint is "the rate at which the objective function will improve as the right-hand-side is increased by a small amount." Since we are minimizing, an improvement corresponds to a *decrease* in cost. Therefore, an increase in the right-hand-side would lower the cost, and conversely a decrease in the right-hand-side would increase the cost.
- The objective function value (cost) will be worsen, i.e., increase, by $250 (= 2.5*100)

e. Suppose that we know in advance that demand for 10 cars must be backlogged in quarter #2. Using the substitution rates found in the tableau, describe how this would change the optimal solution.

Solution: Variable D2 is nonbasic in the optimal solution. The change in a basic variable in the optimal solution is given by the "substitution rate" found in the optimal tableau. For example, the substitution rate of D2 for R2 is 0.005, indicating that each unit of D2 "substitutes for" or replaces 0.005 units of R2 in the solution. Hence R2 would decrease by 10×0.005 = 0.05, from 4 to 3.95.

The substitution rate of D2 for R3, on the other hand, is negative (–0.005), and so each unit increase in D2 will increase R3 by 0.005 units, i.e., from 2.5 to 2.5+10×0.005=2.55. Other changes are given below.

$$\begin{array}{rl} 1 & ART \\ 2 & R1 \\ 3 & R2 \\ 4 & S3 \\ 5 & R3 \\ 6 & B1 \\ 7 & B2 \\ 8 & S4 \\ 9 & ART \\ 10 & SLK10 \\ 11 & C1 \\ 12 & C2 \\ 13 & C3 \\ 14 & C4 \\ 15 & ART \end{array} = \begin{bmatrix} -9750 \\ 3 \\ 4 \\ 1.5 \\ 2.5 \\ 1 \\ 1 \\ 0.5 \\ 0 \\ 0 \\ 600 \\ 800 \\ 500 \\ 400 \\ 0 \end{bmatrix} - \begin{bmatrix} 290 \\ 0 \\ 0.005 \\ 0.01 \\ -0.005 \\ 0 \\ 0.005 \\ -0.005 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \times (10) = \begin{bmatrix} -12650 \\ 3 \\ 3.95 \\ 1.4 \\ 2.55 \\ 1 \\ 0.95 \\ 0.55 \\ 0 \\ 0 \\ 600 \\ 790 \\ 510 \\ 400 \\ 0 \end{bmatrix}$$

```
MIN     500 R1 + 500 R2 + 500 R3 + 500 R4 + 200 I1 + 200 I2 + 200 I3
       + 200 I4 + 5000 B1 + 5000 B2 + 5000 B3 + 5000 B4 - 3000 S1 - 3000 S2
       - 3000 S3 - 3000 S4 + 300 D1 + 300 D2 + 300 D3 + 300 D4
  SUBJECTTO
         2)   R1 - B1 + S1 =    2
         3) - R1 + R2 - B2 + S2 =    0
         4) - R2 + R3 - B3 + S3 =    0
         5) - R3 + R4 - B4 + S4 =    0
         6)   I1 - D1 - C1 =  - 600
         7) - I1 + I2 + D1 - D2 - C2 =  - 800
         8) - I2 + I3 + D2 - D3 - C3 =  - 500
         9) - I3 + I4 + D3 - D4 - C4 =  - 400
        10)   R4 >=   2
        11) - 200 R1 + C1 <=   0
        12) - 200 R2 + C2 <=   0
        13) - 200 R3 + C3 <=   0
        14) - 200 R4 + C4 <=   0
```

```
      15)   D4 =    0
 END
 SLB      R4       2.00000
 SUB      B1       2.00000
 SUB      B2       2.00000
 SUB      B3       2.00000
 SUB      B4       2.00000

      OBJECTIVEFUNCTIONVALUE

      1)     9750.000

 VARIABLE       VALUE        REDUCED COST
       R1        3.000000         0.000000
       R2        4.000000         0.000000
       R3        2.500000         0.000000
       R4        2.000000      3500.000000
       I1        0.000000       190.000000
       I2        0.000000       210.000000
       I3        0.000000       202.500000
       I4        0.000000       200.000000
       B1        1.000000         0.000000
       B2        1.000000         0.000000
       B3        0.000000      2000.000000
       B4        0.000000      2000.000000
       S1        0.000000      2000.000000
       S2        0.000000      2000.000000
       S3        1.500000         0.000000
       S4        0.500000         0.000000
       D1        0.000000       310.000000
       D2        0.000000       290.000000
       D3        0.000000       297.500000
       D4        0.000000       300.000000
       C1      600.000000         0.000000
       C2      800.000000         0.000000
       C3      500.000000         0.000000
       C4      400.000000         0.000000

      ROW  SLACK OR SURPLUS    DUAL PRICES
       2)        0.000000      5000.000000
       3)        0.000000      5000.000000
       4)        0.000000      3000.000000
       5)        0.000000      3000.000000
       6)        0.000000         2.500000
       7)        0.000000        12.500000
       8)        0.000000         2.500000
       9)        0.000000         0.000000
      10)        0.000000         0.000000
      11)        0.000000         2.500000
      12)        0.000000        12.500000
      13)        0.000000         2.500000
      14)        0.000000         0.000000
      15)        0.000000         0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJCOEFFICIENTRANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
      R1     500.000000    62000.000000       500.000000
      R2     500.000000    38000.000000      2500.000000
      R3     500.000000    42000.000000       500.000000
      R4     500.000000        INFINITY      3500.000000
      I1     200.000000        INFINITY       190.000000
      I2     200.000000        INFINITY       210.000000
      I3     200.000000        INFINITY       202.500000
      I4     200.000000        INFINITY       200.000000
      B1    5000.000000    62000.000000       500.000000
      B2    5000.000000      500.000000      2000.000000
      B3    5000.000000        INFINITY      2000.000000
      B4    5000.000000        INFINITY      2000.000000
      S1   -3000.000000        INFINITY      2000.000000
      S2   -3000.000000        INFINITY      2000.000000
```

```
      S3   -3000.000000      500.000000     2000.000000
      S4   -3000.000000     3500.000000      500.000000
      D1     300.000000        INFINITY      310.000000
      D2     300.000000        INFINITY      290.000000
      D3     300.000000        INFINITY      297.500000
      D4     300.000000        INFINITY      300.000000
      C1       0.000000      310.000000      190.000000
      C2       0.000000      190.000000      210.000000
      C3       0.000000      210.000000      202.500000
      C4       0.000000        0.000000       17.500000

                         RIGHTHANDSIDERANGES
     ROW        CURRENT       ALLOWABLE        ALLOWABLE
                  RHS          INCREASE        DECREASE
       2       2.000000        1.000000         1.000000
       3       0.000000        1.000000         1.000000
       4       0.000000        INFINITY         1.500000
       5       0.000000        INFINITY         0.500000
       6    -600.000000      200.000000       200.000000
       7    -800.000000      200.000000       200.000000
       8    -500.000000      100.000000       300.000000
       9    -400.000000        0.000000         0.000000
      10       2.000000        0.000000         INFINITY
      11       0.000000      200.000000       200.000000
      12       0.000000      200.000000       200.000000
      13       0.000000      100.000000       300.000000
      14       0.000000        0.000000         0.000000
      15       0.000000        0.000000         0.000000


 THETABLEAU

     ROW  (BASIS)        R1        R2        R3        R4        I1        I2
       1 ART          0.000     0.000     0.000  3500.000   190.000   210.000
       2      R1      1.000     0.000     0.000     0.000    -0.005     0.000
       3      R2      0.000     1.000     0.000     0.000     0.005    -0.005
       4      S3      0.000     0.000     0.000     0.000     0.005    -0.010
       5      R3      0.000     0.000     1.000     0.000     0.000     0.005
       6      B1      0.000     0.000     0.000     0.000    -0.005     0.000
       7      B2      0.000     0.000     0.000     0.000     0.010    -0.005
       8      S4      0.000     0.000     0.000     1.000     0.000     0.005
       9 ART          0.000     0.000     0.000  -200.000     0.000     0.000
      10 SLK   10     0.000     0.000     0.000    -1.000     0.000     0.000
      11      C1      0.000     0.000     0.000     0.000    -1.000     0.000
      12      C2      0.000     0.000     0.000     0.000     1.000    -1.000
      13      C3      0.000     0.000     0.000     0.000     0.000     1.000
      14      C4      0.000     0.000     0.000  -200.000     0.000     0.000
      15 ART          0.000     0.000     0.000     0.000     0.000     0.000


     ROW        I3        I4        B1        B2        B3        B4        S1
       1   202.500   200.000     0.000     0.000  2000.000  2000.000  2000.000
       2     0.000     0.000     0.000     0.000     0.000     0.000     0.000
       3     0.000     0.000     0.000     0.000     0.000     0.000     0.000
       4     0.005     0.000     0.000     0.000    -1.000     0.000     0.000
       5    -0.005     0.000     0.000     0.000     0.000     0.000     0.000
       6     0.000     0.000     1.000     0.000     0.000     0.000    -1.000
       7     0.000     0.000     0.000     1.000     0.000     0.000     0.000
       8    -0.005     0.000     0.000     0.000     0.000    -1.000     0.000
       9    -1.000     1.000     0.000     0.000     0.000     0.000     0.000
      10     0.000     0.000     0.000     0.000     0.000     0.000     0.000
      11     0.000     0.000     0.000     0.000     0.000     0.000     0.000
      12     0.000     0.000     0.000     0.000     0.000     0.000     0.000
      13    -1.000     0.000     0.000     0.000     0.000     0.000     0.000
      14     0.000     0.000     0.000     0.000     0.000     0.000     0.000
      15     0.000     0.000     0.000     0.000     0.000     0.000     0.000


     ROW        S2        S3        S4        D1        D2        D3        D4
       1  2000.000     0.000     0.000   310.000   290.000   297.500   300.000
       2     0.000     0.000     0.000     0.005     0.000     0.000     0.000
       3     0.000     0.000     0.000    -0.005     0.005     0.000     0.000
       4     0.000     1.000     0.000    -0.005     0.010    -0.005     0.000
```

```
   5     0.000     0.000     0.000     0.000    -0.005     0.005     0.000
   6     0.000     0.000     0.000     0.005     0.000     0.000     0.000
   7    -1.000     0.000     0.000    -0.010     0.005     0.000     0.000
   8     0.000     0.000     1.000     0.000    -0.005     0.005     0.000
   9     0.000     0.000     0.000     0.000     0.000     1.000    -1.000
  10     0.000     0.000     0.000     0.000     0.000     0.000     0.000
  11     0.000     0.000     0.000     1.000     0.000     0.000     0.000
  12     0.000     0.000     0.000    -1.000     1.000     0.000     0.000
  13     0.000     0.000     0.000     0.000    -1.000     1.000     0.000
  14     0.000     0.000     0.000     0.000     0.000     0.000     0.000
  15     0.000     0.000     0.000     0.000     0.000     0.000     1.000

 ROW        C1        C2        C3        C4  SLK   10  SLK   11  SLK   12
   1     0.000     0.000     0.000     0.000     0.000     2.500    12.500
   2     0.000     0.000     0.000     0.000     0.000    -0.005     0.000
   3     0.000     0.000     0.000     0.000     0.000     0.000    -0.005
   4     0.000     0.000     0.000     0.000     0.000     0.000    -0.005
   5     0.000     0.000     0.000     0.000     0.000     0.000     0.000
   6     0.000     0.000     0.000     0.000     0.000    -0.005     0.000
   7     0.000     0.000     0.000     0.000     0.000     0.005    -0.005
   8     0.000     0.000     0.000     0.000     0.000     0.000     0.000
   9     0.000     0.000     0.000     0.000     0.000     0.000     0.000
  10     0.000     0.000     0.000     0.000     1.000     0.000     0.000
  11     1.000     0.000     0.000     0.000     0.000     0.000     0.000
  12     0.000     1.000     0.000     0.000     0.000     0.000     0.000
  13     0.000     0.000     1.000     0.000     0.000     0.000     0.000
  14     0.000     0.000     0.000     1.000     0.000     0.000     0.000
  15     0.000     0.000     0.000     0.000     0.000     0.000     0.000

 ROW  SLK   13  SLK   14
   1     2.500     0.000 -9750.000
   2     0.000     0.000     3.000
   3     0.000     0.000     4.000
   4     0.005     0.000     1.500
   5    -0.005     0.000     2.500
   6     0.000     0.000     1.000
   7     0.000     0.000     1.000
   8    -0.005     0.000     0.500
   9     0.000     1.000     0.000
  10     0.000     0.000     0.000
  11     0.000     0.000   600.000
  12     0.000     0.000   800.000
  13     0.000     0.000   500.000
  14     0.000     1.000   400.000
  15     0.000     0.000     0.000
```

56:171 Operations Research
Homework #5 Solutions -- Fall 2000

1. ***Linear Programming sensitivity.***

a. *Complete the following statement:* the optimal solution is to purchase only newsprint and book paper, process **500** tons of the book paper and **2,500** tons of the newsprint by asphalt dispersion, and the remaining book paper by de-inking. This yields **600** tons of pulp from the newsprint and **1,000** tons of pulp from the book paper. One-half of the pulp from book paper is used in each of grades 1 & 2 paper, and the newsprint is used in grade 3 paper. This plan will use **77.76%** $\underline{\left(=\frac{3000-666.6}{3000}\right)}$ of the de-inking capacity *(since the slack in row 17 is 666.6)* and **100%** of the asphalt dispersion capacity. (Note that BOX is a basic variable, but has a value of zero, categorizing this solution as **degenerate.**

b. How much must tissue drop in price in order that it would enter the solution?
The price of tissue must drop more than $6 which is the ALLOWABLE DECREASE in the Objective coefficient range of TISS.

c. If tissue were to enter the solution (e.g., because of the drop in price you determined in (b)), how much would be purchased? **+¥** *(Hint: use the minimum ratio test!)* ***Note*** *that there is no positive value in the TISS column on which to pivot, so that the cost function becomes unbounded! This is reasonable, since a drop of more than $6 means that TISS would have a negative cost, and there is no constraint which specifies that any tissue acquired must be used.*

d. How much would the cost decrease if 10 additional tons of pulp for grade 1 paper were required?
The cost function value will increase by $833.3

⇒ Dual Price for row (14) is – 83.33
⇒ The increase of 10 tons of pulp for grade 1 paper would change the right-hand-side of row 14 from 500 to 510, i.e., a ***increase*** of 10.
⇒ This is less than the ALLOWABLE INCREASE in the range of RHS of row (14), which is 240, so the dual price (-83.33) remains valid for the entire increase.
⇒ The "dual price" is (as LINDO uses the term) the rate at which the objective function will improve, so a negative dual price means that the cost function will worsen.
⇒ The objective function value (cost) will be increased by $833.3 (= 83.33*10)
⇒ Objective function value is 140,833.3

e. If ten additional tons of pulp for grade 1 paper were required, how would the quantities of raw materials (boxboard, newsprint and book paper) change? **Book paper will be increased by 27.78, but quantities of both newsprint and boxboard will remain unchanged.**

(Hint: if the surplus variable for the row stating the requirement were to increase, what effect would it have on BOX, NEWS, and BOOK?)
Note that if the surplus in row 15 were to change from 0 to 10, the quantity of pulp used (the left hand side of the inequality in row 15) would increase by 10. In the TABLEAU we see that the substitution rate of SLK15 (which is actually the surplus variable!) for BOOK is -2.778, meaning that BOOK will increase as SLK15 increases. The substitution rates for NEWS and BOX are zero, however.

$$\begin{array}{ll} 2 & Book \\ 16 & News \\ 18 & Box \end{array} = \begin{bmatrix} 2833.333 \\ 2500 \\ 0 \end{bmatrix} - \begin{bmatrix} -2.778 \\ 0 \\ 0 \end{bmatrix} \times (10) = \begin{bmatrix} 2860.60 \\ 2500 \\ 0 \end{bmatrix}$$

2. Transportation problem

(a) *Note: You may consider the check types to be the "sources" and the two processing sites as the "destinations", or vice-versa. In the transportation model below, we're using the "vice-versa", i.e., the processing sites are the sources and the check types are the destinations.*

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5	4	2	0	9000
site #2	3	4	5	0	7000
Demand	5000	5000	5000	1000	SUM = 16000

The number of basic variables is 5 (= m+n-1 = 2+4-1)

(b)

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4	2	0	~~9000~~ 4000
site #2	3	4	5	0	7000
Demand	~~5000~~ 0	5000	5000	1000	SUM = 16000

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4000 4	2	0	~~9000~~ ~~4000~~ 0
site #2	3	4	5	0	7000
Demand	~~5000~~ 0	~~5000~~ 1000	5000	1000	SUM = 16000

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4000 4	2	0	~~9000~~ ~~4000~~ 0
site #2	3	1000 4	5	0	~~7000~~ 6000
Demand	~~5000~~ 0	~~5000~~ ~~1000~~ 0	5000	1000	SUM = 16000

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4000 4	2	0	~~9000~~ ~~4000~~ 0
site #2	3	1000 4	5000 5	0	~~7000~~ ~~6000~~ 1000
Demand	~~5000~~ 0	~~5000~~ ~~1000~~ 0	~~5000~~ 0	1000	SUM = 16000

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4000 4	2	0	~~9000~~ ~~4000~~ 0
site #2	3	1000 4	5000 5	1000 0	~~7000~~ ~~6000~~ ~~1000~~ 0
Demand	~~5000~~ 0	~~5000~~ ~~1000~~ 0	~~5000~~ 0	~~1000~~ 0	SUM = 16000

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	4000 4	2	0	9000
site #2	3	1000 4	5000 5	1000 0	7000
Demand	5000	5000	5000	1000	SUM = 16000

Thus, X11 = 5,000, X12 = 4,000, X22 = 1,000, X23 = 5,000, and X24 = 1,000 are initial basic feasible solution with total cost = 70,000

(c)

		Vender	Salary	Personal	Excess Capacity	Supply
	Vj					
	Ui					
site #1	**0**	5000 5	4000 4	2	0	9000
site #2		3	1000 4	5000 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Determining the dual variables (simplex multipliers:

Let's arbitrarily set U1 = 0

Then complementary slackness implies that

U1 + V1 = 0 + V1 = 5 ⇒ V1 = 5

& U1 + V2 = 0 + V2 = 4 ⇒ V2 = 4

		Vender	Salary	Personal	Excess Capacity	Supply
	Vj	**5**	**4**			
	Ui					
site #1	**0**	5000 5	4000 4	2	0	9000
site #2		3	1000 4	5000 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Now we can use Complementary Slackness to obtain

U2 + V2 = U2 + 4 = 4 ⇒ U2 = 0

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		**5**	**4**			
site #1	**0**	5000 5	4000 4	2	0	9000
site #2	**0**	3	1000 4	5000 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Finally, we can use U2 to compute V3 and V4 :

U2 + V3 = 0 + V3 = 5 $\Rightarrow$ V3 = 5

U2 + V4 = 0 + V4 = 0 $\Rightarrow$ V4 = 0

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		**5**	**4**	**5**	**0**	
site #1	**0**	5000 5	4000 4	2	0	9000
site #2	**0**	3	1000 4	5000 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Now let's use the simplex multipliers to compute the reduced costs, using the formula : $\overline{C}_{ij} = C_{ij} - (U_i + V_j)$

Reduced costs : $\overline{C}_{21} = 3 - (0+5) = -2 < 0$

$\overline{C}_{13} = 2 - (0+5) = -3 < 0$

$\overline{C}_{14} = 0 - (0+0) = 0 = 0$

Either X21 or X13 may enter the solution. Let's arbitrarily select X13

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		5	4	5	0	
site #1	0	5000 5	4000$-q$ 4	$+q$ 2	0	9000
site #2	0	3	1000$-q$ [illegible]	50[illegible]0$-q$ 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

As q increased to 4000, the shipment from site#1 to Salary checks (= Salary checks processed at Site #1) becomes zero, preventing any further increase in q .

⇒ X12 leaves the basis.

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	5000 5	0 4	**4000** 2	 0	9000
site #2	 3	5000 4	1000 5	1000 0	7000
Demand	5000	5000	5000	1000	SUM = 16000

The new solution has a total cost of $58,000 , a saving of $12,000 (=3*4000)

Recomputing the dual variables:

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		5	1	2	-3	
site #1	0	5000 5	0 4	**4000** 2	 0	9000
site #2	3	 3	5000 4	1000 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Reduced costs : $\overline{C}_{12} = 4 - (0+1) = 3 > 0$

$\overline{C}_{14} = 0 - (0-3) = 3 > 0$

$\overline{C}_{21} = 3 - (3+5) = -5 < 0$

⇒ X21 may enter the solution.

As q increased to 1000, the shipment from site#2 to Personal checks (= Personal checks processed at Site #2) becomes zero, preventing any further increase in q .

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		5	1	2	-3	
site #1	0	$5000 - q$ 5	0 4	$4000 + q$ 2	 0	9000
site #2	3	$+q$ 3	5000 4	$1000 - q$ 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

⇒ X23 leaves the basis.

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	4000 5	0 4	5000 2	 0	9000
site #2	**1000** 3	5000 4	0 5	1000 0	7000
Demand	5000	5000	5000	1000	SUM = 16000

The new solution has a total cost of $53,000 , a saving of $5,000 (=5*1000)

Recomputing the dual variables:

		Vender	Salary	Personal	Excess Capacity	Supply
	Vj	5	6	2	2	
	Ui					
site #1	0	4000 5	0 4	5000 2	 0	9000
site #2	-2	**1000** 3	5000 4	0 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Reduced costs : $\bar{C}_{12} = 4 - (0+6) = -2 < 0$
$\bar{C}_{14} = 0 - (0+2) = -2 < 0$
$\bar{C}_{23} = 5 - (-2+2) = 5 > 0$

⇒ Either X12 or X14 may enter the solution. Let's arbitrarily select X12

		Vender	Salary	Personal	Excess Capacity	Supply
	Vj	5	6	2	2	
	Ui					
site #1	0	$4000-q$ 5	$0+q$ 4	5000 2	 0	9000
site #2	-2	**1000**$+q$ 3	$5000-q$ 4	0 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

As q increased to 4000, the shipment from site#1 to Vender checks (= Vender checks processed at Site #1) becomes zero, preventing any further increase in q .

⇒ X11 leaves the basis.

	Vender	Salary	Personal	Excess Capacity	Supply
site #1	0 5	4000 4	5000 2	0	9000
site #2	**5000** 3	1000 4	0 5	1000 0	7000
Demand	5000	5000	5000	1000	SUM = 16000

The new solution has a total cost of $45,000 , a saving of $8,000 (=2*4000)

Recomputing the dual variables:

	Ui	Vender	Salary	Personal	Excess Capacity	Supply
Vj		3	4	2	0	
site #1	0	0 5	4000 4	5000 2	0	9000
site #2	0	**5000** 3	1000 4	0 5	1000 0	7000
Demand		5000	5000	5000	1000	SUM = 16000

Reduced costs : $\overline{C}_{11} = 5 - (0+3) = 2 > 0$

$\overline{C}_{14} = 0 - (0+0) = 0 = 0$

$\overline{C}_{23} = 5 - (0+2) = 3 > 0$

All the reduced costs of non-basic variables are nonnegative.

⇒ This solution is optimal !

⇒ Total cost is $45,000 with X12 = 4000 , X13 = 5000 , X21 = 5000 , X22 = 1000 , and X24 = 1000

56:171 Operations Research
Homework #6 Solutions -- Fall 2000

1. Data Envelopment Analysis. The following data are available for each of seven university departments which are to be evaluated by the university administration:

- Number of staff persons
- Academic staff salaries (in thousands of British pounds)
- Support staff salaries (in thousands of British pounds)
- Number of undergraduates
- Number of graduate students
- Number of research papers

Dept	#Staff	Acad-sal	Supp-sal	#UG	#Grad	Papers
1	12	400	20	60	35	17
2	19	750	70	139	41	40
3	42	1500	70	225	68	75
4	15	600	100	90	12	17
5	45	2000	250	253	145	130
6	19	730	50	132	45	45
7	41	2350	600	305	159	97

It was decided to use DEA to compute the relative "efficiencies" of the departments. The results were less than helpful-- all but one department was rated as 100% efficient!

i	Efficiency
1	1
2	1
3	1
4	0.8197
5	1
6	1
7	1

A look at the "prices" assigned by each DMU (department) to each input and output help to explain this result.

i	#UG	#Grad	Papers	#Staff	Acad-salary	Supp-salary
1	0.00613375	0.0179087	0.000304169	0.0791606	0	0.00250366
2	0.0052393	0	0.00679343	0.0472869	0.000135398	0
3	0	0.00257257	0.0110009	0	0.000303082	0.0077911
4	0.00910818	0	0	0.0641504	0.0000629054	0
5	0	0.00280219	0.00456679	0	0.0003988	0.000809599
6	0.00376481	0.0109921	0.000186695	0.0485876	0	0.00153671
7	0.0012067	0.00333687	0.00104531	0.00731881	0.000297842	0

Note, for example, that department #2 places zero value on both the number of graduate students and support staff salaries-- which might be explained by the fact that their support staff salaries (an input) were relatively high and the number of graduate students (an output) were relatively low, compared to the other departments.
This illustrates a limitation of DEA when the number of inputs and outputs is relatively large compared to the number of DMUs being evaluated-- most DMUs are able to find some combination of input & output in which they "shine" and are thereby able to assign appropriate prices in order to earn a 100% efficiency rating.

The analysis which follows used a *single* input-- only the total number of staff-- and used all three of the previous outputs.

a. Write the LP which is solved in order to compute the efficiency of department #5, and solve it with LINDO. What are the values assigned to each of the three outputs? (Enter the efficiency and values assigned to outputs in the tables below.)

Solution:

Dept	Inputs	Outputs		
	# Staff	# UG	# Grad	# Papers
1	12	60	35	17
2	19	139	41	40
3	42	225	68	75
4	15	90	12	17
5	45	253	145	130
6	19	132	45	45
7	41	305	159	97

$$\begin{array}{lll}
\text{Max} & Z = (253\,u1 + 145\,u2 + 130\,u3) / (45\,v1) & \\
\text{Subject to} & (60\,u1 + 35\,u2 + 17\,u3) / (12\,v1) \le 1 & \text{Dept.1} \\
 & (139\,u1 + 41\,u2 + 40\,u3) / (19\,v1) \le 1 & \text{Dept.2} \\
 & (225\,u1 + 68\,u2 + 75\,u3) / (42\,v1) \le 1 & \text{Dept.3} \\
 & (90\,u1 + 12\,u2 + 17\,u3) / (15\,v1) \le 1 & \text{Dept.4} \\
 & (253\,u1 + 145\,u2 + 130\,u3) / (45\,v1) \le 1 & \text{Dept.5} \\
 & (132\,u1 + 45\,u2 + 45\,u3) / (19\,v1) \le 1 & \text{Dept.6} \\
 & (305\,u1 + 159\,u2 + 97\,u3) / (41\,v1) \le 1 & \text{Dept.7} \\
 & vi \ge 0,\ \text{for } i = 1\ ;\ uj \ge 0,\ \text{for } i = 1, 2, 3 &
\end{array}$$

Write the problem above as an LP problem:

$$\begin{array}{lll}
\text{Max} & Z = (253\,u1 + 145\,u2 + 130\,u3) & \\
\text{Subject to} & (60\,u1 + 35\,u2 + 17\,u3) - (12\,v1) \le 0 & \text{Dept.1} \\
 & (139\,u1 + 41\,u2 + 40\,u3) - (19\,v1) \le 0 & \text{Dept.2} \\
 & (225\,u1 + 68\,u2 + 75\,u3) - (42\,v1) \le 0 & \text{Dept.3} \\
 & (90\,u1 + 12\,u2 + 17\,u3) - (15\,v1) \le 0 & \text{Dept.4} \\
 & (253\,u1 + 145\,u2 + 130\,u3) - (45\,v1) \le 0 & \text{Dept.5} \\
 & (132\,u1 + 45\,u2 + 45\,u3) - (19\,v1) \le 0 & \text{Dept.6} \\
 & (305\,u1 + 159\,u2 + 97\,u3) - (41\,v1) \le 0 & \text{Dept.7} \\
 & (45\,v1) = 1 & \\
 & vi \ge 0,\ \text{for } i = 1\ ;\ uj \ge 0,\ \text{for } i = 1, 2, 3 &
\end{array}$$

The Dual LP (which is often preferred, especially when the number of rows of the primal, i.e., the number of DMUs, is much greater than the number of columns, i.e., the combined number of inputs & outputs), is:

$$\begin{array}{l}
\text{Min } Z = Z0 \\
\text{Subject to} \\
60\,L1 + 139\,L2 + 225\,L3 + 90\,L4 + 253\,L5 + 132\,L6 + 305\,L7 >= 253 \\
35\,L1 + 41\,L2 + 68\,L3 + 12\,L4 + 145\,L5 + 45\,L6 + 159\,L7 >= 145 \\
17\,L1 + 40\,L2 + 75\,L3 + 17\,L4 + 130\,L5 + 45\,L6 + 97\,L7 >= 130 \\
45\,Z0 - 12\,L1 - 19\,L2 - 42\,L3 - 15\,L4 - 45\,L5 - 19\,L6 - 41\,L7 >= 0 \\
Li \ge 0,\ \text{for } i = 1, 2, 3, 4, 5, 6, 7\ ;\ Z0 \text{ unrestricted in sign}
\end{array}$$

Lindo Input :

```
Max  253 u1 + 145 u2 + 130 u3
st
```

```
  60 u1 +   35 u2 +   17 u3  -    12 v1   <=  0
 139 u1 +   41 u2 +   40 u3  -    19 v1   <=  0
 225 u1 +   68 u2 +   75 u3  -    42 v1   <=  0
  90 u1 +   12 u2 +   17 u3  -    15 v1   <=  0
 253 u1 +  145 u2 +  130 u3  -    45 v1   <=  0
 132 u1 +   45 u2 +   45 u3  -    19 v1   <=  0
 305 u1 +  159 u2 +   97 u3  -    41 v1   <=  0
 45 v1  = 1
End
```

Lindo Output :

```
        OBJECTIVEFUNCTIONVALUE

        1)     1.000000

  VARIABLE       VALUE        REDUCED COST
        U1       0.000000         0.000000
        U2       0.003247         0.000000
        U3       0.004071         0.000000
        V1       0.022222         0.000000
```

Note: *there is more than one optimal solution for this LP, as can be seen from the fact that U1 is nonbasic with a zero reduced cost.*

The results of the DEA, i.e., the seven LP solutions, are now:

Dept	Efficiency
1	0.7521
2	0.9834
3	0.7383
4	0.8066
5	1.0000
6	0.9692
7	1

Prices:

Dept	#UG	#Grad	Papers
1	0	0.0214885	0
2	0.00707506	0	0
3	0.0015207	0	0.00528225
4	0.00896175	0	0
5	0	0.003247	0.004071
6	0.00336154	0	0.0116766
7	0.00155779	0	0.00541109

Weighted Output Values (%)

i	#UG	#Grad	papers
1	0.0	100.0	0.0
2	100.0	0.0	0.0
3	46.3	0.0	53.7
4	100.0	0.0	0.0
5	35.9	0.0	64.1
6	45.8	0.0	54.2
7	47.5	0.0	52.5

Note that the values for DMU#5 correspond to a different optimal set of prices than those found by LINDO above.

For example, department 6 placed no value on graduate students and assigned values to undergraduate students and research papers so that they accounted for approximately 45% and 55%, respectively.

b. Which department(s) seem to specialize in graduate education, i.e., give the number of graduate students a high priority?
Solution: *Department #1 would appear to be specializing in graduate student production, since it has assigned positive prices only to the "GRAD" output.*

c. Which department(s) seem to specialize in undergraduate education, i.e., give the number of undergraduate students a high priority?
Solution: Likewise, departments #2 & #4 seem to be specializing in undergraduate education, since they give nonzero prices only to the #UG output.

2. ***Assignment Problem.*** An accounting firm has three new clients, each of which is to be assigned a project leader. Based upon the different backgrounds and experiences of the available leaders the various assignments differ in expected completion times, which are (in days):

Project leader	Client A	Client B	Client C
Jackson	10	16	32
Ellis	14	22	40
Smith	22	24	34

Use the Hungarian algorithm to find the optimal assignment.

Solution: First we use row reduction and obtain the following cost matrix.

	Client A	Client B	Client C
Jackson	0	6	22
Ellis	0	8	26
Smith	0	2	12

Then by using column reduction in above cost matrix, we obtain the following cost matrix.

	Client A	Client B	Client C
Jackson	0	4	10
Ellis	0	6	14
Smith	0	0	0

It is obvious that only two lines are needed to cover all zeroes in above cost matrix.
The smallest unlined cost, $\overline{C} = 4$. Subtract this cost from all unlined costs, and add to costs at intersections of lines.

	Client A	Client B	Client C
Jackson	0	0	6
Ellis	0	2	10
Smith	4	0	0

The new cost matrix has 2 zeroes not covered by the previous lines.

	Client A	Client B	Client C
Jackson	0	0	6
Ellis	0	2	10
Smith	4	0	0

The zeroes now require three lines in order to cover all of them.

	Client A	Client B	Client C
Jackson	0	0	6
Ellis	0	2	10
Smith	4	0	0

Hence we stop and can assign as Jackson → Client B, Ellis → Client A , Smith → Client C, and the total cost = 16 + 14 + 34 = 64.

3. ***Assignment Problem.*** A Manufacturer of small electrical devices has purchased an old warehouse and converted it into a primary production facility. The physical dimensions of the existing building left the architect with little leeway for designing locations for the company's five assembly lines and five inspection and storage areas, but these have now been constructed and now exist in fixed areas within the building.

As items are taken off the assembly lines, they are temporarily stored in bins at the end of each line. At 30-minute intervals, the bins are physically transported to one of the five inspection areas. Because different volumes of product are manufactured at each assembly line and different distances must be traversed from each assembly line to each inspection station, different times are required. The company must designate a separate inspection area for each assembly line.

An IE has performed a study showing the times needed to transport finished products from each assembly line to each inspection area in minutes:

	A	B	C	D	E
1	10	4	6	10	12
2	11	7	7	9	14
3	13	8	12	14	16
4	14	16	13	17	17
5	19	11	17	20	19

a. Under the current arrangement, which has been operational since they moved into the building, work on assembly lines 1, 2, 3, 4, and 5 is transported to inspection areas A, B, C, D, and E, respectively. Given that the average worker costs \$12 per hour, what is the annual labor cost for this arrangement, assuming two 8-hour shifts per day, 250 days per year?

Solution: Under the current arrangement, the total time needed to transport finished products from each assembly line to each inspection area is 65 minutes = 65/60 hours. Thus, the annual labor cost for this arrangement is \$ 52,000

$$\left(= \frac{65}{60} \times 12\left(\$/hr\right) \times 16\left(hr/day\right) \times 250\left(days/yr\right) \right)$$

b. Use the Hungarian algorithm to find an optimal assignment of assembly lines to inspection areas.

Assembly Line	Inspection Area
1	

2	
3	
4	
5	

First we use row reduction and obtain the following cost matrix.

	A	B	C	D	E
1	6	0	2	6	8
2	4	0	0	2	7
3	5	0	4	6	8
4	1	3	0	4	4
5	8	0	6	9	8

Then by using column reduction, we obtain the following cost matrix.

	A	B	C	D	E
1	5	0	2	4	4
2	3	0	0	0	3
3	4	0	4	4	4
4	0	3	0	2	0
5	7	0	6	7	4

It is obvious that three lines are needed to cover all zeros in above cost matrix.

The smallest unlined cost, $\overline{C} = 2$. Subtract this cost from all unlined costs, and add to costs at intersections of lines.

	A	B	C	D	E
1	3	0	0	2	2
2	3	2	0	0	3
3	2	0	2	2	2
4	0	5	0	2	0
5	5	0	4	5	2

Again, it is obvious that four lines are needed to cover all zeros in above cost matrix. The smallest unlined cost, $\overline{C} = 2$. Subtract this cost from all unlined costs, and add to costs at intersections of lines.

	A	B	C	D	E
1	1	0	0	0	0
2	3	4	2	0	3
3	0	0	2	0	0
4	0	7	2	2	0
5	3	0	4	3	0

The zeroes now require five lines in order to cover all of them. Hence we stop and can assign. In fact, there are three different zero-cost assignments, all of them optimal for this problem :

Optimal Assignment #1

	A	B	C	D	E
1	1	0	⓪	0	0
2	3	4	2	⓪	3
3	0	0	2	0	⓪
4	⓪	7	2	2	0
5	3	⓪	4	3	0

Optimal Assignment #2

	A	B	C	D	E
1	1	0	⓪	0	0
2	3	4	2	⓪	3
3	⓪	0	2	0	0
4	0	7	2	2	⓪
5	3	⓪	4	3	0

Optimal Assignment #3

	A	B	C	D	E
1	1	0	⓪	0	0
2	3	4	2	⓪	3
3	0	⓪	2	0	0
4	⓪	7	2	2	0
5	3	0	4	3	⓪

Total time required for transportation

Assembly Line	Inspection Area		
	#1	#2	#3
1	C	C	C
2	D	D	D
3	E	A	B
4	A	E	A
5	B	B	E
Total Times	56 (min)	56 (min)	56 (min)

c. What is the annual savings which management could expect if this assignment were made?

Solution: Under the new arrangement obtained from the Hungarian algorithm, the total time is 56 minutes = 56/60 hours. The corresponding annual labor cost for this arrangement is \$44,800$\left(=\frac{56}{60}\times 12\left(\$/hr\right)\times 16\left(hr/day\right)\times 250\left(days/yr\right)\right)$., a savings of \$52,000 - 44,800 = \$7,200 annually.

56:171 Operations Research
Homework #7 Solution -- Fall 2000

1. The campus bookstore must decide how many textbooks to order for a freshman economics course to be offered next semester. The bookstore believes that either seven, eight, nine, or ten sections of the course will be offered, each section consisting of 40 students. The publisher is offering bookstores a discount if they place their orders early. If the bookstore orders too few texts and runs out, the publisher will air express additional books at the bookstore's expense. If it orders too many texts, the store can return unsold texts to the publisher for a partial credit. The bookstore is considering ordering either 280, 320, 360, or 400 textbooks in order to get the discount. Taking into account the discounts, air express expenses, and credits for returned texts, the bookstore manager estimates the following resulting profits:

# books ordered	7 sections	8 sections	9 sections	10 sections
280	$2800	$2720	$2640	$2480
320	$2600	$3200	$3040	$2880
360	$2400	$3000	$3600	$3440
400	$2200	$2800	$3400	$4000

(a.) What is the decision if the manager uses the maximax criterion?

MAXIMAX Criterion

# books ordered	7 sections	8 sections	9 sections	10 sections	Maximum payoff	
280	**$2,800**	$2,720	$2,640	$2,480	$2,800	
320	$2,600	**$3,200**	$3,040	$2,880	$3,200	
360	$2,400	$3,000	**$3,600**	$3,440	$3,600	
400	$2,200	$2,800	$3,400	**$4,000**	$4,000	← Maximum

⇒ ∴ *The bookstore should order 400 books.*

(b.) What is the decision if the manager uses the maximin criterion?

MAXIMIN Criterion

# books ordered	7 sections	8 sections	9 sections	10 sections	Minimum payoff	
280	$2,800	$2,720	$2,640	**$2,480**	$2,480	
320	**$2,600**	$3,200	$3,040	$2,880	$2,600	← Maximum
360	**$2,400**	$3,000	$3,600	$3,440	$2,400	
400	**$2,200**	$2,800	$3,400	$4,000	$2,200	

⇒ ∴ *The bookstore should order 320 books.*

(c.) What is the decision if the manager uses the minimax regret criterion?

Payoff

# books ordered	7 sections	8 sections	9 sections	10 sections
280	**$2,800**	$2,720	$2,640	$2,480
320	$2,600	**$3,200**	$3,040	$2,880
360	$2,400	$3,000	**$3,600**	$3,440
400	$2,200	$2,800	$3,400	**$4,000**

Regret

# books ordered	7 sections	8 sections	9 sections	10 sections
280	$0	($480)	($960)	($1,520)
320	($200)	$0	($560)	($1,120)
360	($400)	($200)	$0	($560)
400	($600)	($400)	($200)	$0

MINIMAX REGRET Criterion

# books ordered	7 sections	8 sections	9 sections	10 sections	Maximum Regret	
280	$0	($480)	($960)	**($1,520)**	$1,520	
320	($200)	$0	($560)	**($1,120)**	$1,120	
360	($400)	($200)	$0	**($560)**	$560	← Minimum
400	**($600)**	($400)	($200)	$0	$600	

⇒ ∴ *The bookstore should order 360 books.*

Suppose now that, based upon conversations held with the chairperson of the economics department, the bookstore manager believes the following probabilities hold:

P{7 sections offered} = 10%
P{8 sections offered} = 30%
P{9 sections offered} = 40%
P{10 sections offered} = 20%

(d.) Using the expected value criterion, determine how many books the manager should purchase in order to maximize the store's expected profit.

Expected Value Criterion

# books ordered	Expected Payoff	
280	0.1($2,800) + 0.3($2,720) + 0.4($2,640) + 0.2($2,480) = $2,648	
320	0.1($2,600) + 0.3($3,200) + 0.4($3,040) + 0.2($2,880) = $3,012	
360	0.1($2,400) + 0.3($3,000) + 0.4($3,600) + 0.2($3,440) = $3,268	← Maximum
400	0.1($2,200) + 0.3($2,800) + 0.4($3,400) + 0.2($4,000) = $3,220	

⇒ ∴ *The bookstore should order 360 books.*

(e.) Based upon the probabilities given, determine the expected value of perfect information and interpret its meaning.

Expected Value Without Information (EVWOI)

# books ordered	7 sections	8 sections	9 sections	10 sections
280	$2,800	$2,720	$2,640	$2,480
320	$2,600	$3,200	$3,040	$2,880
360	$2,400	$3,000	$3,600	$3,440
400	$2,200	$2,800	$3,400	$4,000
probability	0.1	0.3	0.4	0.2

# books ordered	Expected Payoff	
280	0.1($2,800) + 0.3($2,720) + 0.4($2,640) + 0.2($2,480) = $2,648	
320	0.1($2,600) + 0.3($3,200) + 0.4($3,040) + 0.2($2,880) = $3,012	
360	0.1($2,400) + 0.3($3,000) + 0.4($3,600) + 0.2($3,440) = $3,268	← Maximum
400	0.1($2,200) + 0.3($2,800) + 0.4($3,400) + 0.2($4,000) = $3,220	

∴ Maximum Expected Payoff (without information) :
EVWOI = 0.1($2,400) + 0.3($3,000) + 0.4($3,600) + 0.2($3,440) = $3,268

Expected Value With Perfect Information (EVWPI)

# books ordered	7 sections	8 sections	9 sections	10 sections
280	**$2,800**	$2,720	$2,640	$2,480
320	$2,600	**$3,200**	$3,040	$2,880
360	$2,400	$3,000	**$3,600**	$3,440
400	$2,200	$2,800	$3,400	**$4,000**
probability	0.1	0.3	0.4	0.2

i.e., if the manager had a prediction of 4 different sections in advance (possess perfect information), he would purchase

"280 books" if p = 10%
"320 books" if p = 30%
"360 books" if p = 40%
"400 books" if p = 20%

∴ **EVWPI** = 0.1($2,800) + 0.3($3,200) + 0.4($3,600) + 0.2($4,000) = $3,480

⇒ ∴ **Expected Value of Perfect Information (EVPI)**

= **EVWPI – EVWOI** = \$3,480 – \$3,268 = \$212

⇒ That is, possessing knowledge of four different sections before the manager purchases the books will increase his expected return by \$212.

* * *

2. T. Bone Puckett, a corporate raider, has acquired a textile company and is contemplating the future of one of its major plants located in South Carolina. Three alternative decisions are being considered:

   - ☐ Expand the plant and produce light-weight, durable materials for possible sales to the military, a market with little foreign competition.
   - ☐ Maintain the status quo at the plant, continuing production of textile goods that are subject to heavy foreign competition.
   - ☐ Sell the plant now.

   If one the first two alternatives is chosen, the plant will still be sold at the end of a year. The amount of profit that could be earned by selling the plant in a year depends upon foreign market conditions, including the status of a trade embargo bill in Congress. The following payoff table describes this decision situation.

Decision	Good foreign competitive conditions	Poor foreign competitive conditions
Expand	\$800,000	\$500,000
Maintain status quo	\$1,300,000	- \$150,000
Sell now	\$320,000	\$320,000

(a.) Determine the best decision using the following decision criteria:
- Maximax
- Maximin
- Minimax regret

MAXIMAX Criterion

Decision	Good foreign competitive conditions	Poor foreign competitive conditions	Maximum payoff
Expand	\$800,000	\$500,000	\$800,000
Maintain status quo	\$1,300,000	-\$150,000	\$1,300,000 ← Maximum
Sell now	\$320,000	\$320,000	\$320,000

⇒ *∴T. Bone Puckett should maintain status quo at the plant.*

MAXIMIN Criterion

Decision	Good foreign competitive conditions	Poor foreign competitive conditions	Minimum payoff
Expand	\$800,000	\$500,000	\$500,000 ← Maximum
Maintain status quo	\$1,300,000	-\$150,000	-\$150,000
Sell now	\$320,000	\$320,000	\$320,000

⇒ *∴T. Bone Puckett should expand the plant.*

MINIMAX REGRET Criterion

Decision	Good foreign competitive conditions	Poor foreign competitive conditions	Maximum Regret
Expand	\$500,000	\$0	\$500,000 ← Minimum
Maintain status quo	\$0	\$650,000	\$650,000
Sell now	\$980,000	\$180,000	\$980,000

∴T. Bone Puckett should expand the plant.

(b.) Assume it is now possible to estimate a probability of 70% that good foreign competitive conditions will exist and a probability of 30% that poor conditions will exist. Determine the best decision using expected value and expected opportunity loss.

Expected Value Criterion (EVWOI)

Decision	Good foreign competitive conditions	Poor foreign competitive conditions	Expected payoff
Expand	$800,000	$500,000	$710,000
Maintain status quo	$1,300,000	-$150,000	$865,000 ← Maximum
Sell now	$320,000	$320,000	$320,000
probability	0.7	0.3	

\ *Maximum Expected Payoff* = 0.7($1,300,000) + 0.3(–$150,000) = $ 865,000

(c.) Compute the expected value of perfect information.

Expected Value With Perfect Information (EVWPI)

Decision	Good foreign competitive conditions	Poor foreign competitive conditions
Expand	$800,000	**$500,000**
Maintain status quo	**$1,300,000**	-$150,000
Sell now	$320,000	$320,000
probability	0.7	0.3

i.e., if we had a prediction of these two competitive conditions in advance (possess perfect info.), we would "Maintain" (if p = 70%) and "Expand" (if p = 30%)

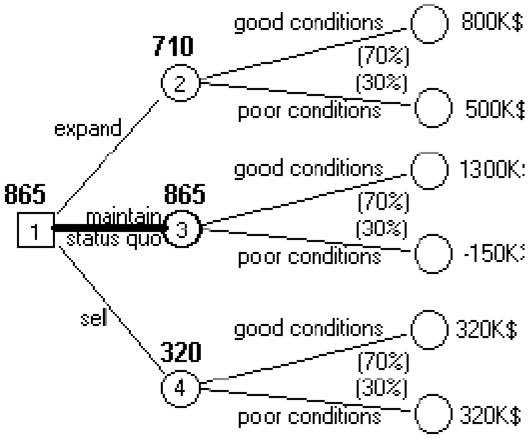
∴ **EVWPI** = 0.7($1,300,000) + 0.3($500,000) = $1,060,000

∴ **Expected Value of Perfect Information:**

⇒ **EVPI = EVWPI – EVWOI** = $1,060,000– $ 865,000= $195,000

That is, possessing knowledge of two competitive conditions before T. Bone Puckett make a decision will increase expected return by $195,000.

(d.) Fold back the decision tree below:



Puckett has hired a consulting firm to provide a report on future political and market situations. The report will be positive (P) or negative (N), indicating either a good (g) or poor (p) future foreign competitive situation. The conditional probability of each report outcome given each state of nature is

P{P|g} = 70%
P{N|g} = 30%
P{P|p} = 20%
P{N|p} = 80%

(e.) Determine the posterior probabilities using Bayes' rule:

P{g| P} = _______ %
P{p| P} = _______ %
P{g| N} = _______ %
P{p| N} = _______ %

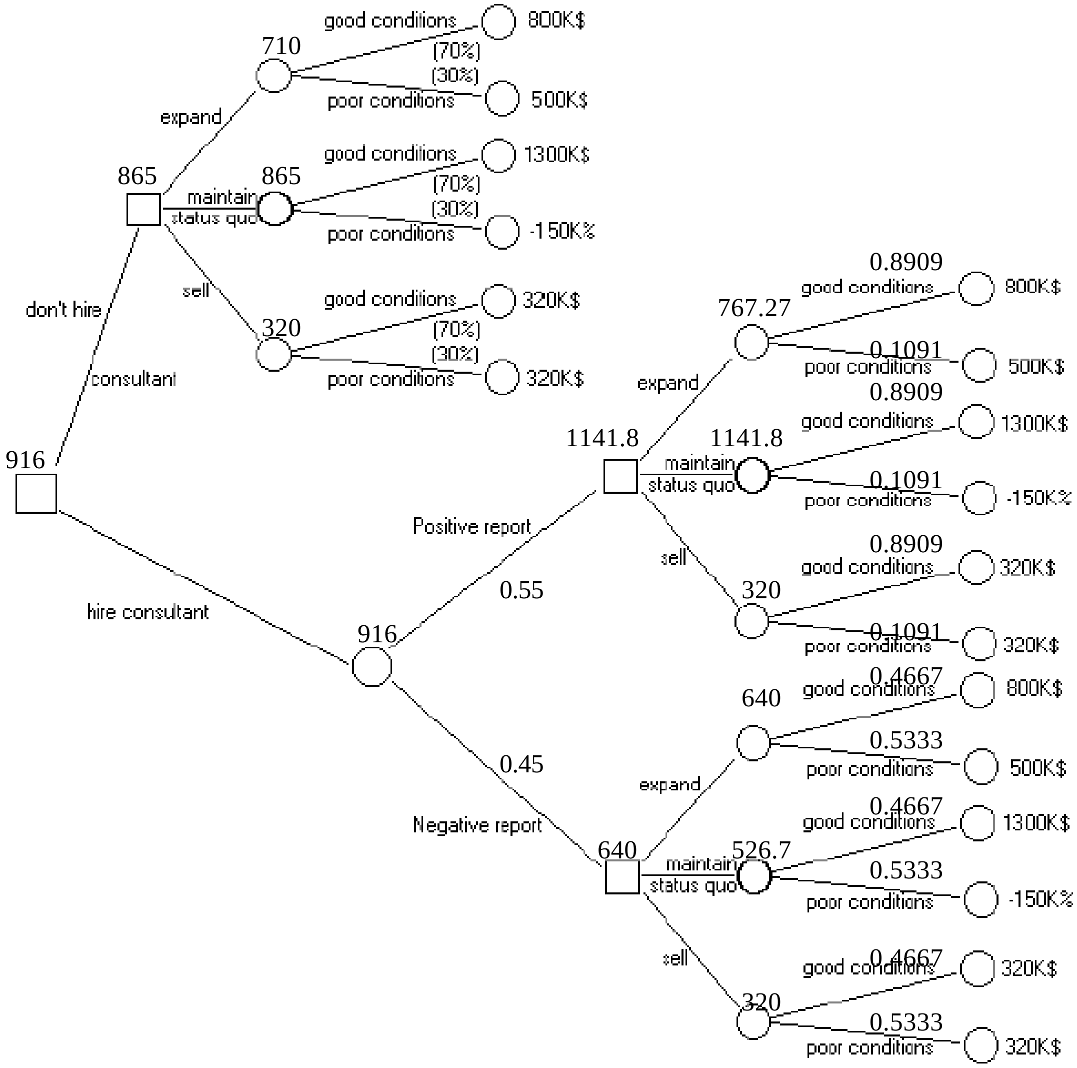
P{g| P} = <u>89.09</u> % $\left(= \dfrac{0.7\times 0.7}{0.7\times 0.7 + 0.3\times 0.2} \right)$

P{p| P} = <u>10.91</u> % $\left(= \dfrac{0.3\times 0.2}{0.7\times 0.7 + 0.3\times 0.2} \right)$

P{g| N} = <u>46.67</u> % $\left(= \dfrac{0.7\times 0.3}{0.7\times 0.3 + 0.3\times 0.8} \right)$

P{p| N} = <u>53.33</u> % $\left(= \dfrac{0.3\times 0.8}{0.7\times 0.3 + 0.3\times 0.8} \right)$

(f.) Perform a decision tree analysis using the posterior probabilities that you have just computed.



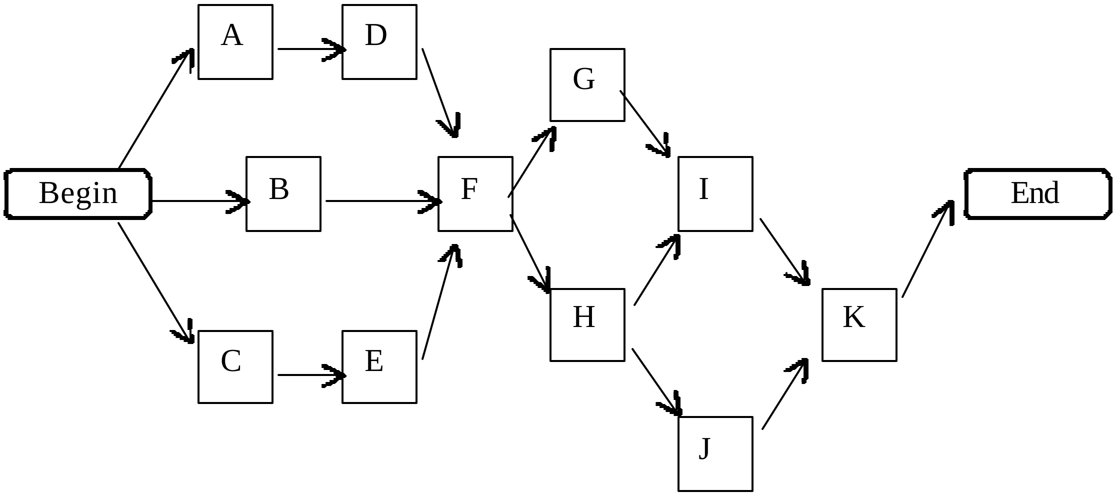
56:171 Operations Research
Homework #8 Solutions -- Fall 2000

It is June 1, and popular recording star Chocolate Cube is planning to add a separate recording studio to his palatial complex in rural Connecticut. The blueprints have been completed, and the following table lists the time estimates of the activities in the construction project. *(Based upon exercise in Applied Mgmt Science, by Lawrence & Pasternack.)*

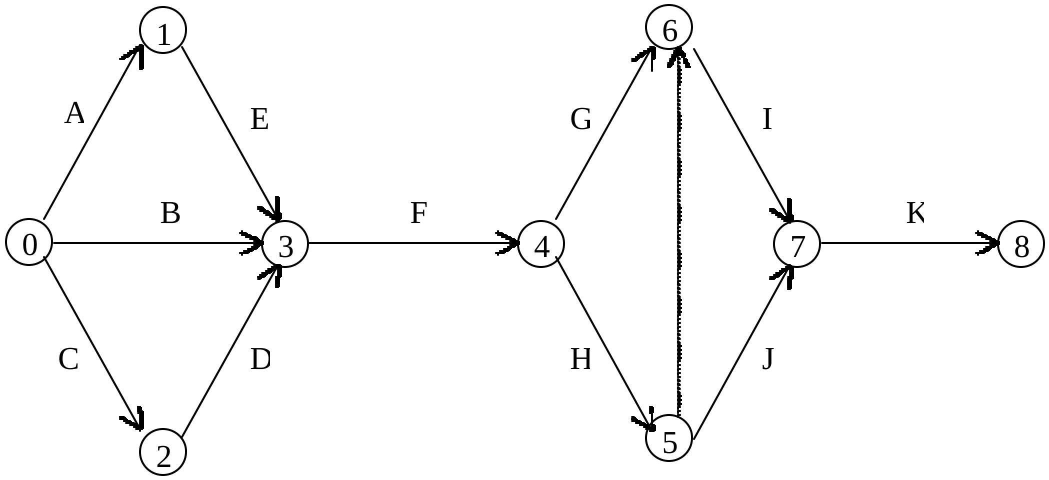
	Activity	Immediate Prede-cessors	Optimistic time (days)	Most likely time (days)	Pessimistic time (days)	Mean μ	Variance σ^2
A	Order materials	none	1	2	9	3	1.778
B	Clear land	none	2.5	4.5	9.5	5	1.361
C	Obtain permits	none	2	5	14	6	4.00
D	Hire subcontractors	C	4	6.5	18	8	5.44
E	Unload/store materials	A	2	4	18	6	7.11
F	Primary structure	B,D,E	22	30	50	32	21.78
G	Install electrical work	F	15	20	37	22	13.44
H	Install plumbing	F	4.5	10	21.5	11	8.03
I	Finish/paint	G,H	12	15	24	16	4.00
J	Complete electrical studio	H	14	14.5	48	20	32.11
K	Clean-up	I,J	5	5	5	5	0.00

1. Compute the expected duration of each activity, based upon the three time estimates.
   Solution: Assume that the activity durations have the *beta* distribution.
   Then Compute $m = \dfrac{a+4m+b}{6}$, $s = \dfrac{b-a}{6}$ where a = "optimistic time", m = "most likely time", and b = "pessimistic time". The results are shown in the table above.

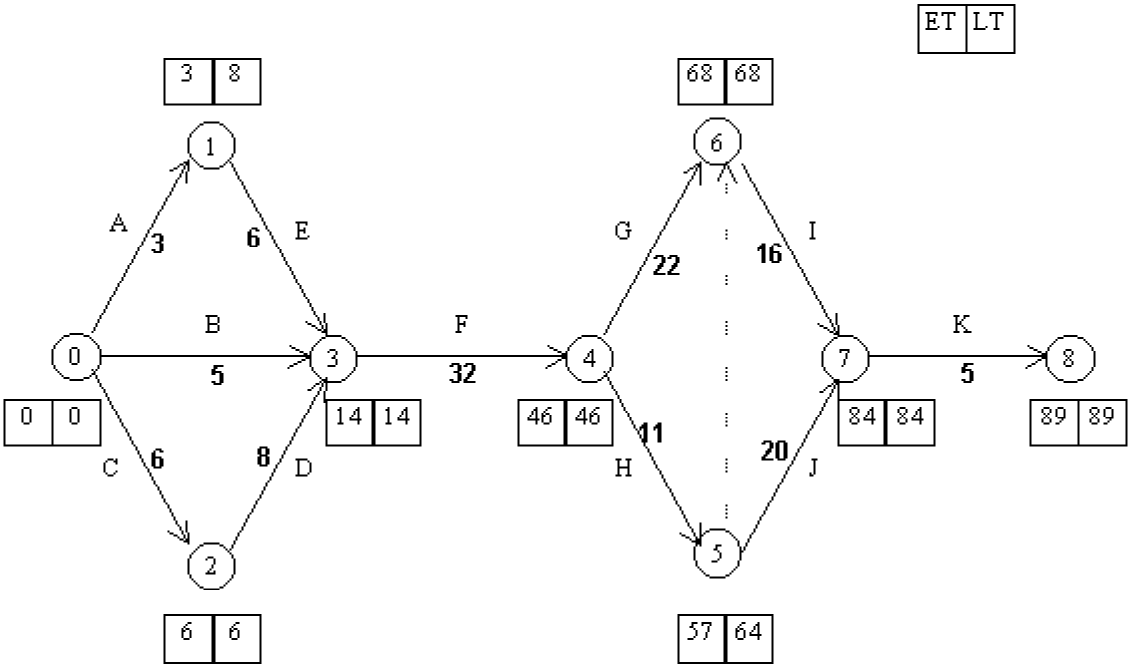
2. Draw the AON (Activity-on-Node) network for the project.



3. Draw the AOA (Activity-on-Arrow) network for the project and label the nodes so that i<j if there is an arrow from node i to node j.



4. For each node (event), compute the ET (early time) and LT (late time), based upon the expected durations.



5. For each activity, compute the ES (early start), EF (early finish), LS (late start), LF (late finish), and TS (total slack).

	Activity	ES	EF	LS	LF	TS
A	Order materials	0	3	5	8	5
B	Clear land	0	5	9	14	9
C	Obtain permits	0	6	0	6	0
D	Hire subcontractors	6	14	6	14	0
E	Unload/store materials	3	9	8	14	5
F	Primary structure	14	46	14	46	0
G	Install electrical work	46	68	46	68	0
H	Install plumbing	46	57	57	68	11
I	Finish/paint	68	84	68	84	0
J	Complete electrical studio	57	77	64	84	7
K	Clean-up	84	89	84	89	0

Note that ES is the ET at the beginning node of the activity, and LF is the LT at the end node of the activity. Then EF = ES + duration and LS = LF - duration.

6. Which activities are on the critical path? Solution: C – D – F – G – I – K (which have zero slack).

7. What is the expected date of completion of this project (assuming a 7-day work week, including July 4 and Labor Day)?

Solution: *Project completion time : 89 days.*

∴ The expected date of completion of this project is August 29 (i.e., 89 days from June 1).

8. Chocolate Cube has committed himself to a recording session beginning September 8 (99 days from now). What is the probability that he will be able to begin recording in his own personal studio on that date?

Solution: *The variance of a sum of random variables is equal to the sum of the variances, and so we sum the variances of the critical activities which were computed in (1) above:*

Activity	μ	σ^2
A	3	1.7778
B	5	1.3611
C	6*** critical	4.0000***
D	8*** critical	5.4444***
E	6	7.1111
F	32*** critical	21.7778***
G	22*** critical	13.4444***
H	11	8.0278
I	16*** critical	4.0000***
J	20	32.1111
K	5*** critical	0.0000***
	SUM =	48.6667

Standard deviation is $\sigma = \sqrt{48.6667} = 6.9762$

The duration of the project is therefore assumed (according to PERT) to have a normal distribution.

∴ The completion time for the project is N(89,6.9762)

The probability that the project is completed within 99 days is therefore

$$\backslash P\{T \leq 99\} = \mathrm{P}\left\{\frac{\mathrm{T}-89}{6.9762} \leq \frac{99-89}{6.9762}\right\}$$

$= \mathrm{P}\{X \leq 1.433\}$ where X is N(0,1).

≈ 92.36 %

(found by consulting standard N(0,1) probability tables)

9. If his studio is not ready in 99 days, Chocolate Cube will be forced to lease his record company's studio, which will cost $120,000. For $3,5000 extra, Eagle Electric, the company hired for the electrical installation (activity G) will work double time; each of the time estimates for this activity will therefore be reduced by 50%. Using an expected cost approach, determine if the $3,500 should be spent.

Solution:

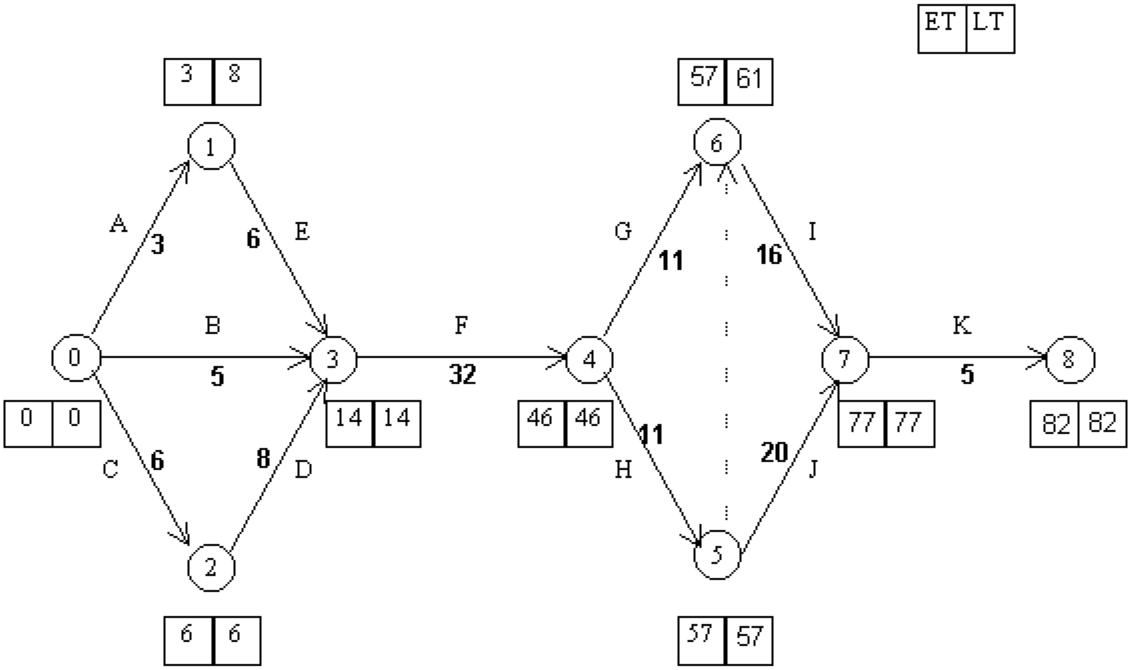
Under the current schedule, the *expected cost* of leasing the studio is (1 - 0.9236)×120000 = $9172.80.

If the electrical installation is expedited using doubletime:

⇒ For activity G, the mean value μ is reduced from 22 days to 11 days,

and the variance σ^2 from 13.444 to $\left(\frac{b-a}{6}\right)^2 = \left(\frac{18.5-7.5}{6}\right)^2 = 3.361$

The critical path analysis is then performed with the revised duration for activity G:



⇒ Critical path will be changed to C – D – F – H – J – K
and Expected completion time will be reduced to 82 days from 89 days.

The variance of the project completion time will now be found by summing
4.000 + 5.4443 + 21.7781 + 8.0276 +32.1115+0.000 = 71.3615.

⇒ Probability that his studio is ready in 99 days is computed as below :

Activity	μ	σ^2
A	3	1.7777
B	5	1.3612
C	6*** critical	4.0000***
D	8*** critical	5.4443***
E	6	7.1113
F	32*** critical	21.7781***
G	11	13.4447
H	11*** critical	8.0276***
I	16	4.0000
J	20*** critical	32.1115***
K	5*** critical	0.0000***
	SUM =	71.3615

The standard deviation of project completion time is

$$\sigma = \sqrt{71.3615} = 8.4476$$

∴ The completion time for the project has (under the assumptions of PERT), the normal distribution
N(82, 8.4476)
Therefore, the probability that the project is completed within 99 days is now

$$\backslash\ P\{T \le 99\} = P\left\{\frac{T-82}{8.4476} \le \frac{99-82}{8.4476}\right\}$$

$$= P\{X \le 2.013\}$$

$$\approx 97.78\ \%$$

- Expected cost of project without paying extra $3,500 :

  Þ (1-0.9236)*($120,000) = $9,168

- Expected cost of project if the electrical installation is done with overtime, costing an extra $3,500 :

  Þ ($3,500) + (1-0.9778)×($120,000) =$3500 + $2664 = $6164.

which is an expected savings of $9168 – $6164 = $3004.

Þ *Therefore, the extra $3,500 should be spent to expedite the electrical installation.*

```
       X22          0.000000        127.500000
       X32          1.000000         80.000000
       X42          0.000000         36.000000
       X52          1.000000         54.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)       125.000000          0.000000
        3)        40.000000          0.000000
        4)        20.000000          0.000000
        5)         2.000000          0.000000
        6)         0.000000          0.000000
        7)         0.000000          0.000000
        8)         0.000000          0.000000
        9)         0.000000          0.000000
       10)         0.000000          0.000000

 NO. ITERATIONS=        7
 BRANCHES=    0 DETERM.=  1.000E    0
```

Optimal decision :

- ***Students from district 1 are sent to school 1 (a distance of 1 mile)***
- ***Students from district 2 are sent to school 1 (a distance of 0.5 mile)***
- ***Students from district 3 are sent to school 2 (a distance of 0.8 mile)***
- ***Students from district 4 are sent to school 1 (a distance of 1.3 mile)***
- ***Students from district 5 are sent to school 2 (a distance of 0.6 mile)***

Corresponding total distance traveled by students is 398.5 miles (which is an average of 0.857 miles for each of the 465 students, ranging from 0.5 mile to 1.3 mile.)

$$+ \ \{(80+30)*2.0\}\ X12 \ + \ \{(70+\ 5)*1.7\}\ X22 \ + \ \{(90+10)*0.8\}\ X32$$
$$+ \ \{(50+40)*0.4\}\ X42 \ + \ \{(60+30)*0.6\}\ X52$$

s.t.

Minimum enrollment at schools:

$$(80+30)\ X11 + (70+5)\ X21 + (90+10)\ X31 + (50+40)\ X41 + (60+30)\ X51 \ \geq\ 150$$

$$(80+30)\ X12 + (70+5)\ X22 + (90+10)\ X32 + (50+40)\ X42 + (60+30)\ X52 \ \geq\ 150$$

Minimum proportion of black students in each school:

$$\frac{30\ X11 + 5\ X21 + 10\ X31 + 40\ X41 + 30\ X51}{(80+30)\ X11 + (70+5)\ X21 + (90+10)\ X31 + (50+40)\ X41 + (60+30)\ X51} \ \geq\ 0.2$$

$$\frac{30\ X12 + 5\ X22 + 10\ X32 + 40\ X42 + 30\ X52}{(80+30)\ X12 + (70+5)\ X22 + (90+10)\ X32 + (50+40)\ X42 + (60+30)\ X52} \ \geq\ 0.2$$

"Multiple choice" constraints: Each district is to be assigned to one of the two schools:

$X11 + X12 = 1$, $X21 + X22 = 1$, $X31 + X32 = 1$, $X41 + X42 = 1$, $X51 + X52 = 1$

LINDO input

```
Min     110 X11 +  37.5 X21 + 80 X31 + 117 X41 + 135 X51
     + 220 X12 + 127.5 X22 + 80 X32 +  36 X42 +  54 X52
s.t.
110 X11 + 75 X21 + 100 X31 + 90 X41 + 90 X51   >= 150
110 X12 + 75 X22 + 100 X32 + 90 X42 + 90 X52   >= 150
8X11 - 10X21 - 10X31 + 22X41 + 12X51 >= 0
8X12 - 10X22 - 10X32 + 22X42 + 12X52 >= 0
X11 + X12 = 1
X21 + X22 = 1
X31 + X32 = 1
X41 + X42 = 1
X51 + X52 = 1

END

INTE 10
```

(Here, zero/one variable (binary) restrictions are imposed by the command INTE.)

LINDO output

```
LP OPTIMUM FOUND AT STEP       6
 OBJECTIVE VALUE =    324.863647

 NEW INTEGER SOLUTION OF    398.500000     AT BRANCH      0 PIVOT        6
 RE-INSTALLING BEST SOLUTION...

        OBJECTIVE FUNCTION VALUE

        1)      398.5000

  VARIABLE        VALUE          REDUCED COST
       X11         1.000000        110.000000
       X21         1.000000         37.500000
       X31         0.000000         80.000000
       X41         1.000000        117.000000
       X51         0.000000        135.000000
       X12         0.000000        220.000000
```

LINDO output

```
         OBJECTIVE FUNCTION VALUE
         1)      12.00000

    VARIABLE        VALUE          REDUCED COST
         RS         1.000000         -6.000000
         BS         0.000000         -5.000000
         DE         1.000000         -3.000000
         ST         1.000000         -3.000000
         TS         0.000000         -2.000000

    NO. ITERATIONS=       6
    BRANCHES=    0 DETERM.=  1.000E    0
```

The Cubs should sign Rick Sutcliffe (RS) , Tim Stoddard (TS) , and Steve Trout (ST) . This would result in 12 victories.

2. **Integer Programming Formulation.** *(#4, p. 547, O.R. text, W. Winston)* A court decision has stated that the enrollment of each high school in Metropolis must be at least 20% black. The numbers of black and white high school students in each of the city's five school districts are shown in the table below.

District	Whitestudents	Black students
1	80	30
2	70	5
3	90	10
4	50	40
5	60	30

The distance (in miles) that a student in each district must travel to each high school is:

District	HS #1	HS #2
1	1	2
2	0.5	1.7
3	0.8	0.8
4	1.3	0.4
5	1.5	0.6

School board policy requires that all the students in a given district attend the same school. Assuming that each school must have an enrollment of at least 150 students, formulate an integer LP that will minimize the total distance that Metropolis students must travel to high school. Find the solution, using LINDO (or equivalent) software.

Solution:

Decision Variables :

$$Xij = \begin{cases} 1 , & \text{if students from district i are sent to school j} \\ 0 , & \text{otherwise} \end{cases}$$

Integer Programming Formulation :

The objective is to minimize the total distance students travel (which would be equivalent to minimizing the average distance traveled), so the coefficient of Xij is the population of district i times the distance from district i to school j.

Min $\{(80+30)*1.0\}$ X11 + $\{(70+ 5)*0.5\}$ X21 + $\{(90+10)*0.8\}$ X31
+ $\{(50+40)*1.3\}$ X41 + $\{(60+30)*1.5\}$ X51

56:171 Operations Research
Homework #9 Solution -- Fall 2000

1. **Integer Programming Formulation** *(#5, page 547, of O.R. text by W. Winston)* The Cubs are trying to determine which of the following free agent pitchers should be signed: Rick Sutcliffe (RS), Bruce Sutter (BS), Dennis Eckersley (DE), Steve Trout (ST), Tim Stoddard (TS). The cost of signing each pitcher and the number of victories each pitcher will add to the Cubs are shown below.

Pitcher	Cost of signing ($million)	Right- or Left-handed?	Victories added to Cubs
RS	$6	Right	6
BS	$4	Right	5
DE	$3	Right	3
ST	$2	Left	3
TS	$2	Right	2

Subject to the following restrictions, the Cubs want to sign the pitchers who will add the most victories to the team.

- At most $12 can be spent.
- If DE and ST are signed, then BS cannot be signed.
- At most two right-handed pitchers can be signed.
- The cubs cannot sign both BS and RS.

Formulate an integer LP to help the Cubs determine whom they should sign. Solve the problem, using LINDO (or equivalent) software.

Solution:

Decision Variables :

$$\text{RS} = \begin{cases} 1\text{ , } \textit{if RS is asigned} \\ 0\text{ , otherwise} \end{cases} \qquad \text{BS} = \begin{cases} 1\text{ , } \textit{if BS is asigned} \\ 0\text{ , otherwise} \end{cases}$$

$$\text{DE} = \begin{cases} 1\text{ , } \textit{if DE is asigned} \\ 0\text{ , otherwise} \end{cases} \qquad \text{ST} = \begin{cases} 1\text{ , } \textit{if ST is asigned} \\ 0\text{ , otherwise} \end{cases}$$

$$\text{TS} = \begin{cases} 1\text{ , } \textit{if TS is asigned} \\ 0\text{ , otherwise} \end{cases}$$

Integer Programming Formulation :

Max 6 RS + 5 BS + 3 DE + 3 ST + 2 TS

s.t. 6 RS + 4 BS + 3 DE + 2 ST + 2 TS <= 12 *(budgetconstraint)*

DE + ST + BS <= 2 *(if DE & ST are signed, then BS cannot be)*

RS + BS + DE + TS <= 2 *(at most two right-handed pitchers)*

BS + RS <= 1 *(cannot sign both BS & RS)*

LINDO input

```
MAX     6 RS + 5 BS + 3 DE + 3 ST + 2 TS
SUBJECT TO
6 RS + 4 BS + 3 DE + 2 ST + 2 TS <=   12
BS + DE + ST <=   2
RS + BS + DE + TS <=   2
RS + BS <=   1
END
INTE     5
```

(Here, zero/one variable (binary) restrictions are imposed by the command INTE)

56:171 Operations Research
Homework #10 Solution -- Fall 2000

1. Markov Chains. *(Based upon Exercise 4, §19.5, page 982 of text by W. Winston)*

At the beginning of each year, my car is in good, fair, or broken-down condition. A good car will be good at the beginning of next year with probability 85%, fair with probability 10%, or broken-down with probability 5%. A fair car will be fair at the beginning of the next year with probability 75%, or broken-down with probability 25%. It costs $9000 to purchase a good car; a fair car can be traded in for $2500; and a broken-down car has no trade-in value and must immediately be replaced by a good car. It costs $1000 per year to operate a good car and $1500 to operate a fair car. Should I replace my car as soon as it becomes a fair car, or should I drive my car until it breaks down? Assume that the cost of operating a car during a year depends on the type of car on hand at the *beginning* of the year (after a new car, if any, arrives).

Define a Markov chain model with three states (Good, Fair, & Broken-down). Assume, as implied by the problem statement, that break-down occurs only at the *end* of a year, and then (at the beginning of the next year) the broken-down car "must immediately be replaced". For each of the two replacement policies mentioned, answer the following questions. *Note: assume that* ***state 1= Good, state 2= Fair, and state 3= Broken-down***.

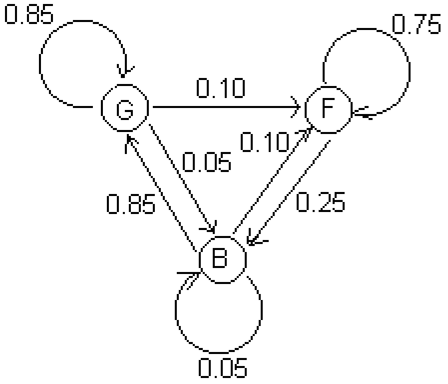
Policy I: "Drive my car until it breaks down"

a. Draw a diagram of the Markov chain and write down the transition probability matrix.

Solution:

Keep in mind that a year passes between observations, so that if the car is observed to be broken down, it is replaced and may deteriorate during the year before it is again observed.

$$P = \begin{bmatrix} .85 & .1 & .05 \\ 0 & .75 & .25 \\ .85 & .1 & .05 \end{bmatrix}$$



b. Write down the equations which could be solved to obtain the steadystate probabilities.

Solution:

The equations $\pi=\pi P$ are

$$\begin{cases} p_1 = 0.85p_1 + 0.85p_3 \\ p_2 = 0.10p_1 + 0.75p_2 + 0.1p_3 \\ p_3 = 0.05p_1 + 0.25p_2 + 0.05p_3 \end{cases}$$

Using any two of these equations and the equation

$$p_1 + p_2 + p_3 = 1$$

c. Solve the equations, either manually or using appropriate computer software.

Solution:

```
|i| name    | P{i}
|1|  GOOD   | 0.60714
|2|  FAIR   | 0.28571
|3|  BROKEN | 0.10714
```

d. Compute the average cost per year for the replacement policy.

Solution:

If the car is in good condition at the end of the year, its operating cost would have been $1000 in that year, and likewise if in fair condition, $1500. If it is broken down, it must be replaced for $9000 and then operated during the next year (in good condition) for $1000. Thus the expected cost is

$$\$1000p_1 + \$1500p_2 + (\$9000 + \$1000)p_3 = \$2107.10$$

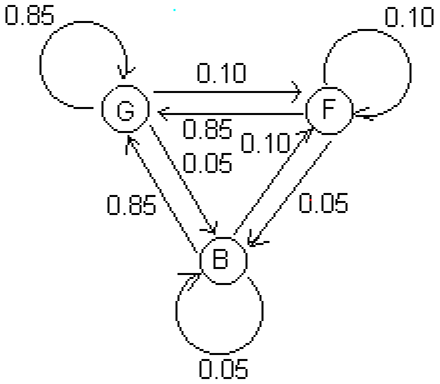
e. What is the expected time between break-downs?
Solution:
The expected time between breakdowns (*mean recurrence time*) is

$$m_{33} = 1/p_3 = 1/0.10714 = 9.333 \text{ years}$$

Policy II: "Replace car when in fair condtion"

a. Draw a diagram of the Markov chain and write down the transition probability matrix.
Solution:
Under this policy, we begin every year with a car in good condition, and so

$$P = \begin{bmatrix} .85 & .1 & .05 \\ .85 & .1 & .05 \\ .85 & .1 & .05 \end{bmatrix}$$



b. Write down the equations which could be solved to obtain the steadystate probabilities.
Solution:
The equations $\pi = \pi P$ are

$$\begin{cases} p_1 = 0.85p_1 + 0.85p_2 + 0.85p_3 \\ p_2 = 0.10p_1 + 0.10p_2 + 0.10p_3 \\ p_3 = 0.05p_1 + 0.05p_2 + 0.05p_3 \end{cases}$$

together with

$$p_1 + p_2 + p_3 = 1$$

c. Solve the equations, either manually or using appropriate computer software.
Solution:
It is obvious that the limiting distribution is

```
|i| name     | P{i}
|1|  GOOD    | 0.85
|2|  FAIR    | 0.10
|3|  BROKEN  | 0.05
```

since the rows of P are identical.

d. Compute the average cost per year for the replacement policy.
Solution:
Since every year begins with a good car, the operating cost every year is assumed to be \$1000. The cost when the car is in fair condition at the end of the year includes the replacement cost minus the trade-in value. Thus:

$$\$1000p_1 + (\$9000 - \$2500 + \$1000)p_2 + (\$9000 + \$1000)p_3 = \$2100$$

f. What replacement policy do you recommend?
Solution:
Based upon this Markov chain model, and taking into consideration only the economic values, the recommendation would be to follow Policy I: "Replace car only when it breaks down".
(Of course, the added \$7/year cost of the other policy is small enough that I would decide to follow Policy II and drive a car in better condition, both for aesthetic reasons as well as to avoid the inconvenience of breakdowns!)

2. Consider a *reorder-point/order-up-to* type of inventory control system, sometimes referred to as *(s,S)*. Suppose that the inventory is counted at the end of the week (Saturday evening), and if s=2 or fewer items remain, enough is ordered to bring the level up to S=8 before the business reopens on Monday morning. The probability distribution of demand is:

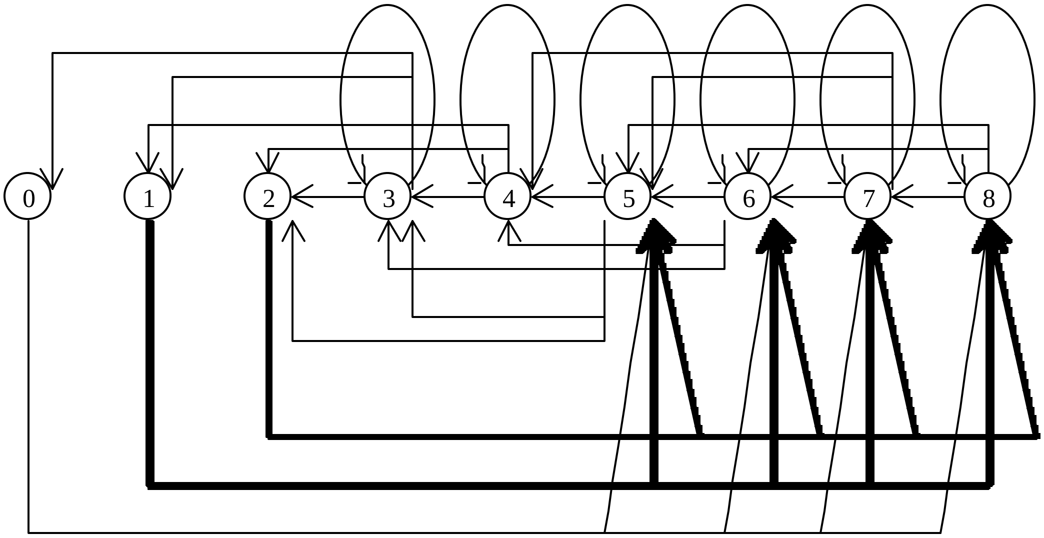
P{D=0}= 0.15 P{D=1}= 0.25 P{D=2}=0.4 P{D=3}= 0.2

a. What are the states in the Markov Chain model of this system? (That is, how many states are there, and what does each state signify?)

The state of the system is defined according to the stock-on-hand (SOH) at the end of the week (Saturday evening) before replenishment occurs, i.e.,

Xn = 0 1 2 3 4 5 6 7 8
SOH = 0 1 2 3 4 5 6 7 8

b. Draw the diagram for this Markov Chain.



c. Write the transition probability matrix.

from\to	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0.2	0.4	0.25	0.15
1	0	0	0	0	0	0.2	0.4	0.25	0.15
2	0	0	0	0	0	0.2	0.4	0.25	0.15
3	0.2	0.4	0.25	0.15	0	0	0	0	0
4	0	0.2	0.4	0.25	0.15	0	0	0	0
5	0	0	0.2	0.4	0.25	0.15	0	0	0
6	0	0	0	0.2	0.4	0.25	0.15	0	0
7	0	0	0	0	0.2	0.4	0.25	0.15	0
8	0	0	0	0	0	0.2	0.4	0.25	0.15

For the following questions, consult the computations below.

d. Over a long period of time, what is the percent of the weeks in which you would expect there to be a stockout (zero inventory)? 3 % (From steady state probability π_0 = 0.0310024)

e. What will be the average end-of-week inventory level?

$$\sum_{i=0}^{8} i \cdot p_i \;=\; 0\cdot(0.0310024) + 1\cdot(0.0907444) + \cdots\cdots + 8\cdot(0.0440368) \;=\; 4$$

f. How often (i.e. once every how many weeks?) will the inventory be full at the end of the week?

i.e., what is the average number of weeks between SOH = 8

Since the steady state probability that SOH = 8 is $p_8 = 0.0440368$

The frequency that the system visits the state #8 is therefore the reciprocal of this probability, i.e., $\frac{1}{0.0440368} = 22.71$

g. How often will the inventory be restocked?

i.e., what is the average number of weeks between restocking?

Since the steady state probability that the inventory is restocked is

$$\sum_{i=0}^{2} p_i = 0.0310024+0.0907444+0.127795=0.2495418$$

The frequency that the system visits the set of states {0,1,2} is therefore the reciprocal of this probability, i.e., $\frac{1}{0.2495418} = 4.007$

h. What is the expected number of weeks, starting with a full inventory, until a stockout occurs?

$m_{80} = 32.256$ (weeks)

i. Starting with a full inventory, what is the expected number of stockouts during the first 20 weeks?

$$\sum_{n=0}^{20} P_{80}^{(n)} = 0.569$$

What is the expected number of times that the inventory is restocked?

$$\sum_{n=0}^{20} P_{80}^{(n)} + \sum_{n=0}^{20} P_{81}^{(n)} + \sum_{n=0}^{20} P_{82}^{(n)} = 0.569 + 1.678 + 2.411 = 4.658$$

j. This inventory system was simulated ten times for 20 weeks, starting in state 8.

- In each simulated history, what is the number of stockouts during the first 20 weeks?
- In each simulated history, how many times did restocking occur?
- Compute the average number of stockouts and restocking of the inventory during the ten 20-week intervals which were simulated. How do these values compare with the answers you found in (i)?

Simulation#	*# of stockouts*	*# of restocking*
1	*0*	*6*
2	*0*	*4*
3	*0*	*5*
4	*1*	*5*
5	*0*	*4*
6	*1*	*6*
7	*0*	*4*
8	*0*	*5*
9	*1*	*5*
10	*0*	*3*

From the simulation

the average # of stockouts = 0.3

the average # of restocking = 4.7

We can say these simulation results are similar to the computational results we found in (i), there is 47% gap in the average # of stockouts, though.

P^2 = square of transition probability matrix:

```
  \to  0    1    2      3      4      5      6      7      8
From|
  0 |0    0    0.04   0.16   0.26   0.26   0.1825 0.075  0.0225
  1 |0    0    0.04   0.16   0.26   0.26   0.1825 0.075  0.0225
  2 |0    0    0.04   0.16   0.26   0.26   0.1825 0.075  0.0225
  3 |0.03 0.06 0.0375 0.0225 0      0.17   0.34   0.2125 0.1275
  4 |0.05 0.13 0.1225 0.075  0.0225 0.12   0.24   0.15   0.09
  5 |0.08 0.21 0.23   0.1825 0.075  0.0625 0.08   0.05   0.03
  6 |0.04 0.16 0.26   0.26   0.1825 0.075  0.0225 0      0
  7 |0    0.04 0.16   0.26   0.26   0.1825 0.075  0.0225 0
  8 |0    0    0.04   0.16   0.26   0.26   0.1825 0.075  0.0225
```

Sum of first 20 powers of P:

```
from\to  0     1     2     3     4     5     6     7     8
 0     0.569 1.678 2.411 2.999 2.847 3.251 3.457 1.841 0.947
 1     0.569 1.678 2.411 2.999 2.847 3.251 3.457 1.841 0.947
 2     0.569 1.678 2.411 2.999 2.847 3.251 3.457 1.841 0.947
 3     0.768 2.042 2.555 2.993 2.678 3.066 3.265 1.739 0.895
 4     0.591 1.914 2.774 3.109 2.805 3.011 3.208 1.709 0.880
 5     0.632 1.812 2.670 3.316 2.880 3.084 3.101 1.653 0.851
 6     0.608 1.807 2.541 3.195 3.099 3.154 3.174 1.598 0.823
 7     0.579 1.728 2.506 3.050 2.993 3.372 3.249 1.725 0.798
 8     0.569 1.678 2.411 2.999 2.847 3.251 3.457 1.841 0.947
```

Steady state distribution:

```
|i| name   | P{i}
|0|  SOH.0 | 0.0310024
|1|  SOH.1 | 0.0907444
|2|  SOH.2 | 0.127795
|3|  SOH.3 | 0.155012
|4|  SOH.4 | 0.143698
|5|  SOH.5 | 0.157814
|6|  SOH.6 | 0.163551
|7|  SOH.7 | 0.0863466
|8|  SOH.8 | 0.0440368
```

First Passage Probabilities

n	$f_{80}^{(n)}$
1	0
2	0
3	0.032
4	0.0459
5	0.0295
6	0.023933
7	0.028956
8	0.02948
9	0.026245
10	0.024937
11	0.025001
12	0.02431
13	0.023174
14	0.022383
15	0.02178
16	0.021073
17	0.020333
18	0.019668
19	0.019042
20	0.018418
sum:	0.45613

Mean First Passage array:

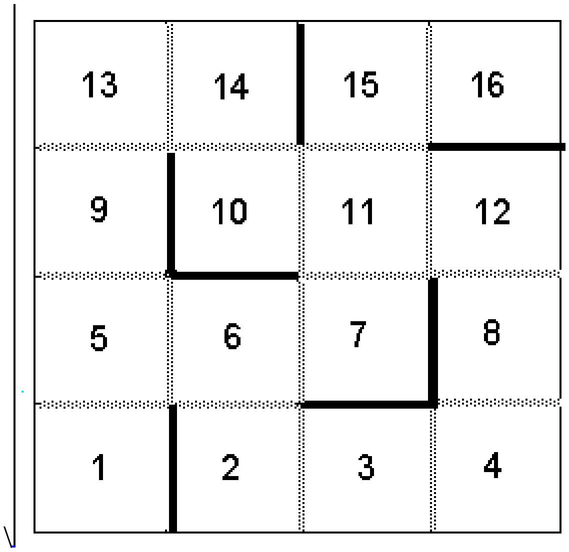
```
 \to    0        1       2       3      4      5      6      7      8
From|
  0 |32.256 11.02   7.825  6.4131 6.6646 5.2769 4.383  10.239 22.708
  1 |32.256 11.02   7.825  6.4131 6.6646 5.2769 4.383  10.239 22.708
  2 |32.256 11.02   7.825  6.4131 6.6646 5.2769 4.383  10.239 22.708
  3 |25.842  7.0106 6.7    6.4511 7.8411 6.4534 5.5595 11.415 23.885
  4 |31.546  8.4243 4.9883 5.7033 6.959  6.7994 5.9055 11.761 24.231
  5 |30.205  9.5462 5.7966 4.3629 6.4345 6.3366 6.5609 12.417 24.886
  6 |30.986  9.5981 6.8052 5.1436 4.9139 5.8946 6.1143 13.049 25.519
  7 |31.927 10.474  7.0795 6.0844 5.6498 4.51   5.6535 11.581 26.094
  8 |32.256 11.02   7.825  6.4131 6.6646 5.2769 4.383  10.239 22.708
```

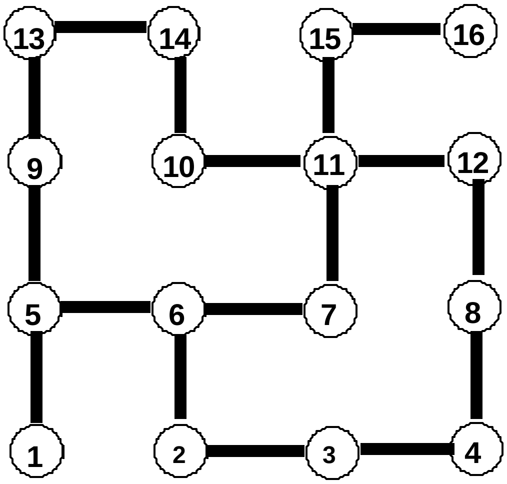
Results of simulation:

```
r                               1 1 1 1 1 1 1 1 1 1 2
u    0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0
n
 1)  8 5 3 1 8 5 2 6 3 2 6 5 3 2 6 5 2 7 5 3 1
 2)  8 7 5 4 3 1 5 3 2 8 5 3 1 6 3 2 8 8 7 6 3
 3)  8 7 5 4 3 1 6 6 4 1 6 4 2 6 5 3 1 8 6 3 2
 4)  8 5 4 2 7 5 3 0 7 4 3 2 6 4 2 6 5 3 1 8 5
 5)  8 6 3 1 8 7 6 3 1 8 7 5 3 2 6 3 3 1 6 5 3
 6)  8 5 3 0 8 8 6 3 1 7 4 1 7 4 2 5 3 2 7 5 2
 7)  8 7 5 4 1 8 5 2 7 4 3 1 6 5 5 5 3 3 1 7 5
 8)  8 6 4 2 5 2 6 3 2 8 7 5 4 3 1 6 4 2 6 4 3
 9)  8 8 5 4 2 8 6 4 3 3 1 5 3 1 6 3 0 6 5 2 7
10)  8 7 7 7 6 5 3 3 3 3 1 7 5 4 3 1 5 3 1 7 7
```

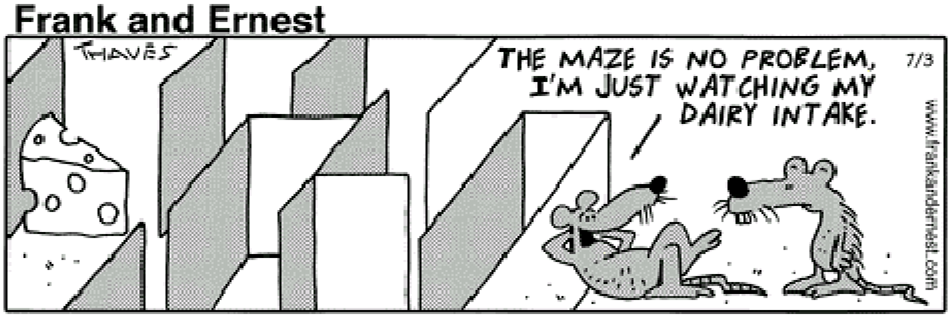
56:171 Operations Research
Homework #11 Solution -- Fall 2000

We wish to model the passage of a rat through a maze. Consider a maze in the form of a 4x4 array of boxes, such as the one below on the left:





The solid lines represent walls, the shaded lines represent doors. We will assume that a rat is placed into box #1. While in any box, the rat is assumed to be equally likely to choose each of the doors leaving the box (including the one by which he entered the box). For example, when in box #2 above, the probability of going next to boxes 3 and 6 are each $^{1}/_{2}$, regardless of the door by which he entered the box. *This assumption implies that no learning takes place if the rat tries the maze several times!*



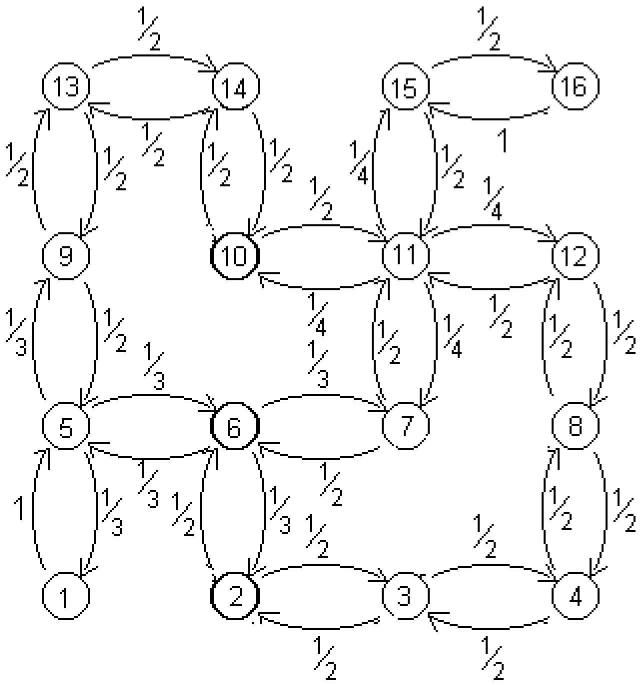
Copyright (c) 1998 by Thaves. Distributed from www.thecomics.com.

Transition probabilities:

```
    1     2     3   4   5     6     7      8   9    10   11  12   13  14  15   16
 1) 0     0     0   0   1     0     0      0   0     0    0   0    0   0   0    0
 2) 0     0     0.5 0   0     0.5   0      0   0     0    0   0    0   0   0    0
 3) 0     0.5   0   0.5 0     0     0      0   0     0    0   0    0   0   0    0
 4) 0     0     0.5 0   0     0     0      0.5 0     0    0   0    0   0   0    0
 5) 0.333 0     0   0   0     0.333 0      0   0.333 0    0   0    0   0   0    0
 6) 0     0.333 0   0   0.333 0     0.333  0   0     0    0   0    0   0   0    0
 7) 0     0     0   0   0     0.5   0      0   0     0    0.5 0    0   0   0    0
 8) 0     0     0   0.5 0     0     0      0   0     0    0   0.5  0   0   0    0
 9) 0     0     0   0   0.5   0     0      0   0     0    0   0    0.5 0   0    0
10) 0     0     0   0   0     0     0      0   0     0    0.5 0    0   0.5 0    0
11) 0     0     0   0   0     0     0.25   0   0     0.25 0   0.25 0   0   0.25 0
12) 0     0     0   0   0     0     0      0.5 0     0    0.5 0    0   0   0    0
13) 0     0     0   0   0     0     0      0   0.5   0    0   0    0   0.5 0    0
14) 0     0     0   0   0     0     0      0   0     0.5  0   0    0.5 0   0    0
15) 0     0     0   0   0     0     0      0   0     0    0.5 0    0   0   0    0.5
16) 0     0     0   0   0     0     0      0   0     0    0   0    0   0   1    0
```

a. On the diagram representing the Markov chain, write the transition probabilities on each transition in each direction.

Solution:



b. Write one of the equations (other than $\sum_i p_i = 1$) that determines the steady-state distribution of the rat's location.

Solution: The equations are of the form:

π_i = inner product of π and column i of the matrix P, for example:

$p_1 = 0.333p_5$, and $p_2 = 0.5p_3 + 0.333p_6$

c. In steady state, which box will be visited most frequently by the rat?

Solution:

The state with the maximum steady-state probability is #11, with π_{11} = 11.8%. That is, we expect that the rat will be in box #11 after 11.8% of his moves.

d. Suppose that in box #16 a reward (e.g. food) is placed. What is the expected number of moves of the rat required to reach this reward from box #1?

Solution:

The mean first passage time $m_{1,16}$ is 87.3; that is, he will require an average of 87.3 moves to reach the food.

e. Count the minimum number of moves (M) required to reach the reward from box #1. What is the probability that the rat reaches the reward in **exactly** this number of moves?

Solution:

The shortest path from box #1 to box #16 consists of six moves. The first passage probability $f_{1,16}^{(6)}$ is 0.694%.

f. What is the probability that the rat reaches the reward with no more than 4 unnecessary moves?

Solution:

$$\sum_{n \le 10} f_{1,16}^{(n)} = 0.00694 + 0.0126 + 0.0165 = 0.03604$$

g. If the food were placed in box #7, into which box should we place the rat, if we want the largest expected number of moves to find the food?

Solution:

The mean first passage time $m_{i,7}$ is maximum (26) for i=13.

Mean first passage times:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1)	34	31	43.6	49.1	1	11.3	23.1	47.6	19.6	33	23.3	39	31.1	35.6	54.3	87.3
2)	59.7	17	18.3	29.5	26.7	8.67	20.5	33.6	39.6	36	20.7	30.7	45.5	44.3	51.7	84.7
3)	65.2	11.2	17	15.7	32.2	15.3	23.7	24.4	44	37	20.5	26	48.7	46.4	51.5	84.5
4)	68.7	20.4	13.7	17	35.7	20	25	13.2	46.4	36	18.4	19.3	50	46.5	49.4	82.4
5)	33	30	42.6	48.1	11.3	10.3	22.1	46.6	18.6	32	22.3	38	30.1	34.6	53.3	86.3
6)	52.1	20.8	34.5	41.2	19.1	11.3	15.2	40.8	33.2	33	18.8	33.3	40.2	40.1	49.8	82.8
7)	60.7	29.4	39.7	43	27.7	12	17	39.2	38.4	28	10.4	28.3	42	38.5	41.4	74.4
8)	70.3	27.6	25.5	16.3	37.3	22.7	24.3	17	46.8	33	14.3	10.7	49.3	44.7	45.3	78.3
9)	43.9	35.2	46.7	51.1	10.9	16.7	25.1	48.4	17	27	21.9	38.7	16.1	25.1	52.9	85.9
10)	64.5	38.8	46.9	47.9	31.5	23.7	21.9	41.8	34.2	17	8.47	28.7	29.9	18.5	39.5	72.5
11)	67.3	36	42.9	42.8	34.3	22	16.8	35.6	41.6	21	8.5	21.3	41.8	34.9	31	64
12)	69.8	32.8	35.2	30.5	36.8	23.3	21.5	18.8	45.2	28	8.13	17	46.5	40.8	39.1	72.1
13)	52.7	38.4	48.7	52	19.7	21	26	48.2	13.4	20	19.4	37.3	17	13.5	50.4	83.4
14)	59.6	39.6	48.8	50.9	26.6	23.3	24.9	46	24.8	11	14.9	34	15.9	17	45.9	78.9
15)	70.3	39	45.9	45.8	37.3	25	19.8	38.6	44.6	24	3	24.3	44.8	37.9	17	33
16)	71.3	40	46.9	46.8	38.3	26	20.8	39.6	45.6	25	4	25.3	45.8	38.9	1	34

First-passage probabilities:

n	$I_{1,16}$	$p_{1,16}$
---	----	---------
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0.00694	0.00694
7	0	0
8	0.0126	0.0161
9	0	0
10	0.0165	0.025
11	0	0
12	0.0189	0.0326
13	0	0
14	0.0203	0.0388
15	0	0
16	0.021	0.0437
17	0	0
18	0.0213	0.0474
19	0	0
20	0.0213	0.0503
21	0	0
22	0.0211	0.0524
23	0	0
24	0.0208	0.0541
25	0	0
26	0.0204	0.0553
27	0	0
28	0.02	0.0562
29	0	0
30	0.0196	0.0568
sum=	0.241	0.536

Steadystate distribution:

i	πi
1	0.0294
2	0.0588
3	0.0588
4	0.0588
5	0.0882
6	0.0882
7	0.0588
8	0.0588
9	0.0588
10	0.0588
11	0.118
12	0.0588
13	0.0588
14	0.0588
15	0.0588
16	0.0294

* * *

2. Coldspot manufactures refrigerators. The company has issued a warranty on all refrigerators that requires free replacement of any refrigerator that fails before it is three years old.

- 4% of all new refrigerators fail during their first year of operation.
- 6% of all 1-year-old refrigerators fail during their second year of operation.
- 7% of all 2-year-old refrigerators fail during their third year of operation.
- 8% of all 3-year-old refrigerators fail during their fourth year of operation.

Replacement refrigerators are not covered by the warranty.

Define a discrete-time Markov chain, with states

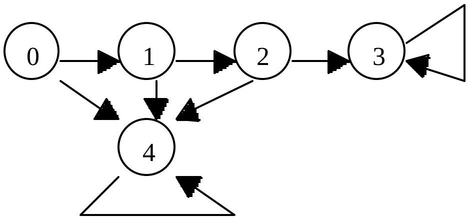
(0.) new refrigerators
(1.) 1-year-old refrigerators
(2.) 2-year-old refrigerators
(3.) refrigerators that have passed their third anniversary
(4.) replacement refrigerators

Define the stages to be years, with the refrigerator observed on the anniversaries of its purchase.

a. Draw the transition diagram and write the transition probability matrix.

Solution: P=

```
 | 0  1    2    3    4
0| 0 0.96 0    0    0.04
1| 0 0    0.94 0    0.06
2| 0 0    0    0.93 0.07
3| 0 0    0    1    0
4| 0 0    0    0    1
```



b. Which states are transient, and which are absorbing?

Solution: States 0, 1, and 2 are transient, while states 3 & 4 are absorbing.

c. Identify the matrices Q (probabilities of transitions between transient states) and R (probabilities of transitions from transient states to absorbing states).

Solution:

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}, \text{where } Q = \begin{bmatrix} 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0.04 \\ 0 & 0.06 \\ 0.93 & 0.07 \end{bmatrix}$$

d. Calculate the matrix E (expected number of visits) and A (absorption probabilities).

Solution:

Expected No. Visits to Transient States

$$E = \left(I - Q\right)^{-1} =$$

```
 \ 0   1    2
0| 1 0.96 0.9024
1| 0 1    0.94
2| 0 0    1
```

Absorption Probabilities

A = ER =

```
\      3       4
0| 0.83923 0.16077
1| 0.8742  0.1258
2| 0.93    0.07
```

e. What fraction of the refrigerators will Coldspot expect to replace?

Solution: a_{04} = 16.077%

f. Suppose that it costs $500 to replace a refrigerator, and that the company sells 10,000 units per year. What is the expected annual replacement cost?

Solution: 0.16077×(10000 refrigerators/year)×$500/refrigerator = $803,850/year.

g. They are considering extending the warranty period to four years. Assuming that this would have no effect on sales, what would be the increased replacement costs?

Solution:

Use a Markov chain with six states, where states 0, 1, and 2 are defined as above, and

(3.) 3-year-old refrigerators
(4.) past fourth anniversary
(5.) replacement

Computing the matrices E and A as before, we obtain:

E =

```
 | 0   1     2      3
0| 1 0.96 0.9024 0.83923
1| 0 1    0.94   0.8742
2| 0 0    1      0.93
3| 0 0    0      1
```

A =

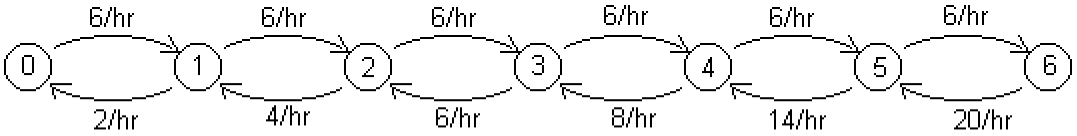
```
 |    4       5
0| 0.77209 0.22791
1| 0.80426 0.19574
2| 0.8556  0.1444
3| 0.92    0.08
```

With the extended 4-year warranty, Coldspot should expect to replace a_{05} = 22.79% of its refrigerators, with an expected annual cost of 0.22791×(10000 refrigerators/year)×$500/refrigerator = $1,139,550/year, an increase of $335,700/year

3. Birth-death model of queue. A parking lot consists of four spaces. Cars making use of these spaces arrive according to a Poisson process at the rate of six cars per hour. Parking time is exponentially distributed with mean of 30 minutes. Cars who cannot find an empty space immediately on arrival may temporarily wait inside the lot until a parked car leaves, but may get impatient and leave before a parking space opens up. Assume that the time that a driver is willing to wait has exponential distribution with an average of 10 minutes. The temporary space can hold only two cars. All other cars that cannot park or find a temporary waiting space must go elsewhere.

a. Model this system as a birth-death process, with states 0, 1, … 6.

Solution:



The "death" (departure) rate for state 5 is the sum of the departure rate of the cars from the parking spaces (2/hour for each space, times 4 spaces) plus the rate at which the waiting driver becomes impatient and leaves (6/hour), i.e., 14/hr. For state 6, there are two waiting cars, and so the departure rate is 8/hr +2×6/hr = 20/hr.

b. Find the steadystate probability distribution of the number of cars in the system.

Solution:

$$\frac{1}{p_0} = 1 + \frac{6}{2} + \left(\frac{6}{2}\times\frac{6}{4}\right) + \left(\frac{6}{2}\times\frac{6}{4}\times\frac{6}{6}\right) + \left(\frac{6}{2}\times\frac{6}{4}\times\frac{6}{6}\times\frac{6}{8}\right) + \left(\frac{6}{2}\times\frac{6}{4}\times\frac{6}{6}\times\frac{6}{8}\times\frac{6}{14}\right) + \left(\frac{6}{2}\times\frac{6}{4}\times\frac{6}{6}\times\frac{6}{8}\times\frac{6}{14}\times\frac{6}{20}\right)$$

= 1+3+4.5+4.5+3.375+1.4464+0.43393 = 18.255

So $p_0 = \frac{1}{18.255} = 0.054778$. Then $\pi_1 = 3\times\pi_0$, $\pi2 = \pi3 = 4.5\times\pi_0$, $\pi4 = 3.375\times\pi_0$, $\pi5 = 1.4464\times\pi_0$, and $\pi6 = 0.43393\times\pi_0$. This gives us:

n	0	1	2	3	4	5	6
π_n	0.054778	0.16434	0.2465	0.2465	0.18488	0.079233	0.02377

c. What is the fraction of the time that there is at least one empty space?

Solution: $\pi_0 + \pi_1 + \pi_2 + \pi_3$= 71.2 %

d. What is the average number of cars in the lot? … the average number of cars waiting?

Solution: $L = \sum_{n=0}^{6} np_n = 2.6751$ & $L_q = 1p_5 + 2p_2 = 0.12677$

e. What is the average arrival rate (keeping in mind that the arrival rate is zero when n=6)?

Solution: $\overline{l} = 6p_0 + 6p_1 + 6p_2 + \cdots + 6p_5 + 0p_6 = 5.8574/\text{hour}$

f. According to Little's Law, what is the average time that a car waits for a parking space?

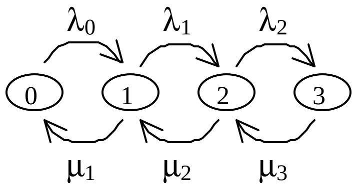
Solution: $L_q = \overline{l}W_q \Rightarrow W_q = {}^{L_q}\!/_{l} = \dfrac{0.12677}{5.8574/hr} = 0.021643$ hour = 1.29 minutes

56:171 Operations Research
Homework #12 Solutions -- Fall 2000

1. Birth/Death Process. A small service station has one gasoline pump. Cars wanting gasoline arrive according to a Poisson process at a mean rate of 15/hour. However, if the pump is already being used, these potential customers may balk (drive on to another service station). The probability that an arriving customer will balk is n/3 for n=1,2,3, where n = number of cars in the station (including the one using the pump.) The time required by a customer to fill a tank and pay the cashier has exponential distribution with a mean of 4 minutes.

(a.) Construct the diagram showing the birth & death rates.

Solution:



Here, the "birth rates", i.e., the rates at which the population of cars increases or enter the station, are $\lambda_0 = 15$ / hr, $\lambda_1 = \left(1-\frac{1}{3}\right)\times$ (15 / hr) = 10 / hr ,

$\lambda_2 = \left(1-\frac{2}{3}\right)\times$ (15 / hr)= 5 / hr, and $\lambda_3 = \left(1-\frac{3}{3}\right)\times$ (15 / hr)= 0,

while the "death rates", i.e., the rates at which the population of cars decreases or leaves the station, are $\mu_1 = \mu_2 = \mu_3 = 15$ / hr.

(b.) Compute the steady state probability distribution of the number of cars in the station.

Solution: The steady state distribution is found by first computing

$$\frac{1}{p_0} = 1 + \left(\frac{l_0}{m_1}\right) + \left(\frac{l_0}{m_1}\times\frac{l_1}{m_2}\right) + \left(\frac{l_0}{m_1}\times\frac{l_1}{m_2}\times\frac{l_2}{m_3}\right)$$

$\Rightarrow$ $\pi_0 = 0.346$ and then $\pi_1 = \left(\frac{15}{15}\right)\pi_0 = 0.346$, $\pi_2 = \left(\frac{15}{15}\times\frac{10}{15}\right)\pi_0 = 0.231$,

$\pi_3 = \left(\frac{15}{15}\times\frac{10}{15}\times\frac{5}{15}\right)\pi_0 = 0.077$

(c.) Compute the average number of cars *waiting* in the station.

Solution: The average # of cars waiting in the station $= L_q = \pi_2 + 2\,\pi_3 = 0.3847$

(d.) Compute the average arrival rate $\underline{\lambda}$.

Solution: $\overline{l} = \sum_{i=0}^{3} l_i\, p_i = 9.8083$/hr.

(e.) How many customers are expected per day if the station is open 12 hours?

Solution: The expected number of customers per day is the average arrival rate times the length of the day, i.e., (9.8083/hr)×12 hr. = 117.7

(f.) What is the average time that a customer waits for use of the pump?

Solution: $L_q = \bar{l} W_q \Rightarrow W_q = \frac{L_q}{\bar{l}} = \frac{0.3847}{9.8083/\text{hr}} = 0.1059 \text{ hr} = 6.354 \text{ minutes}$.

2. Deterministic Dynamic Programming Model: Power Plant Capacity Planning (see class notes):

This DP model schedules the construction of powerplants over a six-year period, given

R[t] = cumulative number of plants required at the end of year t (t=1,2,…6)
C[t] = cost per plant (in $millions) during year t

Year t	C_t	R_t
1	5.4	1
2	5.5	2
3	5.6	4
4	5.7	6
5	5.8	7
6	5.9	8

Eight plants are to be constructed during the six-year period, with a restriction that no more than three may be built during each one-year period. Any year in which a plant is built, a cost of $1.2 million is incurred (independent of number of plants built). A discount factor β = 85% is used to account for the time value of money, i.e., $1 spent a year from today is equivalent to $0.85 spent today.

In addition to a difference in the cost data, the computer output below differs from that in the notes in that the stages are numbered in *increasing* order, i.e., t=1 is the first year and t=6 is the final year.

Complete the computations for stage 1 (values in boxes):

Solution:

(a.) 35.6428 That is, if the *state* of the system (that is, the number of plants built at the beginning of year 1) is 0 and the *decision* (that is, the number of plants to be built in year 1) is 2, then the present value of total cost for years 1 through 6 will be the construction cost in year 1 plus the discount factor times the minimum present value of cost in years 2 through 6 if 2 plants have been built at the beginning of year 2.
This is 1.2 + 2*5.4 + 0.85*f(2) , where f(2) = 27.468

(b.) 35.3478 That is, $f_1(0)$ which is the minimum cost of years 1 through 6 if 0 plants have been built prior to year 1. This is simply the smallest cost in row 0 of the table for stage 1.

(c.) 2 This is the value of X for which the minimum cost (35.3478) is achieved.

(d.) 2 This is the state that results when the optimal decision (2) has been selected.

(e) What is the present value of the minimum total cost?
Solution: $f_1(0)$ = $35.3478 million

(f) What is the optimal construction schedule? (X_t = # plants to be constructed in year t.)
Solution: We have seen that the optimal value of X_1 is 2 and the resulting state is 0+2=2. The table for stage 2 then indicates that if the state of the system is 2, the

optimal decision X_2 is 3 and the resulting state is 2+3=5. The table for stage 3 then indicates that if the state is 5 the optimal decision is X_3=0, etc.

Year t	X_t
1	2
2	3
3	0
4	2
5	0
6	1

Note: 999.999 in the output below represents +¥ to prevent an infeasible choice of state & decision combination.

```
                   ---Stage 6---
         s \ x:   0        1

         7 | 999.9999   7.1000
         8 |   0.0000 999.9999

                   ---Stage 5---
         s \ x:   0        1        2

         6 | 999.9999  13.0350  12.8000
         7 |   6.0350   7.0000 999.9999
         8 |   0.0000 999.9999 999.9999

                   ---Stage 4---
         s \ x:   0        1        2        3

         4 | 999.9999 999.9999  23.4800  23.4297
         5 | 999.9999  17.7800  17.7298  18.3000
         6 |  10.8800  12.0298  12.6000 999.9999
         7 |   5.1297   6.9000 999.9999 999.9999
         8 |   0.0000 999.9999 999.9999 999.9999

                   ---Stage 3---
         s \ x:   0        1        2        3

         2 | 999.9999 999.9999  32.3153  33.0703
         3 | 999.9999  26.7153  27.4703  27.2480
         4 |  19.9153  21.8703  21.6480  22.3603
         5 |  15.0703  16.0480  16.7603  18.0000
         6 |   9.2480  11.1603  12.4000 999.9999
         7 |   4.3603   6.8000 999.9999 999.9999
         8 |   0.0000 999.9999 999.9999 999.9999

                   ---Stage 2---
         s \ x:   0        1        2        3

         1 | 999.9999  34.1680  34.9080  34.6280
         2 |  27.4680  29.4080  29.1280  30.5097
         3 |  22.7080  23.6280  25.0097  25.5608
         4 |  16.9280  19.5097  20.0608  21.4062
         5 |  12.8097  14.5608  15.9062  17.7000
         6 |   7.8608  10.4062  12.2000 999.9999
```

```
         7 |   3.7062   6.7000 999.9999 999.9999
         8 |   0.0000 999.9999 999.9999 999.9999

                    ---Stage 1---
         s \ x:   0        1        2        3
____________________________________________________

         0 | 999.9999 (a)________  35.3478  36.7018

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
Stage 6:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
         7   7.1000         1               8
         8   0.0000         0               8
------------------------------------------------------------
Stage 5:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
         6  12.8000         2               8
         7   6.0350         0               7
         8   0.0000         0               8
------------------------------------------------------------
Stage 4:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
         4  23.4297         3               7
         5  17.7298         2               7
         6  10.8800         0               6
         7   5.1297         0               7
         8   0.0000         0               8
------------------------------------------------------------
Stage 3:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
         2  32.3153         2               4
         3  26.7153         1               4
         4  19.9153         0               4
         5  15.0703         0               5
         6   9.2480         0               6
         7   4.3603         0               7
         8   0.0000         0               8
------------------------------------------------------------
Stage 2:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
         1  34.1680         1               2
         2  27.4680         0               2
         3  22.7080         0               3
         4  16.9280         0               4
         5  12.8097         0               5
         6   7.8608         0               6
         7   3.7062         0               7
         8   0.0000         0               8
------------------------------------------------------------
Stage 1:
            Optimal        Optimal    Resulting
 State      Values        Decisions     State
```

0 (b)________ (c)____ (d)____

56:171 Operations Research
Homework #13 Solution -- Fall 2000

1. ***Production Planning*** We wish to plan production of an expensive, low-demand item for the next nine months (January through September).

- the cost of production is $5 for setup, plus $5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is $2 per unit, based upon the level at the beginning of the month.
- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand D is random, with the same probability distribution each month:

demand d	0	1	2
P{D=d}	0.2	0.5	0.3

- there is a penalty of $10 per unit for any demand which cannot be immediately satisfied. A maximum of 2 units may be backordered.
- the initial inventory (i.e., the inventory at the end of December) is 1.
- a salvage value of $4 per unit is received for any inventory remaining at the end of the last month (September)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 9= January, stage 8= February, etc.** (i.e., n = # months remaining in planning period.) We define

S_n = inventory position at stage n, i.e., stock on hand if positive, # backordered if negative.

$f_n(S_n)$ = minimum total cost for the last n months if at the beginning of stage n the inventory position is S_n.

(a.) What is the optimal production quantity for January? __0____

(b.) What is the total expected cost for the nine months, if there is one unit of stock on hand initially?
__$89.44___ (= $f_9(1)$)

(c.) If, during January, the demand is 1 unit, how many units should be produced in February? __3___

(d.) Three values have been blanked out in the computer output, What are they?

- the cost associated with the decision to produce 1 unit in February when the inventory is 1 at the end of January. $81.01 *(Note: this may or may not be the optimal decision!)*
Computation*:* storage cost (for 1 unit) + production cost (for 1 unit) + $P\{d=0\}f_7(2)$ + $P\{d=1\}f_7(1)+P\{d=2\}f_7(0)$ = $2 + $10 + (0.2)(62.68) + (0.5)(68.80) + (0.3)(73.57) = $81.01.
- the optimal value $f_7(0)$, i.e., the minimum total cost of the last 7 months (March through September) if there is no stock on hand (& no backorders) at the beginning of March. *$73.57*
- the corresponding optimal decision $X_7^*(0)$ _produce 3_

```
                    ---Stage  1---
   s \ x:    0         1         2         3         4
  -2 |     55.00     62.00     55.50     45.20     41.40
  -1 |     42.00     40.50     30.20     26.40     27.40
   0 |     20.50     15.20     11.40     12.40     14.20
   1 |      7.20      8.40      9.40     11.20     15.00
   2 |      0.40      6.40      8.20     12.00     17.00
   3 |     -1.60      5.20      9.00     14.00     19.00
```

etc.

```
                    ---Stage  7  (March)---
   s \ x:    0         1         2         3         4
  -2 |    113.36    120.34    114.78    106.80    103.68
  -1 |    100.34     99.78     91.80     88.68     88.57
   0 |     79.78     76.80     73.68     73.57     75.18
   1 |     68.80     70.68     70.57     72.18     76.25
   2 |     62.68     67.57     69.18     73.25     78.25
   3 |     59.57     66.18     70.25     75.25     80.25

                    ---Stage  8  (February)---
   s \ x:    0         1         2         3         4
  -2 |    123.68    130.66    125.11    117.12    114.01
  -1 |    110.66    110.11    102.12     99.01     98.90
```

```
  0 |    90.11    87.12    84.01    83.90    85.51
  1 |    79.12    _____    80.90    82.51    86.57
  2 |    73.01    77.90    79.51    83.57    88.57
  3 |    69.90    76.51    80.57    85.57    90.57

                  ---Stage 9 (January)---
  s \ x:   0        1        2        3        4
  1 |    89.44    91.33    91.22    92.83    96.90
```

The values of f_n and X_n are:

```
Stage 9 (January)
                    Optimal  Optimal
 State              Values   Decision
   1 Stock1 |      89.44 |    0 Idle
-------------------------
Stage  (February)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |     114.01 |    4 Prod 4
  -1 Back1  |      98.90 |    4 Prod 4
   0 Empty  |      83.90 |    3 Prod 3
   1 Stock1 |      79.12 |    0 Idle
   2 Stock2 |      73.01 |    0 Idle
   3 Stock3 |      69.90 |    0 Idle
-------------------------
Stage 7 (March)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |     103.68 |    4 Prod 4
  -1 Back1  |      88.57 |    4 Prod 4
   0 Empty  |      _____ |    ________
   1 Stock1 |      68.80 |    0 Idle
   2 Stock2 |      62.68 |    0 Idle
   3 Stock3 |      59.57 |    0 Idle
-------------------------
Stage 6 (April)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      93.36 |    4 Prod 4
  -1 Back1  |      78.25 |    4 Prod 4
   0 Empty  |      63.25 |    3 Prod 3
   1 Stock1 |      58.47 |    0 Idle
   2 Stock2 |      52.36 |    0 Idle
   3 Stock3 |      49.25 |    0 Idle
-------------------------
Stage 5 (May)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      83.04 |    4 Prod 4
  -1 Back1  |      67.93 |    4 Prod 4
   0 Empty  |      52.93 |    3 Prod 3
   1 Stock1 |      48.15 |    0 Idle
   2 Stock2 |      42.04 |    0 Idle
   3 Stock3 |      38.93 |    0 Idle
-------------------------
Stage 4 (June)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      72.73 |    4 Prod 4
  -1 Back1  |      57.60 |    4 Prod 4
   0 Empty  |      42.60 |    3 Prod 3
   1 Stock1 |      37.83 |    0 Idle
   2 Stock2 |      31.73 |    0 Idle
   3 Stock3 |      28.60 |    0 Idle
-------------------------
Stage 3 (July)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      62.36 |    4 Prod 4
  -1 Back1  |      47.28 |    4 Prod 4
   0 Empty  |      32.28 |    3 Prod 3
   1 Stock1 |      27.54 |    0 Idle
   2 Stock2 |      21.36 |    0 Idle
   3 Stock3 |      18.28 |    0 Idle
-------------------------
Stage 2 (August)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      52.10 |    4 Prod 4
  -1 Back1  |      37.04 |    4 Prod 4
   0 Empty  |      22.04 |    3 Prod 3
   1 Stock1 |      17.06 |    0 Idle
   2 Stock2 |      11.10 |    0 Idle
   3 Stock3 |       8.04 |    0 Idle
-------------------------
Stage 1 (September)
                    Optimal  Optimal
 State              Values   Decision
  -2 Back2  |      41.40 |    4 Prod 4
  -1 Back1  |      26.40 |    3 Prod 3
   0 Empty  |      11.40 |    2 Prod 2
   1 Stock1 |       7.20 |    0 Idle
   2 Stock2 |       0.40 |    0 Idle
   3 Stock3 |      -1.60 |    0 Idle
```

2. *Stochastic Machine Replacement*-- A component of a machine has an active life, measured in weeks, that is a random variable T, where

t	P{T=t}
1	0.1
2	0.2
3	0.4
4	0.3

Note that the component never survives more than four weeks! Suppose that we start with a fresh component, and wish to plan the replacement strategy for the next eight weeks, after which the machine will be retired. At the beginning of each week, the component is inspected and determined to be either operational or broken down. (That is, the component is not continuously monitored, and so the broken-down condition is only discovered at the beginning of the week.) At the beginning of the week, after determining the condition of the component, we may decide to replace it with a fresh component or, if it is still operational, to continue with the current component. The machine earns $500 in revenues each week that it is operational with no breakdowns. Assume that no revenues are earned in a week if the component fails. A replacement for the component costs $300. We began last week with a fresh component.

We define the stages to be the weeks, with the number of the stage being the number of weeks *remaining* in the eight-week period. S_n = state = age (in weeks) of the component at the beginning of stage n, where S_n = 4 means either that it is 4 weeks old or has broken down. The decisions are X_n= 0 (keep) or 1 (replace). The optimal value $f_n(S_n)$is the maximum expected value (revenues minus replacement costs) for the last n weeks, if the current state of the component is S_n.

(a.) What is the failure probability for a 2-week-old component? _0.2223_

Solution: *Survival* probability for a 2-week-old component is

$$P\{Z=0 \mid S=2 \text{ \& X=0}\} = P\{T \geq 3 \mid T \geq 2\} = \frac{0.4+0.3}{0.2+0.4+0.3} = 0.7777$$

$\Rightarrow \therefore$ The *failure* prob. for a 2-week-old component is (1–0.7777) =***0.2223***

(b.) What is the failure probability for a 3-week-old component? 0.5714

Solution: *Survival* probability for a 3-week-old component is

$$P\{Z=0 \mid S=3 \text{ \& X=0}\} = P\{T \geq 4 \mid T \geq 3\} = \frac{0.3}{0.4+0.3} = 0.4286$$

$\Rightarrow \therefore$ The *failure* prob. for a 2-week-old component is (1–0.4286) =***0.5714***

(c.) What is the expected total value for the seventh and eighth weeks if, in the seventh week (i.e., n=2), the part is 3 weeks old and is kept? $364.30

= $214.30 (for the current week) + $150 (for the remaining weeks).

Solution: current week: *(0.4286*500+0.5714*0) = 214.30*

remaining weeks: (0.4286 $f_1(4)$+0.5714* $f_1(4)$) = (0.4286* 150 +0.5714* 150)*

(d.) In the third week (i.e., n=6 weeks to go), if the component is replaced the total expected value will be $1891.17 = $150 (for the current week) + $1741.17 (for the remaining weeks).

Solution: *current week: – 300+0.9*500+0.1*0*

remaining weeks: 0.9 $f_5(1)$+0.1* $f_5(4)$ = 0.9* 1764.185 +0.1* 1534.08*

(e.) What is the maximum total expected value for the eight week period, if the component is one week old at the beginning of the period? $2717.14 = $f_8(1)$

(f.) If the part survives the first week, should it be replaced? _NO_ = $X_7(2)$

(g.) In general, the optimal replacement rule seems to be "replace when _3_ weeks old or when it has failed", except in the final week, when the rule is "replace if four weeks old (or broken down) "

```
==============================================================

               ---Stage 1---                              ---Stage 3---
       s \ x:   0        1                       s \ x:   0        1
---------------------------                 ---------------------------
       1 |   450.00   150.00                     1 |  1037.00   940.50
       2 |   388.89   150.00                     2 |   958.89   940.50
       3 |   214.29   150.00                     3 |   784.29   940.50
       4 |     0.00   150.00                     4 |   570.00   940.50

               ---Stage 2---                              ---Stage 4---
       s \ x:   0        1                       s \ x:   0        1
---------------------------                 ---------------------------
       1 |   815.00   570.00                     1 |  1407.05  1177.35
       2 |   588.89   570.00                     2 |  1329.39  1177.35
       3 | (c)______  570.00                     3 |  1154.79  1177.35
       4 |   150.00   570.00                     4 |   940.50  1177.35
```

```
              ---Stage 5---
         s \ x:   0       1
----------------------------
         1 |  1764.19  1534.08
         2 |  1566.24  1534.08
         3 |  1391.64  1534.08
         4 |  1177.35  1534.08

              ---Stage 6---
         s \ x:   0       1
----------------------------
         1 |  2013.02  (d)______
         2 |  1922.97  (d)______
         3 |  1748.37  (d)______
         4 |  1534.08  (d)______
```

```
              ---Stage 7---
         s \ x:   0       1
----------------------------
         1 |  2369.79  2150.84
         2 |  2280.06  2150.84
         3 |  2105.46  2150.84
         4 |  1891.17  2150.84

              ---Stage 8---
         s \ x:   0       1
----------------------------
         1 |  2717.14  2497.89
         4 |  2150.84  2497.89
```

```
====================================================================
```

The optimal values (f_n) & decisions (X_n):

```
-------------------------
Stage 8 (i.e., initial week)
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 2717.1409 | 0 keep
4)failed   | 2497.89   | 1 replace
-------------------------
Stage 7
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 2369.7895 | 0 keep
2)2 wk old | 2280.0634 | 0 keep
3)3 wk old | 2150.8382 | 1 replace
4) failed  | 2150.8382 | 1 replace
-------------------------
Stage 6
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 2013.0230 | 0 keep
2)2 wk old | 1922.9689 | 0 keep
3)3 wk old | 1891.1745 | 1 replace
4) failed  | 1891.1745 | 1 replace
-------------------------
Stage 5
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 1764.1850 | 0 keep
2)2 wk old | 1566.2389 | 0 keep
3)3 wk old | 1534.0800 | 1 replace
4) failed  | 1534.0800 | 1 replace
```

```
Stage 4
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 1407.0500 | 0 keep
2)2 wk old | 1329.3889 | 0 keep
3)3 wk old | 1177.3500 | 1 replace
4) failed  | 1177.3500 | 1 replace
-------------------------
Stage 3
           Optimal       Optimal
State      Values         Decision
1)1 wk old | 1037.0000 | 0 keep
2)2 wk old |  958.8889 | 0 keep
3)3 wk old |  940.5000 | 1 replace
4) failed  |  940.5000 | 1 replace
-------------------------
Stage 2
           Optimal       Optimal
State      Values         Decision
1)1 wk old |  815.0000 | 0 keep
2)2 wk old |  588.8889 | 0 keep
3)3 wk old |  570.0000 | 1 replace
4) failed  |  570.0000 | 1 replace

-------------------------
Stage 1
           Optimal       Optimal
State      Values         Decision
1)1 wk old |  450.0000 | 0 keep
2)2 wk old |  388.8889 | 0 keep
3)3 wk old |  214.2857 | 0 keep
4) failed  |  150.0000 | 1 replace
```