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Operations Research -- Sample Homework Assignments

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❍❍❍❍❍❍❍❍❍❍ Homework #1 ❍❍❍❍❍❍❍❍❍

Linear Programming Model Formulation: Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available on the HP-UX workstations, or you may use the software packaged with the textbook.) Be sure to state precisely the definitions of your decision variables, and explain in a few words the purpose of each type of constraint. Write a few words to state what the optimal solution is (i.e., without making use of variable names). *(For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

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1. Exercise #4, page 113 *(Walnut Orchard Farms)*

"Walnut Orchard has two farms that grow wheat and corn. Because of differing soil conditions, there are differences in the yields and costs of growing crops on the two farms. The yields and costs are

	Farm 1	Farm 2
Corn yield/acre	500 bushels	650 bushels
Cost/acre of corn	$100	$120
Wheat yield/acre	400 bushels	350 bushels
Cost/acre of wheat	$90	$80

Each farm has 100 acres available for cultivation; 11,000 bushels of wheat and 7000 bushels of corn must be grown. Determine a planting plan that will minimize the cost of meeting these demands.

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2. Investment planning

I now have $100. The following investments are available at the beginning of each of the next five years:

Investment A: Every dollar invested yields $0.10 a year from now and $1.30 three years after the original investment, a total of $1.40.

Investment B: Every dollar invested yields $1.35 two years later.

Investment C: Every dollar invested yields \$0.45 at the end of each of the following three years, a total return of \$1.35.

During each year, uninvested cash can be placed in money market funds, which yield 6% interest per year. At most \$50 may be placed in an investment in any one year. (It is permitted, for example, to invest \$50 in A during two consecutive years, so that we have invested a total of more than \$50 in A at some point in time-- this restriction limits only the size of each investment in A.) Formulate an LP to maximize my cash on hand <u>five</u> years from now.

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1. Exercise #4, page 113 *(Walnut Orchard Farms)*

Solution:

Note: The author of this exercise was evidently ignorant of crop yields, since even prime Iowa farmland seldom yields more than 200 bushels/acre. Perhaps the original units of land were hectares rather than acres, and were changed for an American student audience?

<u>Decision variables</u>:

C1 = # of acres of Farm **1** planted in **corn**
W1 = # of acres of Farm **1** planted in **wheat**
C2 = # of acres of Farm **2** planted in **corn**
W2 = # of acres of Farm **2** planted in **wheat**

Constraints:

- We must impose a constraint which restricts the number of acres of each farm which are planted in crops. Thus, we impose:

  $C1 + W1 \leq 100$

  $C2 + W2 \leq 100$

  Note that we are <u>not</u> requiring that all of the land be planted in crops!
- We must also impose a constraint which restricts us to produce a minimum quantity of each crop. Thus, we impose:

  $500C1 + 650C2 \geq 11000$

  $400W1 + 350W2 \geq 7000$

  For example, the left side of the first of the two above constraints states that the total production of corn, namely,

  (500 bushels/acre)(# acres of farm 1 planted in corn) +
  (650 bushels/acre)(# acres of farm 2 planted in corn)

  must be <u>at least</u> 11000 bushels.

  Note that we assume that all crops which are produced can be sold, i.e., we must produce at least the required amounts, but may, if doing so resulted in lower costs (unlikely here), produce an excess.
- Lastly, we must impose a nonnegativity constraint on each of the four variables.

  Note that these nonnegativity constraints are not entered explicitly into LINDO, which assumes that all the user-defined variables are nonnegativity!

Objective:
We wish to minimize the costs of producing the required quantities of each crop, i.e.,

```
MAX  100 C1 + 120 C2 + 90 W1 + 80 W2
```

The complete LP model is therefore

```
MAX  100 C1 + 120 C2 + 90 W1 + 80 W2
subject to
   C1 + W1   100
   C2 + W2   100
   500C1 + 650C2   11000
   400W1 + 350W2   7000
   C1 0, C2 0, W1 0, W2 0
```

LINDO output:

```
 MIN     100 C1 + 120 C2 + 90 W1 + 80 W2
  SUBJECT TO
         2)   C1 + W1 <=   100
         3)   C2 + W2 <=   100
         4)   500 C1 + 650 C2 >=   11000
         5)   400 W1 + 350 W2 >=   7000
  END

 LP OPTIMUM FOUND AT STEP      4

        OBJECTIVE FUNCTION VALUE

        1)      3605.769

  VARIABLE        VALUE          REDUCED COST
        C1          .000000          7.692307
        C2        16.923080           .000000
        W1        17.500000           .000000
        W2          .000000          1.250000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        82.500000           .000000
        3)        83.076920           .000000
        4)         -.000000          -.184615
        5)         -.000000          -.225000
```

That is, the optimal plan is to plant 17.5 acres of wheat on farm #1 and 16.923 acres of corn on farm #2. (Most of each farm is left unplanted.)

2. Investment planning

Solution:

The variables may be defined as follows:

At = $ invested in A at the beginning of year t, t=1,2,3

Bt = $ invested in B at the beginning of year t, t=1,2,3,4

Ct = $ invested in C at the beginning of year t, t=1,2,3
Rt = $ invested in the money market fund at the beginning of year t, t=1,2,3,4,5

Note that I'm not considering the possibility of investing in A4, A5, B5, C4, and C5 because in these cases, not all of the returns on these investments will be received in time to be included in the objective function, which is the total accumulation at the beginning of year 6 (i.e., end of year 5).

The constraints will include a restriction for each year, stating that the return obtained at the end of the previous year must equal the total amount invested at the beginning of the current year. To assist in writing the constraints, consider the following table:

Begin yr.	Investment A1	A2	A3	B1	B2	B3	B4	C1	C2	C3	R1	R2	R3	R4	R5
1	-1			-1				-1			-1				
2	0.1	-1			-1			0.45	-1		1.06	-1			
3		0.1	-1	1.35		-1		0.45	0.45	-1		1.06	-1		
4	1.3		0.1		1.35		-1	0.45	0.45	0.45			1.06	-1	
5		1.3				1.35			0.45	0.45				1.06	-1
6			1.3				1.35			0.45					1.06

That is, each column of the Investment section of the table gives the cash flow for that investment. For example, **A2** has a cash flow of -1 at the beginning of year 2 (representing the cash going into that investment, a 0.1 at the beginning of year 3 (i.e., the end of year 2), and 1.3 at the beginning of year 5 (end of year 4). The cash flow balance equation for the beginning of each year may easily be written by using the coefficients in the corresponding row of the table. The LP model, as displayed by **LINDO**, is as follows.

```
MAX     1.3 A3 + 1.35 B4 + 1.06 R5 + 0.45 C3
SUBJECT TO
       2)   A1 + B1 + C1 + R1 =    100
       3)   0.1 A1 + 0.45 C1 + 1.06 R1 - A2 - B2 - C2 - R2 =    0
       4) - A3 + 1.35 B1 + 0.45 C1 + 0.1 A2 + 0.45 C2 + 1.06 R2 - B3 -
C3
     - R3 =    0
       5)   0.1 A3 - B4 + 1.3 A1 + 0.45 C1 + 1.35 B2 + 0.45 C2 + 0.45
C3
     + 1.06 R3 - R4 =    0
       6) - R5 + 1.3 A2 + 0.45 C2 + 1.35 B3 + 0.45 C3 + 1.06 R4 =  0
END
```

The $50 upper limits on the investments have not been specified yet. For curiosity's sake, let's first find the solution of this less restrictive LP, so that we may compare to the more restrictive LP. **LINDO**'s solution of this less restricted problem is as follows:

```
 LP OPTIMUM FOUND AT STEP      4

        OBJECTIVE FUNCTION VALUE

        1)      211.8150

  VARIABLE        VALUE          REDUCED COST
        A3          .000000           .099500
        B4       126.000000           .000000
        R5        20.250000           .000000
        A1          .000000           .180900
```

```
       B1          .000000           .046575
       C1       100.000000           .000000
       R1          .000000           .186300
       A2          .000000           .291050
       B2        45.000000           .000000
       C2          .000000           .047475
       R2          .000000           .195930
       B3          .000000           .103500
       C3        45.000000           .000000
       R3          .000000           .103500
       R4          .000000           .226400
```

That is, at the beginning of year 6 (i.e. end of year 5), $211.815 will have been accumulated. Notice that all $100 is initially invested in **C1**, which will return $45 at the beginning of each of the years 2, 3, and 4. At the beginning of year 2, this $45 is invested in **B2**, while at the beginning of year 3, the $45 received from **C1** is reinvested in **C3**. At the beginning of year 4, cash returns will be received from **C1**, **B2**, and **C3**. This ($126) is all invested in **B4**. At the beginning of year 5, only **C3** returns any cash, and this cash ($20.25) is put into the money-market fund for that year. At the beginning of year 6, then, cash is received from both **B4** and the money market fund (**R5**), a total of $211.815.

Next we add the *simple upper bound* (SUB) command for each of the variables **A1, A2, A3, B1, B2, B3, B4, C1, C2**, and **C3** (but not **R1** through **R5**!). The revised LP model as displayed by **LINDO** appears below:

```
  MAX     1.3 A3 + 1.35 B4 + 1.06 R5 + 0.45 C3
  SUBJECT TO
         2)   A1 + B1 + C1 + R1 =    100
         3)   0.1 A1 + 0.45 C1 + 1.06 R1 - A2 - B2 - C2 - R2 =    0
         4) - A3 + 1.35 B1 + 0.45 C1 + 0.1 A2 + 0.45 C2 + 1.06 R2 - B3 -
C3
       - R3 =    0
         5)   0.1 A3 - B4 + 1.3 A1 + 0.45 C1 + 1.35 B2 + 0.45 C2 + 0.45
C3
       + 1.06 R3 - R4 =    0
         6) - R5 + 1.3 A2 + 0.45 C2 + 1.35 B3 + 0.45 C3 + 1.06 R4 =  0
  END
  SUB       A3       50.00000
  SUB       B4       50.00000
  SUB       A1       50.00000
  SUB       B1       50.00000
  SUB       C1       50.00000
  SUB       A2       50.00000
  SUB       B2       50.00000
  SUB       C2       50.00000
  SUB       B3       50.00000
  SUB       C3       50.00000
```

The solution of this problem with the upper limits on amounts to be invested is as follows:

```
 LP OPTIMUM FOUND AT STEP      5

       OBJECTIVE FUNCTION VALUE
```

```
       1)      202.0675

 VARIABLE        VALUE          REDUCED COST
       A3          .125000           .000000
       B4        50.000000         -.226400
       R5       105.570740          .000000
       A1          .000000          .246866
       B1        50.000000         -.037322
       C1        50.000000          .000000
       R1          .000000          .154091
       A2          .000000          .098946
       B2          .000000          .101322
       C2        22.500000          .000000
       R2          .000000          .121080
       B3        50.000000         -.018640
       C3        50.000000         -.020260
       R3          .000000          .221344
       R4         5.137498          .000000
```

The solution may be stated as follows:

1. The original $100 is evenly divided between investments **B1** and **C1** in the first year.

2. The $22.50 (= [0.45][50]) return from **C1** at the end of the first year is re-invested in **C2**.

3. At the end of the second year, the investor receives $67.50 (= [1.35][50]) return from his investment in **B1**, another $22.50 from **C1**, plus $10.125 (= [0.45][22.50]) from the investment **C2**, a total of $100.125. Of this total, $50 of this money is invested in each of **B3** and **C3**, leaving $0.125 which is invested in **A3**.

4. At the end of the third year, the investment **C1** returns $22.50, C2 return another $10.125, **C3** returns another $22.50, and **A3** returns $0.0125 (= [0.1][0.125]), a total of $55.1375. Of this total, $50 is invested in **B4**, and the remaining $5.1375 is invested in the money-market fund **R4**.

5. At the end of the fourth year, **B3** returns $67.50, **C2** returns $10.125, **C3** returns $22.50, and the money-market fund **R4** returns $5.44575 (= [1.06][5.1375]), a total of $105.57075. The only possible investment for this money is the money-market fund, so all $105.57075 is invested in **R5**.

6. At the end of the fifth year, the money-market fund **R5** yields $111.904995 (= [1.06][105.57075]). Also, **A3** returns $0.1625 (= [1.3][0.125]), **B4** returns $67.50, and **C3** returns $22.50. Thus, a total of $111.904995 + $0.1625 + $67.50 + $22.50 = $202.067495 has been accumulated, *more than double* the original holdings of $100.

Notice that placing the **SUB** restrictions reduced the final accumulated cash from \$211.815   to \$202.0675, a reduction of \$9.7475 (approximately 4.6% reduction).

❍❍❍❍❍❍❍❍❍❍ **Homework #2** ❍❍❍❍❍❍❍❍❍

Linear Programming Model Formulation: Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available on the HP-UX workstations, or you may use the software packaged with the textbook.) Be sure to state precisely the definitions of your decision variables, and explain in a few words the purpose of each type of constraint. Write a few words to state what the optimal solution is (i.e., without making use of variable names). *(For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

1. Two alloys, A and B, are made from four metals, labelled I, II, III, and IV, according to the following specifications:

Alloy	Specifications	Selling price (\$/ton)
A	At most 80% of I At most 30% of II At least 50% of IV	200
B	Between 40 and 60% of II At least 30% of III At most 70% of IV	300

The four metals, in turn, are extracted from three ores according to the following data:

Ore	Max Quantity (tons)	I	II	III	IV	Others	Price (\$/ton)
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

a. How much of each type of alloy should be produced?
b. How much of each ore should be allocated to the production of each alloy?

(Hint: Let x_{ij} be tons of ore i allocated to alloy j, and define w_j as tons of alloy j to be produced.)

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2. Consider the LP

$$\text{Minimize } z = x_1 + 2x_2 - 3x_3 + x_4$$

subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$
$$x_1, x_2, x_3, x_4 \quad 0$$

a. A basic solution of the constraint equations of this problem has how many basic variables, in addition to -z? _____
b. What is the maximum number of basic solutions (either feasible or infeasible) which might exist? (That is, how many ways might you select a set of basic variables from the four variables x_1 through x_4?) _____
c. Find and list all of the basic solutions of the constraint equations.
d. Is the number of basic solutions in (c) equal to the maximum possible number which you specified in (b)? ______
e. How many of the basic solutions in (c) are feasible (i.e. nonnegative)?
f. By evaluating the objective function at each basic solution, find the optimal solution.

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3. Consider the LP

$$\text{Maximize } z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$
$$\text{subject to}$$
$$x_1 + 4x_2 - 2x_3 + 8x_4 \quad 2$$
$$-x_1 + 2x_2 + 3x_3 + x_4 \quad 1$$
$$x_1, x_2, x_3, x_4 \quad 0$$

a. Introduce slack variables to convert the inequality constraints to equations.
b. Form the simplex tableau for this LP.
c. What is the beginning basic solution? Is it feasible? What is its objective function value?

d. Perform as many iterations of the simplex algorithm as required to find an optimum solution.

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1. **Solution:**

This problem (which I obtained from a recent edition of the O.R. text by Hamdy Taha) is rather ambiguous and not well stated. The hint above suggests what might be an unlikely interpretation of the problem, which is that this is a traditional "blending" problem in which the ores are blended together to produce the alloy. If this is the case, is each alloy the sum of the ores which are allocated to it? (What about the impurities, i.e., the "other"? Can they be "skimmed" off when the ores are smelted?) Must all of the metals contained in the ores allocated to an alloy also appear in the alloy, or do the metals first get extracted from the ore and then selected for the alloy mix (but only the alloy to which the ore was allocated)?

A more likely situation in reality is that the metals are extracted from the ores, and then the metals are blended together to obtain the alloys. (This second interpretation gives the manager more flexibility and the possibility of a higher profit.) I will give a formulation for each interpretation below.

I. Ores are allocated directly to the alloys.

Defining the decision variables as in the hint above, the objective function becomes

```
Maximize  200WA + 300WB - 30(X1A + X1B) - 40(X2A + X2B) - 50(X3A + X3B)
```

i.e., the difference between the revenue obtained from sales of the alloys and the cost of the ores.

The constraints are:

Availability of ores:

X1A+X1B   1000
X2A+X2B   2000
X3A+X3B   3000

Composition of alloys:

0.2X1A + 0.10X2A + 0.05X3A	0.8WA	*(At most 80% of alloy A is metal I)*
0.10X1A + 0.20X2A + 0.05X3A	0.3WA	*(At most 30% of alloy A is metal II)*
0.30X1A + 0.30X2A + 0.20X3A	0.5WA	*(At least 50% of alloy A is metal IV)*
0.10X1B + 0.20X2B + 0.05X3B	0.4WB	*(At least 40% of alloy B is metal I)*
0.10X1B + 0.20X2B + 0.05X3B	0.6WB	*(At most 60% of alloy B is metal I)*
0.30X1B + 0.30X2B + 0.70X3B	0.3 WB	*(At least 30% of alloy B is metal III)*
0.30X1B + 0.30X2B + 0.20X3B	0.7 WB	*(At most 70% of alloy B is metal IV)*

Ia. ***Assuming that the impurities ("other") and all metals in the ores remain present in the alloys****, we get the constraint:*

WA = X1A + X2A + X3A
WB = X1B + X2B + X3B

LINDO solution:

```
 MAX     200 WA + 300 WB - 30 X1A - 30 X1B - 40 X2A - 40 X2B - 50
X3A
       - 50 X3B
  SUBJECT TO
         2)   X1A + X1B <=   1000
         3)   X2A + X2B <=   2000
         4)   X3A + X3B <=   3000
         5) - 0.8 WA + 0.2 X1A + 0.1 X2A + 0.05 X3A <=   0
         6) - 0.3 WA + 0.1 X1A + 0.2 X2A + 0.05 X3A <=   0
         7) - 0.5 WA + 0.3 X1A + 0.3 X2A + 0.2 X3A >=   0
         8) - 0.4 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B >=   0
         9) - 0.6 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B <=   0
        10) - 0.3 WB + 0.3 X1B + 0.3 X2B + 0.7 X3B >=   0
        11) - 0.7 WB + 0.3 X1B + 0.3 X2B + 0.2 X3B <=   0
        12) - WA + X1A + X2A + X3A =    0
        13) - WB + X1B + X2B + X3B =    0
  END

 LP OPTIMUM FOUND AT STEP      5

        OBJECTIVE FUNCTION VALUE

        1)      .0000000

  VARIABLE        VALUE          REDUCED COST
        WA          .000000           .000000
        WB          .000000           .000000
```

```
         X1A           .000000           .000000
         X1B           .000000        120.000000
         X2A           .000000         10.000000
         X2B           .000000           .000000
         X3A           .000000        105.000010
         X3B           .000000        205.000000


         ROW   SLACK OR SURPLUS     DUAL PRICES
          2)      1000.000000           .000000
          3)      2000.000000           .000000
          4)      3000.000000           .000000
          5)          .000000           .000000
          6)          .000000           .000000
          7)         -.000000       -850.000100
          8)         -.000000      -1300.000000
          9)          .000000           .000000
         10)          .000000           .000000
         11)          .000000           .000000
         12)          .000000        225.000030
         13)          .000000        220.000000
```

Note that the optimal solution is to produce nothing whatsoever. This apparently is because there is no way to mix the ores while satisfying the restrictions on the metal contents.

Ib. ***Assuming the the impurities are somehow removed during the processing:***

WA = 0.9X1A + 0.9X2A + X3A

WB = 0.9X1B + 0.9X2B + X3B

LINDO solution:

```
        MAX      200 WA + 300 WB - 30 X1A - 30 X1B - 40 X2A - 40 X2B - 50
      X3A
             - 50 X3B
        SUBJECT TO
               2)   X1A + X1B <=   1000
               3)   X2A + X2B <=   2000
               4)   X3A + X3B <=   3000
               5) - 0.8 WA + 0.2 X1A + 0.1 X2A + 0.05 X3A <=   0
               6) - 0.3 WA + 0.1 X1A + 0.2 X2A + 0.05 X3A <=   0
               7) - 0.5 WA + 0.3 X1A + 0.3 X2A + 0.2 X3A >=   0
               8) - 0.4 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B >=   0
               9) - 0.6 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B <=   0
              10) - 0.3 WB + 0.3 X1B + 0.3 X2B + 0.7 X3B >=   0
              11) - 0.7 WB + 0.3 X1B + 0.3 X2B + 0.2 X3B <=   0
              12) - WA + 0.9 X1A + 0.9 X2A + X3A =    0
              13) - WB + 0.9 X1B + 0.9 X2B + X3B =    0
        END

       LP OPTIMUM FOUND AT STEP      6

              OBJECTIVE FUNCTION VALUE

              1)      .0000000

        VARIABLE        VALUE          REDUCED COST
              WA           .000000           .000000
```

```
          WB           .000000           .000000
         X1A           .000000           .000000
         X1B           .000000        133.750000
         X2A           .000000         10.000000
         X2B           .000000           .000000
         X3A           .000000        150.000030
         X3B           .000000        253.125010


         ROW   SLACK OR SURPLUS     DUAL PRICES
          2)      1000.000000           .000000
          3)      2000.000000           .000000
          4)      3000.000000           .000000
          5)          .000000           .000000
          6)          .000000           .000000
          7)         -.000000      -1000.000100
          8)         -.000000      -1437.500000
          9)          .000000           .000000
         10)          .000000           .000000
         11)          .000000           .000000
         12)          .000000        300.000100
         13)          .000000        275.000030
```

The optimal solution for this model is the same as that above, i.e., produce nothing, apparently because it is not possible to construct a mix of the ores which can satisfy the restrictions on the metal contents.

Ic. ***Assuming that the ores are first allocated to the alloys, but that the metals are extracted from the ores and may then be added to the alloy mix as desired.*** That is, not all of the metals obtained from the ore (nor the impurities) need to be added to the alloy mix.
The last two equality contraints in the previous model are replaced by inequality constraints:

```
WA = 0.9X1A + 0.9X2A + X3A
WB = 0.9X1B + 0.9X2B + X3B
```

LINDO solution:

```
MAX     200 WA + 300 WB - 30 X1A - 30 X1B - 40 X2A - 40 X2B - 50 X3A
     - 50 X3B
SUBJECT TO
       2)   X1A + X1B <=   1000
       3)   X2A + X2B <=   2000
       4)   X3A + X3B <=   3000
       5) - 0.8 WA + 0.2 X1A + 0.1 X2A + 0.05 X3A <=   0
       6) - 0.3 WA + 0.1 X1A + 0.2 X2A + 0.05 X3A <=   0
       7) - 0.5 WA + 0.3 X1A + 0.3 X2A + 0.2 X3A >=   0
       8) - 0.4 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B >=   0
       9) - 0.6 WB + 0.1 X1B + 0.2 X2B + 0.05 X3B <=   0
      10) - 0.3 WB + 0.3 X1B + 0.3 X2B + 0.7 X3B >=   0
      11) - 0.7 WB + 0.3 X1B + 0.3 X2B + 0.2 X3B <=   0
      12) - WA + 0.9 X1A + 0.9 X2A + X3A >=   0
      13) - WB + 0.9 X1B + 0.9 X2B + X3B >=   0
END

LP OPTIMUM FOUND AT STEP      3
```

```
      OBJECTIVE FUNCTION VALUE

      1)      400000.0

VARIABLE        VALUE          REDUCED COST
      WA      1800.000000            .000000
      WB      1000.000000            .000000
     X1A      1000.000000            .000000
     X1B          .000000          45.000003
     X2A          .000000          30.000000
     X2B      2000.000000            .000000
     X3A      3000.000000            .000000
     X3B          .000000          42.500000


     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)          .000000          90.000010
      3)          .000000         110.000000
      4)          .000000          30.000001
      5)      1090.000000            .000000
      6)       290.000030            .000000
      7)         -.000000        -400.000000
      8)         -.000000        -750.000000
      9)       200.000010            .000000
     10)       300.000000            .000000
     11)        99.999961            .000000
     12)      2100.000000            .000000
     13)       799.999930            .000000
```

The optimal solution in this case is nontrivial, i.e., alloys are in fact produced and a profit obtained. The optimal plan is to allocate all available quantities of Ores #1&3 to Alloy A, and all available Ore #2 to Alloy B. Not all of the metals extracted from these ores are used in the alloys, since we see that rows 12 and 13 have "surplus" 2100 tons and 800 tons, respectively. The production quantities of alloys A and B are <u>1800 tons</u> and <u>1000 tons</u>, respectively, giving the firm a profit of <u>$400,000</u>.

II. ***Assume metals are first extracted from ores and then blended to produce alloys***

Define the decision variables differently:

Yij = quantity (tons) of metal I added to alloy j (i=I,II,III,IV & j=A,B)
Wj = quantity (tons) of alloy j produced (j=A,B)
Zk = quantity (tons) of ore k processed

The objective function is, as before, the difference between the revenue for the alloys and the cost of the ores:

```
Maximize  200WA + 300WB - 30Z1 - 40Z2 - 50Z3
```

The constraints are:

Availability of ores:

```
Z1   1000
Z2   2000
Z3   3000
```

Metals extracted from ores:

```
Y1A + Y1B = 0.20Z1 + 0.10Z2 + 0.05Z3
Y2A + Y2B = 0.10Z1 + 0.20Z2 + 0.05Z3
Y3A + Y3B = 0.30Z1 + 0.30Z2 + 0.70Z3
Y4A + Y4B = 0.30Z1 + 0.30Z2 + 0.20Z3
```

Composition of alloys:

```
Y1A    0.8WA         (At most 80% of alloy A is metal I)
Y2A   0.3WA          (At most 30% of alloy A is metal II)
Y4A   0.5WA          (At least 50% of alloy A is metal IV)

Y1B   0.4WB          (At least 40% of alloy B is metal I)
Y1B   0.6WB          (At most 60% of alloy B is metal I)
Y3B   0.3 WB         (At least 30% of alloy B is metal III)
Y4B   0.7 WB         (At most 70% of alloy B is metal IV)
```

Assuming that only the four metals are used in production of the alloys

```
WA = Y1A + Y2A + Y3A + Y4A
WB = Y1B + Y2B + Y3B + Y4B
```

LINDO solution:

```
MAX     200 WA + 300 WB - 30 Z1 - 40 Z2 - 50 Z3
  SUBJECT TO
         2) - 0.2 Z1 - 0.1 Z2 - 0.05 Z3 + Y1A + Y1B =    0
         3) - 0.1 Z1 - 0.2 Z2 - 0.05 Z3 + Y2A + Y2B =    0
         4) - 0.3 Z1 - 0.3 Z2 - 0.7 Z3 + Y3A + Y3B =    0
         5) - 0.3 Z1 - 0.3 Z2 - 0.2 Z3 + Y4A + Y4B =    0
         6) - 0.8 WA + Y1A <=   0
         7) - 0.3 WA + Y2A <=   0
         8) - 0.5 WA + Y4A >=   0
         9) - 0.4 WB + Y1B >=   0
        10) - 0.6 WB + Y1B <=   0
        11) - 0.3 WB + Y3B >=   0
        12) - 0.7 WB + Y4B <=   0
        13) - WA + Y1A + Y2A + Y3A + Y4A =    0
        14) - WB + Y1B + Y2B + Y3B + Y4B =    0
  END
  SUB       Z1     1000.00000
  SUB       Z2     2000.00000
  SUB       Z3     3000.00000

 LP OPTIMUM FOUND AT STEP     15

        OBJECTIVE FUNCTION VALUE

        1)      564210.6

  VARIABLE        VALUE          REDUCED COST
        WA      1884.211000           .000000
        WB      1026.316000           .000000
        Z1      1000.000000       -268.421100
        Z2      2000.000000       -147.894740
        Z3       210.526300           .000000
       Y1A          .000000       1105.263100
       Y1B       410.526300           .000000
       Y2A       510.526330           .000000
       Y2B          .000000           .000000
       Y3A       431.579000           .000000
       Y3B       615.789500           .000000
       Y4A       942.105300           .000000
       Y4B          .000000        684.210510


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000        963.157900
```

```
     3)          .000000       -142.105300
     4)          .000000       -142.105300
     5)          .000000        542.105300
     6)      1507.369000           .000000
     7)        54.736880           .000000
     8)         -.000000       -684.210510
     9)         -.000000      -1105.263100
    10)       205.263200           .000000
    11)       307.894710           .000000
    12)       718.421020           .000000
    13)          .000000        142.105300
    14)          .000000        142.105300
```

That is, 1884.211 tons of alloy A and 1026.316 tons of alloy B should be produced. All available supplies of ores #1 and #2, together with 210.5263 tons of ore #3 should be processed to extract the metals. 410.5263 tons of metal I will be extracted, all of which is to be allocated to alloy A. 510.5263 tons of metal II will be extracted, all of which is allocated to alloy A. 1047.385 tons of metal III will be extracted, of which 431.579 tons are allocated to alloy A and 615.7895 tons to alloy B. Finally, 942.1053 tons of metal IV will be extracted, all of which is allocated to alloy A. The profit resulting from this production plan is $564210.60,
which is $164,210 higher than was obtained in the previous model, since in this model the firm has more flexibility in the allocation of the metals. (Recall that in model **Ic**, the metals from an ore could only be used in the alloy to which it was originally allocated.)

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2. Solution

a. *A basic solution of the constraint equations of this problem has how many basic variables, in addition to -z?* **Solution:** __two (one for each equation)__

b. *What is the maximum number of basic solutions (either feasible or infeasible) which might exist?* (That is, how many ways might you select a set of basic variables from the four variables x_1 through x_4?) **Solution:** __6__ ,that is,

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{24}{2 \times 2}$$

c. *Find and list all of the basic solutions of the constraint equations.*
Solution: For each possible pair of basic variables, set the nonbasic variables equal to zero, and solve for the basic variables:

	Basic	Nonbasic
1)	$x_1= ?$, $x_2= ?$	$x_3= 0$, $x_4= 0$
2)	$x_1= 4$, $x_3= 0$	$x_2= 0$, $x_4= 0$
3)	$x_1= 4$, $x_4= 0$	xv= 0, $x_3= 0$
4)	$x_2= 2$, $x_3= 0$	$x_1= 0$, xv= 0
5)	$x_2= 2$, $x_4= 0$	$x_1= 0$, $x_3= 0$
6)	$x_3= -{}^4/_7$, x4= ${}^{16}/_7$	$x_1= 0$, $x_4= 0$

Note that x_1 and x_2 are not determined when x_3 and x_4 are set equal to zero, i.e., x_1 and x_2 do not form a valid basis. (The 2x2 basis matrix must be nonsingular, i.e., possess an inverse!)

d. Is the number of basic solutions in (c) equal to the maximum possible number which you specified in (b)? _no_

e. How many of the basic solutions in (c) are feasible (i.e. nonnegative)?
Solution: Solutions (2), (3), (4), and (5) are feasible, although these are only two distinct solutions.

f. By evaluating the objective function at each basic solution, find the optimal solution.
Solution: The objective function equals 4 at solutions (2) & (3), and also at (4) & (5). We thus have an unusual case in which every basic feasible solution of the LP is optimal!

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3. Consider the LP

$$\text{Maximize } z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$
subject to
$$x_1 + 4x_2 - 2x_3 + 8x_4 \quad 2$$
$$-x_1 + 2x_2 + 3x_3 + x_4 \quad 1$$
$$x_1, x_2, x_3, x_4 \quad 0$$

a. Introduce slack variables to convert the inequality constraints to equations.
Solution: $$\text{Maximize } z = 2x_1 - 4x_2 + 5x_3 - 6x_4$$
subject to
$$x_1 + 4x_2 - 2x_3 + 8x_4 + x_5 = 2$$
$$-x_1 + 2x_2 + 3x_3 + x_4 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \quad 0$$

b. Form the simplex tableau for this LP.

Solution:

-z	x1	x2	x3	x4	x5	x6	rhs
1	2	-4	5	-6	0	0	0
0	1	4	-2	8	1	0	2
0	-1	2	3	1	0	1	1

c. What is the beginning basic solution? Is it feasible? What is its objective function value?
Solution: We may use -z, x_5 and x_6 as the initial basic variables, which is feasible (-z=0, x_5=2, x_6=1)

d. Perform as many iterations of the simplex algorithm as required to find an optimum solution.

Solution: Tableau #1:

1	2	-4	5	-6	0	0	0
0	1	4	−2	8	1	0	2
0	−1	2	3	1	0	1	1

We can choose as a pivot column *any* which has a positive value (relative profit) in the objective row. There are two such columns. If we choose x_3 (which has the largest relative profit) to enter the basis (i.e., increase), then we must pivot in the bottom row (the only row with a positive substitution rate), so that x_3 will replace x_6 in the basis. This results in the tableau #2:

1	3.667	−7.333	0	−7.667	0	−1.667	−1.667
0	0.3333	5.333	0	8.667	1	0.6667	2.667
0	−0.3333	0.6667	1	0.3333	0	0.3333	0.3333

Note that the objective has increased by 1.667 (i.e., -z=0 previously, but now -z = -1.667, or z=+1.667). There is now only one positive relative profit in the objective row, namely 3.667, and so the next pivot must be in the x_1 column. Again, there is only a single positive substitution rate in this column, and so we must pivot in that row, so that x_1 will replace x_5 in the basis. This results in the tableau #3:

1	0	−66	0	−103	−11	−9	−31
0	1	16	0	26	3	2	8
0	0	6	1	9	1	1	3

Since no variable now has a positive relative profit, the tableau #3 is optimal, i.e., the basic solution x_1=8, x_3=3, x_2=x_4=x_5=x_6=0 is optimal, with an objective value of z=31 (-z=-31)

❍❍❍❍❍❍❍❍❍❍ Homework #3 ❍❍❍❍❍❍❍❍❍

1. ***Simplex Method.*** Classify each simplex tableau below, using the following classifications, and write the appropriate letter on the right of the tableau. If class B, D, or E, indicate, by circling,the additional information requested.

A. UNIQUE OPTIMUM.
B. OPTIMAL, but with ALTERNATE optimal solutions. *Indicate (by circling) a pivot element which would yield an alternate basic optimal solution.*
C. INFEASIBLE
D. FEASIBLE but NOT OPTIMAL. *Indicate (by circling) a pivot element which would yield an improved solution.*
E. FEASIBLE but UNBOUNDED. *Indicate a variable which, by increasing without limits, will improve the objective without limit.*

Take careful note of whether the LP is being **min**imized or **max**imized! Note also that (-z), rather than z, appears in the first column (i.e., corresponding to the approach used in my notes instead of that in the text by Winston).

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	2	0	4	-2	-3	0	1	0	-10	
	0	0	0	2	-4	0	0	-1	1	3	____
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	0	0	4	2	3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	3	____
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	-2	0	-4	-2	-3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	3	____
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MAX	1	2	0	4	-2	-3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	-3	____
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS
MAX	1	-2	0	-4	-2	-3	0	1	0	-10

0	0	0	2	1	0	0	-1	1	3	_____
0	-3	1	0	-1	2	0	2	0	6	
0	2	0	3	0	5	1	1	0	2	

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2. LP Formulation During each 6-hour period of the day, the Bloomington Police Department needs at least the number of police officers shown below:

Time Period	# of Officers Required
12 midnight- 6 a.m.	12
6 a.m. - 12 noon	8
12 noon - 6 p.m.	6
6 p.m. - 12 midnight	15

Officers can be hired to work either 12 consecutive hours or 18 consecutive hours. Officers are paid \$12 per hour for each of the first 12 hours a day they work, and are paid \$18 per hour for each of the next 6 hours they work in a day. Formulate and solve an LP that can be used to minimize the cost of meeting Bloomington's daily police requirements.

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3. Revised Simplex Method We wish to solve the LP problem

$$\begin{aligned} \text{Maximize } \; z &= 10X_1 + 6X_2 + 4X_3 \\ \text{subject to: } \; X_1 + X_2 + X_3 &\le 100 \\ 10X_1 + 4X_2 + 5X_3 &\le 600 \\ 2X_1 + 2X_2 + 6X_3 &\le 300 \\ X_j &\ge 0, \; j=1,2,3 \end{aligned}$$

After several iterations, we obtain the tableau below (in which some values have been omitted):

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
				$-10/3$			
			$5/6$	$5/3$	$-1/6$	0	$200/3$
	1	0	$1/6$	$-2/3$	$1/6$	0	
				-2	0	1	100

a. What is the "substitution rate" of X_4 for X_1?_______

b. If X_4 increases by 1 unit, X_1 will (**in**crease/**de**crease) *(circle one)* by ________ units.

c. What are the values of the simplex multipliers ( ) for this tableau? ____________

d. Using the results of (c), what is the relative profit of X_3? __________

e. Complete the missing portions of the tableau above.

f. Is the above tableau optimal? ________ If not, circle a pivot element which would improve the objective.

❍❍❍❍❍❍❍❍❍❍ Homework #3 Solutions ❍❍❍❍❍❍❍❍❍

1. ***Simplex Method.*** Classify each simplex tableau below, using the following classifications, and write the appropriate letter on the right of the tableau. If class B, D, or E, indicate, by circling,the additional information requested.

A. UNIQUE OPTIMUM.
B. OPTIMAL, but with ALTERNATE optimal solutions. *Indicate (by circling) a pivot element which would yield an alternate basic optimal solution.*
C. INFEASIBLE
D. FEASIBLE but NOT OPTIMAL. *Indicate (by circling) a pivot element which would yield an improved solution.*
E. FEASIBLE but UNBOUNDED. *Indicate a variable which, by increasing without limits, will improve the objective without limit.*

Take careful note of whether the LP is being **min**imized or **max**imized! Note also that (-z), rather than z, appears in the first column (i.e., corresponding to the approach used in my notes instead of that in the text by Winston).

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	2	0	4	-2	-3	0	1	0	-10	
	0	0	0	2	-4	0	0	-1	1	3	__E_
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

The tableau is unbounded, since increasing X_4 will lower the objective value (based upon the reduced cost, which equals -2) but there is not eligible pivot row in this column, i.e., increasing X_4 will also increase the two basic variables X_8 (in first row) and X_2 (in second row), and cause no change on X_6 (bottom row). Thus, increasing X_4 will not force any basic variable toward its lower limit of zero. *Note that one might pivot in the column for X_5 (in the bottom row), which will cause a finite improvement, but nevertheless the problem is unbounded, and eventually one would be left with X_4 as the only column with a negative reduced cost, but no eligible pivot row.*

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	0	0	4	2	3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	3	__B_
	0	-3	1	0	-1	2	0	2	0	6	
	0	$\boxed{2}$	0	3	0	5	1	1	0	2	

The tableau is optimal, since there is no negative reduced cost in the objective row. However, another optimal basic solution exists, which is evident from the fact that X_1 is nonbasic but has a zero reduced cost, indicating that increasing X1 will cause no change in the objective value.

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MIN	1	-2	0	-4	-2	-3	0	1	0	-10	
	0	0	0	2	$\boxed{1}$	0	0	-1	1	3	D
	0	-3	1	0	-1	2	0	2	0	6	
	0	$\boxed{2}$	0	$\boxed{3}$	0	$\boxed{5}$	1	1	0	2	

Note: This was originally stated as a MAX problem, and was identical to the last tableau in this exercise. In the revised exercise it was changed to a MIN problem.

The tableau is not optimal, since there are four different variables having negative reduced costs in the objective row. Pivoting in any one of these columns (as indicated) will accomplish an improvement in the objective function.

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MAX	1	2	0	4	-2	-3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	-3	C
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	1	0	2	

The tableau is infeasible, since the value of the basic variable X_8 is negative.

	-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
MAX	1	-2	0	-4	-2	-3	0	1	0	-10	
	0	0	0	2	1	0	0	-1	1	3	D
	0	-3	1	0	-1	2	0	2	0	6	
	0	2	0	3	0	5	1	$\boxed{1}$	0	2	

The tableau is not optimal, since one variable (X_7) has a positive relative profit. The minimum ratio test indicates that the pivot should be in the bottom row, so that X_7 will replace X_6 in the basis.

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2. LP Formulation Solution:

Number the four 6-hour time intervals by integers 1,2,3, & 4. Define decision variables

X_t = number of officers assigned to 12-hour shift beginning in time interval t (t=1,2,3,4)

Y_t = number of officers assigned to 18-hour shift beginning in time interval t (t=1,2,3,4)

Thus, for example, Y_3 is the number of officers who begin their shift at noon and complete their shift at 6 a.m. the following morning.

The daily salary for a 12-hour shift is \$144 (\$12/hour x 12 hours), while the daily salary for an 18-hour shift is \$252 (\$144 for the first 12 hours, plus \$18/hour x 6 hours for the additional 6 hours).
The **objective function** then becomes

$$\text{MIN}\quad 144X_1 + 144X_2 + 144X_3 + 144X_4 + 252Y_1 + 252Y_2 + 252Y_3 + 252Y_4$$

For each 6-hour period of the day, we impose a restriction that the number of officers on duty must equal (or exceed) the number required, namely

$$\begin{array}{llllllllll}
X_1 & & & + X_4 & + Y_1 & & + Y_3 & + Y_4 & & 12 \\
X_1 & + X_2 & & & + Y_1 & + Y_2 & & + Y_4 & & 8 \\
 & X_2 & + X_3 & & + Y_1 & + Y_2 & & + Y_3 & & 6 \\
 & & X_3 & + X_4 & & + Y_2 & + Y_3 & + Y_4 & & 15
\end{array}$$

For example, those on duty between noon and 6 p.m. will be those working 12-hour shifts beginning either at 6 a.m. or noon, plus those working 18-hour shifts beginning either at midnight, 6 a.m., or noon (as in the 3rd constraint above).
In addition, of course, we impose both nonnegativity constraints and integer constraints on the variables

$$X_i \in \{0, 1, 2, 3, \ldots\} \text{ and } Y_i \in \{0, 1, 2, 3, \ldots\},\ \ i=1,2,3,4$$

, i.e., the number of officers working each shift must be a nonnegative integer. Ignoring the integer restriction, LINDO gives us the solution:

```
MIN     144 X1 + 144 X2 + 144 X3 + 144 X4 + 252 Y1 + 252 Y2 + 252 Y3
       + 252 Y4
  SUBJECT TO
         2)   X1 + X4 + Y1 + Y3 + Y4 >=   12
         3)   X1 + X2 + Y1 + Y2 + Y4 >=   8
         4)   X2 + X3 + Y1 + Y2 + Y3 >=   6
         5)   X3 + X4 + Y2 + Y3 + Y4 >=   15
  END

 LP OPTIMUM FOUND AT STEP      4

        OBJECTIVE FUNCTION VALUE

        1)      3132.000

  VARIABLE        VALUE          REDUCED COST
        X1         3.000000           .000000
        X2          .000000           .000000
        X3         1.000000           .000000
        X4         9.000000           .000000
        Y1          .000000         72.000000
        Y2         5.000000           .000000
        Y3          .000000         72.000000
        Y4          .000000           .000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         -.000000        -36.000000
        3)         -.000000       -108.000000
        4)         -.000000        -36.000000
        5)         -.000000       -108.000000

 NO. ITERATIONS=       4

 RANGES IN WHICH THE BASIS IS UNCHANGED:
```

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	144.000000	.000000	36.000000
X2	144.000000	INFINITY	.000000
X3	144.000000	.000000	.000000
X4	144.000000	.000000	.000000
Y1	252.000000	INFINITY	72.000000
Y2	252.000000	.000000	72.000000
Y3	252.000000	INFINITY	72.000000
Y4	252.000000	INFINITY	.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	12.000000	5.000000	1.000000
3	8.000000	1.000000	5.000000
4	6.000000	5.000000	.500000
5	15.000000	1.000000	5.000000

X1	3.000000	.000000
X2	.000000	.000000
X3	1.000000	.000000
X4	9.000000	.000000
Y1	.000000	72.000000
Y2	5.000000	.000000
Y3	.000000	72.000000
Y4	.000000	.000000

Note that the LP solution, ignoring the integer restrictions, happens to be integer! (Such is not in general the case, of course-- the requirements were selected here in such a way as to give this result.)
The optimal solution is to assign 3 officers to begin a 12-hour shift beginning at midnight, one to a 12-hour shift beginning at noon, 9 to a 12-hour shift beginning at 6 p.m., and 5 to an 18-hour shift beginning at 6 p.m. This will cost the police department $3132 in daily salaries.

Note that X2 and Y4 are both zero with reduced costs of zero. Both are nonbasic, since there are 4 positive basic variables (X1, X3, X4, & Y2), one for each constraint. Therefore, there are **alternate optimal solutions** for this problem.

The substitution rates are given by the TABL (tableau) command of LINDO:

THE TABLEAU

ROW	(BASIS)	X1	X2	X3	X4	Y1	Y2
1	ART	.000	.000	.000	.000	72.000	.000
2	X1	1.000	1.000	.000	.000	2.000	.000
3	X4	.000	-1.000	.000	1.000	-1.000	.000
4	Y2	.000	.000	.000	.000	-1.000	1.000
5	X3	.000	1.000	1.000	.000	2.000	.000

ROW	Y3	Y4	SLK 2	SLK 3	SLK 4	SLK 5	
1	72.000	.000	36.000	108.000	36.000	108.000	-3132.000
2	1.000	.000	-1.000	.000	-1.000	1.000	3.000
3	.000	1.000	.000	.000	1.000	-1.000	9.000
4	-1.000	1.000	1.000	-1.000	1.000	-1.000	5.000

```
        5     2.000   -1.000    -1.000     1.000    -2.000     1.000     1.000
```

From this we see that each officer assigned to the 12-hour shift beginning at 6 a.m. (X2) would replace 1 officer on each of the 12-hour shifts beginning at midnight (X1) and noon (X3), and require an additional officer to be added to the 12-hour shift beginning at 6 p.m. (X4). The minimum-ratio test limits the number to be added to this shift (12-hour shift beginning at 6 a.m.) to a single officer, since X3 is only 1.000.

Another alternative is to add officers to the 18-hour shift beginning at 6 p.m. (Y4), each officer replacing one officer each from the 12-hour shift beginning at 6 p.m. (X3) and the 18-hour shift beginning at 6 a.m. (Y2), and requiring another officer to be added to the 12-hour shift beginning at noon (X3). Up to 5 officers may be added to the 18-hour shift beginning at 6 p.m. (Y4), at which point Y2 becomes zero and leaves the basis.

3. Revised Simplex Method We wish to solve the LP problem

$$\begin{aligned} \text{Maximize } & z=10X_1 + 6X_2 + 4X_3 \\ \text{subject to: } & X_1 + X_2 + X_3 \le 100 \\ & 10X_1 + 4X_2 + 5X_3 \le 600 \\ & 2X_1 + 2X_2 + 6X_3 \le 300 \\ & X_j \ge 0,\ j=1,2,3 \end{aligned}$$

After several iterations, we obtain the tableau below (in which some values have been omitted):

-Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
				$-10/3$			
			$5/6$	$5/3$	$-1/6$	0	$200/3$
	1	0	$1/6$	$-2/3$	$1/6$	0	
				-2	0	1	100

a. What is the "substitution rate" of X_4 for X_1? $\boxed{-2/3}$

b. If X_4 increases by 1 unit, X_1 will (<u>**in**crease</u>/**de**crease) *(circle one)* by $\boxed{2/3}$ units.

c. What are the values of the simplex multipliers ( ) for this tableau? $\boxed{10/3,\ 2/3,\ 0}$

Given the information in the tableau above, the basis must consist of (besides -z) X_2, X_1, and X_6 (in that order!). The basis inverse matrix must be the 3x3 submatrix which appears in the constraint rows in the columns of the slack variables X_4, X_5, and X_6, i.e.,

$$\left(A^B\right)^{-1} = \begin{bmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Therefore the simplex multiplier vector is

$$\pi = c_B\left(A^B\right)^{-1} = \left[c_2,\ c_1,\ c_6\right]\left(A^B\right)^{-1}$$

$$= \left[6,\ 10,\ 0\right] \times \begin{bmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

d. Using the results of (c), what is the relative profit of X_3? $\boxed{-3/3}$

$$\bar{c}_3 = c_3 - \pi A^3 = 4 - \left[10/3\ ,\ 2/3\ ,\ 0\right]\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} = 4 - 20/3$$

e. Complete the missing portions of the tableau above.

1	0	0	$-8/3$	$-10/3$	$-2/3$	0	$-2200/3$
0	0	1	$5/6$	$5/3$	$-1/6$	0	$200/3$
0	1	0	$1/6$	$-2/3$	$1/6$	0	$100/3$
0	0	0	4	-2	0	1	100

f. Is the above tableau optimal? _Yes_ (since the problem is a MAXimization and there are no positive relative profits in the objective row!)

❍❍❍❍❍❍❍❍❍❍ Homework #4 ❍❍❍❍❍❍❍❍❍

1. Obtaining a dual problem of a given LP Consider the following LP problem *(the primal):*

Maximize $z = 2x_1 + 6x_2 + 9x_3$
subject to

$x_1 + x_3 \le 3$

$x_2 + 2x_3 \le 5$

$x_j \ge 0$ for i=1,2,3

a. Write the dual LP problem formulation.
b. Plot the feasible region of the dual problem.
c. Determine the optimal solution of the dual problem, by comparing its objective value at the various basic feasible ("corner-point") solutions.
d. Which variables are basic in the optimal dual solution?
e. Compute the simplex multiplier vector (ð) for the optimal basis of the dual problem.
f. From your results above, what must be the optimal solution of the primal problem?
g. If one were to use the revised simplex method, what would be the sizes of the basis inverse matrix for the primal and dual problems above?

2. Sensitivity Analysis. Consider the following LP problem:

Brady Corporation produces wooden cabinets. Each week, they require 90,000 cu ft of processed lumber. They may obtain processed lumber in two ways. First, they may purchase lumber from an outside supplier and then dry it at their kiln. Second, they may chop down logs on their land, cut them into lumber at their sawmill, and finally dry the lumber at their kiln.

Brady can purchase grade 1 or grade 2 lumber. Grade 1 lumber costs \$3 per cu ft and when dried yields 0.7 cu ft of useful lumber. Grade 2 lumber costs \$7 per cu ft and when dried yields 0.9 cu ft of useful lumber. It costs the company \$3 per cu ft to chop down a log. After being cut and dried, one cubic foot of log yields 0.8 cu ft of lumber. Brady incurs costs of \$4 per cu ft of lumber dried.

It costs \$2.50 per cu ft of logs sent through their sawmill. Each week, the sawmill can process up to 35,000 cu ft of lumber. Each week, up to 40,000 cu ft of grade 1 lumber and up to 60,000 cu ft of grade 2 lumber can be purchased.

Each week, 40 hours of time are available for drying lumber. The time it takes to dry 1 cu ft of grade 1 lumber, grade 2 lumber, or logs is as follows:

grade 1:	2 seconds
grade 2:	0.8 seconds
log:	1.3 seconds

Brady has formulated an LP to minimize the weekly cost of meeting the demand for processed lumber:

Define the decision variables

G1 = the # of cu ft /week of grade 1 lumber purchased and used,
G2 = the # of cu ft /week of grade 2 lumber purchased and used,
LOG = the # of cu ft /week of the corporation's own lumber used.

LP Model :

```
Min   3G1+7G2          (purchase cost)
      +4(G1+G2+LOG)    (dry cost for lumber)
```

```
        +3LOG       (cost for chopping)
        +2.5LOG           (cost for sawmill)
  s.t.
        0.7G1+0.9G2+0.8LOG 90000      (constraint for demand)
        2G1+0.8G2+1.3LOG 144000 (available hours for drying)
        G1 40000          (available cu ft of grade 1 per week)
        G2 60000          (available cu ft of grade 2 per week)
        LOG 35000         (available cu ft of own lumber per week)
        G1,G2,LOG 0
```

LINDO output : *(Note that the upper bounds on G1, G2, and LOG were entered into LINDO using the SUB (simple upper bound) command.)*

```
      MIN     7 G1 + 11 G2 + 9.5 LOG
        SUBJECT TO
               2)   0.7 G1 + 0.9 G2 + 0.8 LOG >=   90000
               3)   2 G1 + 0.8 G2 + 1.3 LOG <=   144000
        END
        SUB       G1    40000.00000
        SUB       G2    60000.00000
        SUB      LOG    35000.00000

       LP OPTIMUM FOUND AT STEP      3

              OBJECTIVE FUNCTION VALUE

              1)      1033585.

        VARIABLE        VALUE          REDUCED COST
              G1     40000.000000          -.905659
              G2     55471.703000           .000000
             LOG     15094.340000           .000000


             ROW   SLACK OR SURPLUS     DUAL PRICES
              2)          -.000000        -12.641510
              3)           .000000           .471699

       NO. ITERATIONS=       3

       RANGES IN WHICH THE BASIS IS UNCHANGED:

                                 OBJ COEFFICIENT RANGES
       VARIABLE          CURRENT        ALLOWABLE        ALLOWABLE
                          COEF          INCREASE         DECREASE
             G1         7.000000         .905659         INFINITY
             G2        11.000000         .695651          .312500
            LOG         9.500000         .277778          .387096

                                 RIGHTHAND SIDE RANGES
            ROW          CURRENT        ALLOWABLE        ALLOWABLE
                           RHS          INCREASE         DECREASE
              2    90000.000000      1846.151400     13187.500000
              3   144000.000000     11722.222000      2999.996000

THE TABLEAU
    ROW (BASIS)      G1       G2        LOG     SLK 2   SLK 3
    1 ART           .91    0.69E-06 0.72E-06  13.      .47         -0.10E+07
    2 LOG         -2.340   0.000    1.000      1.509   1.698   15094.34
    3 G2           1.302   1.000     .000     -2.453  -1.509   55471.70
```

Note: It might seem surprising that G1 is nonbasic, since it has a positive value. I chose to use the SUB command of LINDO to enter the upper bounds on the variables, which LINDO handles by what is called the Upper Bounding Technique, a variation of the simplex method in which nonbasic variables may be at either their upper of lower bounds (unlike the original simplex method, in which nonbasic variables are equal to their lower bounds, namely zero.) You can see that the optimal value of G1 is its upper bound.

a. How many cubic feet of each grade of lumber should be purchased and processed?
Grade 1: _______ cu ft/week; Grade 2: _______ cu ft/week

b. How many cubic feet of logs should Brady cut from its own forest? _______ cu ft/week

c. Suppose that the company can increase the amount of time available for drying by 2 hours (= _______ seconds) per week, through use of overtime, which will cost $1500 per hour including additional labor and energy costs. Should they schedule the overtime? ______ (Explain why or why not!)

Should they schedule all 2 hours which are available? ______

d. Whether or not you answered "yes" to both questions in part (c), suppose that 2 additional hours are available on the dryer. Using the substitution rates, compute the modifications to the optimal values of the variables which would result from the use of this additional time.
G1 ___________ increase? decrease? (circle)
G2 ___________ increase? decrease? (circle)
LOG___________ increase? decrease? (circle)

e. If the cost of purchasing Grade 2 lumber were to increase by 50¢/cu ft, will the optimal solution change? ____________ Why or why not?

f. If the cost of cutting and processing logs from its own land were to increase to $10/cu ft, will the optimal solution change? ____________ Why or why not?

g. If the company could deliver less than the demanded amount of lumber by forfeiting a penalty of $10/cu ft, should they do so? ___________
If so, how much should they deliver? ____________ cu ft.

❍❍❍❍❍❍❍❍❍❍ Homework #4 Solutions ❍❍❍❍❍❍❍❍❍

1. Obtaining a dual problem of a given LP Consider the following LP problem *(the primal):*

Maximize $z = 2x_1 + 6x_2 + 9x_3$
subject to

$x_1 + x_3 \leq 3$

$x_2 + 2x_3 \leq 5$

$x_j \geq 0$ for i=1,2,3

a. Write the dual LP problem formulation.

Solution: Note that this LP is in the form which appears in the symmetric primal-dual pair (i.e., Max with nonnegative variables and " " constraints). The dual LP is

Minimize $z = 3y_1 + 5y_2$
subject to

$y_1 \geq 2$

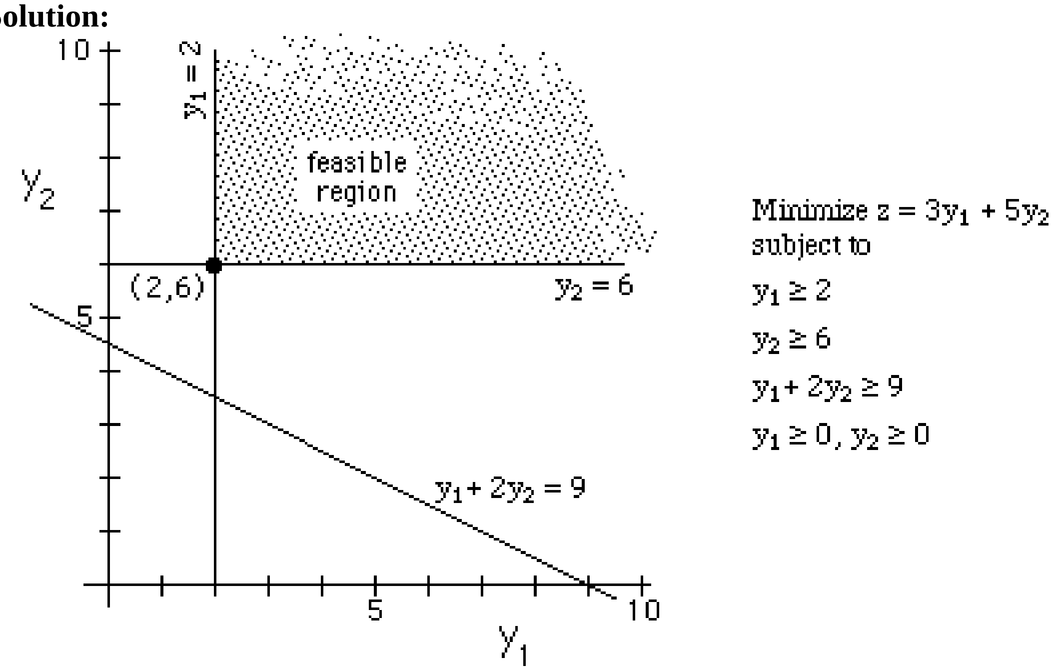
$y_2 \geq 6$

$y_1 + 2y_2 \geq 9$

$y_1 \geq 0,\ y_2 \geq 0$

b. Plot the feasible region of the dual problem.

Solution:



Note that the third inequality constraint is slack everywhere in the feasible region, i.e., the constraint is superfluous and may be discarded without any effect on the feasible solutions!

c. Determine the optimal solution of the dual problem, by comparing its objective value at the various basic feasible ("corner-point") solutions.

Solution: There is only one corner-point of the dual feasible region, namely y = (2,6),. It is easily seen that the objective function in the dual is bounded below (by zero, for example), so that this solution mUst be optimal.

d. Which variables are basic in the optimal dual solution?

Solution: y_1, y_2, and S_3 (the surplus variable in the third constraint) which has the positive value 2 + 2(6) - 9 = 5 at the solution.

e. Compute the simplex multiplier vector (ð) for the optimal basis of the dual problem.

Solution: Using the formula $\pi = c_B\left(A^B\right)^{-1}$, where A is the coefficient matrix of the dual problem, namely

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \end{bmatrix},$$

B is the optimal basis, i.e., the index set of the columns of basic variables, namely [1, 2, 5], from (d) above, we obtain:

$$\pi = [3, 5, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}^{-1}$$

$$= [3, 5, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= [3, 5, 0]$$

Note that the inverse of the basis matrix is identical to the basis matrix in this case.

f. From your results above, what must be the optimal solution of the primal problem?

Solution: Since the simplex multipliers of an optimal LP basis solve the dual of that problem, this means that x* = ð = [3, 5, 0], i.e., x_1=3, x_2=5, x_3=0. Note that the objective values of the two problems at the two optimal solutions are identical, i.e., cx* = 2(3) + 6(5) + 9(0) = 36 & by* = 3(2) + 5(6) = 36.

g. If one were to use the revised simplex method, what would be the sizes of the basis inverse matrix for the primal and dual problems above?

Solution: Since the primal (max) problem tableau has 2 rows, its basis matrix (and its basis inverse) are 2 x 2 (a total of 4 entries); the dual (min) tableau has 3 rows, so that its basis matrix (and its inverse) will be 3 x 3 (a total of 9 entries, more than twice that of the primal!)

2. Sensitivity Analysis Solution:

a. How many cubic feet of each grade of lumber should be purchased and processed? Grade 1: 40,000 cu ft/week; Grade 2: 55,471.703 cu ft/week

b. How many cubic feet of logs should Brady cut from its own forest? 15,094.34 cu ft/week

c. Suppose that the company can increase the amount of time available for drying by 2 hours (= 7200 seconds) per week, through use of overtime, which will cost $1500 per hour including additional labor and energy costs. Should they schedule the overtime? __YES_
Should they schedule all 2 hours which are available? __YES_

Solution: The cost of overtime is \$1500/hour = \$0.41667/second. The "DUAL PRICE" of row 3 (which imposes the restriction on the number of seconds available for drying lumber) is \$0.4717/second, i.e., increasing the right-hand-side of row 3 will improve (lower) the objective function (cost) at the rate of \$0.4717/second, so long as the current basis remains unchanged. Consulting the RHS range information, we see that the "ALLOWABLE INCREASE" of the right-hand-side of row 3 is 1722.222 seconds. Thus the 7200 second increase which is being considered is well within the range and each of those seconds of additional drying time improves our objective by \$0.4717/second. Since the reduction in cost exceeds the cost of the overtime (\$0.41667/sec.), the net effect on cost will be a reduction by \$0.4717 - 0.4167 = \$ 0.055/sec. or \$396.22 for 2 hours.

d. Whether or not you answered "yes" to both questions in part (c), suppose that 2 additional hours are available on the dryer. Using the substitution rates, compute the modifications to the optimal values of the variables which would result from the use of this additional time.

G1 _no change_

G2 _10864.8 cu ft__ decrease?

LOG_12225.6 cu ft increase

Solution: The available drying time constraint is

```
3)   2 G1 + 0.8 G2 + 1.3 LOG <=   144000
```

or, after converting to equation form,

```
3)   2 G1 + 0.8 G2 + 1.3 LOG + SLK3 =   144000
```

In order to increase the drying time (2 G1 + 0.8 G2 + 1.3 LOG) by 7200 seconds, we must decrease SLK3 from its current value (0) by 7200 to -7200. The substitution rates of SLK3 for the basic variables are, according to the tableau,

$$\begin{array}{r} \text{LOG:} \\ \text{G2:} \end{array} \begin{bmatrix} 1.698 \\ -1.509 \end{bmatrix},$$

i.e., an increase in SLK3 would decrease the basic variable LOG at the rate of 1.698 cu ft/sec and would increase the basic variable G2 at the rate of 1.509 cu ft/sec. Conversely, reducing SLK3 will increase LOG and decrease G2 at those rates. Thus, a decrease in SLK of 7200 seconds will increase LOG by 1.698 cu ft/sec (7200 sec.) = 12,225.6 cu ft. and will decrease G2 by 1.509 cu ft/sec (7200 sec.) = 10,864 cu ft. The nonbasic variables (including G1) will remain unchanged. *Note: It might seem surprising that G1 is nonbasic, since it has a positive value. I chose to use the SUB command of LINDO to enter the upper bounds on the variables, which LINDO handles by what is called the Upper Bounding Technique, a variation of the simplex method in which nonbasic variables may be at either their upper of lower bounds (unlike the original simplex method, in which nonbasic variables are equal to their lower bounds, namely zero.) You can see that the optimal value of G1 is its upper bound.*

e. If the cost of purchasing Grade 2 lumber were to increase by 50¢/cu ft, will the optimal solution change? ___NO_ Why or why not?

Solution: The "ALLOWABLE INCREASE" in the objective coefficient of the variable G2 is \$0.6956/cu ft. Therefore, if an increase of \$0.50/cu ft were to occur, the optimal basis will not change, and hence also the values of the basic variables will not change.

f. If the cost of cutting and processing logs from its own land were to increase to \$10/cu ft, will the optimal solution change? ___YES___ Why or why not?

Solution: The "ALLOWABLE INCREASE" in the objective coefficient of the variable LOG is \$0.277778/cu ft, which is less than the hypothetical change of \$0.50/cu ft (i.e., \$9.50/cu ft - \$10/cu ft = - \$0.50/cu ft). Therefore, the basis will change if the increase in cost of LOG were to occur.

g. If the company could deliver less than the demanded amount of lumber by forfeiting a penalty of \$10/cu ft, should they do so? ___YES___
If so, how much should they deliver? 90000 - 13187.5 = _76812.5_ cu ft.

Solution: The "DUAL PRICE" of row 2 (which imposes the constraint that 90,000 cu ft of lumber be provided to the cabinet-making unit of the company) is - \$12.64151 / cu ft. That is, every unit (cubic foot) of increase in the right-hand-side of row 2 will result in a cost saving of approximately \$12.65. If we were to pay a penalty of \$10/cu ft in order to accomplish this, it will still leave us with a net reduction in cost of \$2.65. The DUAL PRICE used above is valid throughout the range of right-hand-sides within which the current basis is optimal, i.e., as much as 13187.5 cubic feet ("ALLOWABLE DECREASE").

❍❍❍❍❍❍❍❍❍❍ Homework #5 ❍❍❍❍❍❍❍❍❍

1. Obtaining a dual problem of a given LP Write the dual LP of the following LP problem *(the primal)* :

Minimize $-2x_1 + 3x_2 + 5x_3$
subject to

$-2x_1 + x_2 + 3x_3 + x_4 \geq 5$

$2x_1 + x_3 \leq 4$

$2x_2 + x_3 + x_4 = 6$

$x_1 \leq 0$

$x_2,\ x_3 \geq 0$

x_4 unrestricted in sign

2. Sensitivity Analysis. Consider the LP problem and LINDO output in the Hypercard stack, "Lizzie's Dairy".

a. Suppose that the evaporator malfunctions during the day, and is able to process only 1500 lb. of milk. If possible, determine the resulting loss of profit: \$____________

b. Suppose that there is a 10% increase in the minimum requirement for cream cheese. If possible, determine the resulting change in profit: \$___________ (increase or decrease?)

c. In the situation of (b) above, determine (if possible) the change, if any, of the optimal quantity of
   # pounds of high-fat milk to be purchased: ____________
   # pounds of low-fat milk to be purchased: ____________

d. Suppose that, due to a shortage of cattle feed, the costs of high-fat and low-fat milk will increase by 5% each. Would the optimal solution change? ________ (yes/no?) What if the cost increase is 10% ? _____________ (yes/no?) *Explain your answers!*

e. Suppose that, due to a misunderstanding, 100 pounds of low-fat milk was put through the evaporator. Determine, if possible,
   - the resulting loss in profit, if any: $____________
   - the change in the quantity of high-fat milk to be put through the evaporator: _______ lb.
   - the optimal quantities of high-fat milk and low-fat milk to be purchased:
     High-fat milk: ____________ pounds
     Low-fat milk: ____________ pounds

3. Operations Research resources on the WWW: Go to the class home web page at the URL

http://www.uiowa.edu/~ie171

From here, click on the link "O.R. on the WWW", and find the link to "NEOS Guide to Optimization Software".

a. What does "NEOS" abbreviate? ____

Click on the link to "Linear Programming".

b. How many "LP Solver" (software) links are listed here? _________

Find the link to LINDO

c. What is the maximum problem size which can be handled by the largest version of LINDO ("Extended LINDO")? ____________ constraints and __________ variables.

Find the link to "What's Best?".

d. Describe briefly the "What's Best?" software.

Go back to the "O.R. on the WWW" page on the class web site, and find the link to "List of Optimization Software in the Public Domain".

e. How many LP solvers (for general LP problems) are listed here? _______

❍❍❍❍❍❍❍❍❍❍❍ Homework #5 Solutions ❍❍❍❍❍❍❍❍❍❍

1. Obtaining a dual problem of a given LP

Primal:
$$\left\{\begin{array}{l} \text{Minimize } -2x_1 + 3x_2 + 5x_3 \\ \text{subject to} \\ -2x_1 + x_2 + 3x_3 + x_4 \geq 5 \\ 2x_1 + x_3 \leq 4 \\ 2x_2 + x_3 + x_4 = 6 \\ x_1 \leq 0 \\ x_2\,,\ x_3 \geq 0 \\ x_4 \text{ unrestricted in sign} \end{array}\right.$$

Dual:
$$\left\{\begin{array}{l} \text{Maximize } 5y_1 + 4y_2 + 6y_3 \\ \text{subject to} \\ -2y_1 + 2y_2 \geq -2 \\ y_1 + 2y_3 \leq 3 \\ 3y_1 + y_2 + y_3 \leq 5 \\ y_1 + y_3 = 0 \\ y_1 \geq 0,\ y_2 \leq 0, \\ y_3 \text{ unrestricted in sign} \end{array}\right.$$

Note: one may also convert the primal to the form of the "minimize" problem in the symmetric primal-dual pair, by

- negating both sides of the " " to convert it to a " " constraint
- replacing the equality constraint by a pair of inequalities
- replacing the nonpositive variable x_1 by its negative (which would be nonnegative)
- replacing x_4 by the difference of two nonnegative variables

and then writing the symmetric dual of the result.

$$\left\{\begin{array}{l} \text{Minimize } 2\overline{x}_1 + 3x_2 + 5x_3 \\ \text{subject to} \\ 2\overline{x}_1 + x_2 + 3x_3 + \left(x'_4 - x''_4\right) \geq 5 \\ 2\overline{x}_1 - x_3 \geq -4 \\ 2x_2 + x_3 + \left(x'_4 - x''_4\right) \geq 6 \\ -2x_2 - x_3 - \left(x'_4 - x''_4\right) \geq -6 \\ \overline{x}_1 \geq 0 \\ x_2\ \&\ x_3 \geq 0 \\ x'_4\ \&\ x''_4 \geq 0 \end{array}\right.$$

This gives a dual problem which is equivalent to the dual problem above:

$$\left\{\begin{array}{l} \text{Maximize } 5y_1 - 4\overline{y}_2 + 6y'_3 - 6y''_3 \\ \text{subject to} \\ 2y_1 + 2\overline{y}_2 \leq 2 \\ y_1 + 2y'_3 - 2y''_3 \leq 3 \\ 3y_1 - \overline{y}_2 + y'_3 - y''_3 \leq 5 \\ y_1 + y_3 \leq 0 \\ -y_1 - y_3 \leq 0 \\ y_1 \geq 0,\ \overline{y}_2 \geq 0,\ y'_3 \geq 0,\ y''_3 \geq 0 \end{array}\right.$$

By making the substition $y_2 = -\overline{y}_2$, $y_3 = y'_3 - y''_3$, and replacing the pair of inequalities

$$y_1 + y_3 \leq 0\ \&\ -y_1 - y_3 \leq 0$$

with the single equation $y_1 + y_3 = 0$, we would obtain the previous dual problem.

2. Sensitivity Analysis. Consider the LP problem and LINDO output in the Hypercard stack, "Lizzie's Dairy".

a. Suppose that the evaporator malfunctions during the day, and is able to process only 1500 lb. of milk. If possible, determine the resulting loss of profit: $_no change_

Solution: Row #6 imposes the capacity restriction on the evaporator:

6) HFE + LFE 2000

The dual price for row #6 is 0, with an ALLOWABLE DECREASE of 962.962 pounds. Since the decrease in capacity resulting from the malfunction is only 500 (< 962.962), there will be **no change** in the profit.

b. Suppose that there is a 10% increase in the minimum requirement for cream cheese. If possible, determine the resulting change in profit: $_10 decrease

Solution: The minimum requirement for cream cheese is imposed by the restriction in row #4:

4) P1 1000

The dual price for row #4 is - $0.10/lb, with an ALLOWABLE INCREASE of 500 lb. Hence an increase of 100 lb. in the requirement of cream cheese will cause an "improvement" in the objective ("profit") of (-$0.10/lb)(100 lb) = - $10, i.e., a decrease in the profit of $10.

c. In the situation of (b) above, determine (if possible) the change, if any, of the optimal quantity of

pounds of high-fat milk to be purchased: __increase 106.7 lb., to 1691.88 lb. __

pounds of low-fat milk to be purchased: ___increase 20 lb., to 960.74 lb.__

Solution: Row #4, if converted to an equation, would be

4) P1 - SLK4 = 1000

Hence, to increase P1 to 1100 lb. requires that the surplus (SLK4, which is nonbasic) be increased from 0 to 100 lb. Looking at the SLK4 column of the tableau, we determine that the substitution rate of SLK4 for the number of pounds of high-fat milk to be purchased (variable HF) is -1.067. This means that, if SLK4 is increased by 100 lb., the basic variable HF will also increase, by the amount 100(1.067) = 106.7 pounds.

On the other hand, the variable LF (= # pounds of low-fat milk to be purchased) has a substitution rate of -0.200, so that it will increase by only 100(0.200) = 20 lb.

d. Suppose that, due to a shortage of cattle feed, the costs of high-fat and low-fat milk will increase by 5% each. Would the optimal solution change? ___NO__

What if the cost increase is 10% ? ___NO__ *Explain your answers!*

Solution: Since the objective coefficient is being changed simultaneously for more than a single variable, this requires use of the "100% Rule". (See the text by Winston!) The increase in cost of high-fat milk (variable HF) is 0.05($0.80) = $0.04/lb. Since the ALLOWABLE INCREASE is $0.40/lb, the increase of 4¢ is 4/40 = 10% of the allowable increase. Similarly, the increase in cost of low-fat milk (variable LF) is 0.05($0.40) = $0.02/lb, while the ALLOWABLE INCREASE is also $0.40/lb, this is 5% of the allowable increase. The total increases, as a percent of the

ALLOWABLE INCREASEs, is 10% + 5% = 15%. Since this is less than 100%, the basis will not change as a result of the 5% price increases, and so the values of the basic variables are unchanged also.

In the case of the 10% price increases, the actual increases will be $0.08 = 20% of ALLOWABLE INCREASE and $0.04 = 10% of ALLOWABLE INCREASE for HF and LF, respectively, a sum of 30%. Again, since this is less than 100%, the basis, as well as the values of the basic variables, remain unchanged.

e. Suppose that, due to a misunderstanding, 100 pounds of low-fat milk was put through the evaporator. Determine, if possible,
- the resulting loss in profit, if any: $__20.00_____
- the change in the quantity of high-fat milk to be put through the evaporator: decrease 50 lb.
- the optimal quantities of high-fat milk and low-fat milk to be purchased:

High-fat milk: 1585.18 - 50 = 1535.18 pounds

Low-fat milk: 940.74 + 100 = 1040.74 pounds

Solution: The nonbasic variable LFE (# lb of low-fat milk put through the evaporator) has a reduced cost of $0.20/lb, so that increasing it by 100 lb will worsen the objective by $0.20/lb (100lb) = $20. To find the changes in the basic variables, we use the substitution rates from the tableau the nonzero substitution rates are:

SLK 6:	0.500
HF	0.500
LF	-1.000
HFE	0.500

This means that if LFE is increased by 100 lb., SLK6, HF, and HFE will all decrease by 0.500(100lb) = 50 lb, while LF will increase by 1.000(100lb) = 100 lb.

3. Operations Research resources on the WWW: *Go to the class home web page at the URL*

http://www.uiowa.edu/~ie171

From here, click on the link "O.R. on the WWW", and find the link to "NEOS Guide to Optimization Software".

a. What does "NEOS" abbreviate? __**N**etwork-**E**nabled **O**ptimization **S**ystem_

Click on the link to "Linear Programming".

b. *How many "LP Solver" (software) links are listed here?* __23__

Find the link to LINDO

c. What is the maximum problem size which can be handled by the largest version of LINDO ("Extended LINDO")? __32,000__ constraints and _100,000_ variables.

Find the link to "What's Best?".

d. Describe briefly the "What's Best?" software.

"What's Best" is a spreadsheet interface to LINDO, supporting Quattro Pro, Lotus 1-2-3, Excel for Macintosh, and Symphony."

Go back to the "O.R. on the WWW" page on the class web site, and find the link to "List of Optimization Software in the Public Domain".

e. *How many LP solvers (for general LP problems) are listed here?* __8___

❍❍❍❍❍❍❍❍❍❍ Homework #6 ❍❍❍❍❍❍❍❍❍

1. Transportation Model for Production Planning: (Exercise #3 of Review Problems, page 371 of text by Winston.) A company must meet the following demands for a product:

January	30
February	30
March	20

Demand may be backlogged at a cost of $5/unit/month. (That is, demand need not be satisfied on-time, but there is a penalty for lateness.) Of course, all demand must be met by the end of March. Thus, if 1 unit of January demand is met during March, a backlogging cost of $5(2) = $10 is incurred. Monthly production capacity and unit production cost during each month are:

Month	Production Capacity	Unit Production Cost
January	35	$400
February	30	$420
March	35	$410

A holding cost of $20/unit is assessed on the inventory at the end of each month.

a. Formulate a balanced transportation problem that could be used to determine how to minimize the total cost (including backlogging, holding, and production costs) of meeting demand. (It is sufficient to display a transportation tableau with rows for sources and columns for destinations.)

b. Use *either* Vogel's method *or* the Northwest-corner method to find a basic feasible solution of the transportation problem.

c. Use the transportation simplex method to determine how to meet each month's demand. Make sure to give an interpretation of your optimal solution. (For example, 20 units of month 2 demand is met from month 1 production, etc.)

2. Assignment Problem: (Exercise 2 of Review Problems, page 371 of text by Winston) Five workers are available to perform four jobs. The time (in hours) which it takes each worker to perform each job is:

	Job 1	Job 2	Job 3	Job 4
Worker #1	10	15	10	15
Worker #2	12	8	20	16
Worker #3	12	9	12	18
Worker #4	6	12	15	18
Worker #5	16	12	8	12

The goal is to assign workers to jobs so as to minimize the total time required to perform the four jobs. Use the Hungarian method to solve the problem. *(Hint: define a "dummy" job which requires zero time when assigned to any worker.)*

3. O.R. resources on the WWW. From the course web page, go to the page of O.R. resources on the WWW (as in last week's homework assignment). Find the link to INFORMS

a. What does the acronym INFORMS abbreviate?

Find the link to "INFORMS Education and Student Affairs", and from there to "INFORMS Student Union", and from there to the "Career Center". Click on the link to "Career Planning".

b. According to the Bureau of Labor Statistics survey of 1994, how many Operations Research analysts are employed? __________

c. What is the projected change (as a percent) in employment during the period 1994-2005? _______ % (increase *or* decrease?)

d. This ranks at # ____ of the top 25 fastest growing occupations.

❍❍❍❍❍❍❍❍❍❍ Homework #6 Solutions ❍❍❍❍❍❍❍❍❍

1. Transportation Model for Production Planning: (Exercise #3 of Review Problems, page 371 of text by Winston.)

a. *Formulate a balanced transportation problem that could be used to determine how to minimize the total cost (including backlogging, holding, and production costs) of meeting demand. (It is sufficient to display a transportation tableau with rows for sources and columns for destinations.)*

Solution:

Demand

Production	JAN	FEB	MAR	dummy	
JAN	400	420	440	0	35
FEB	425	420	440	0	30
MAR	420	415	410	0	35
	30	30	20	20	100

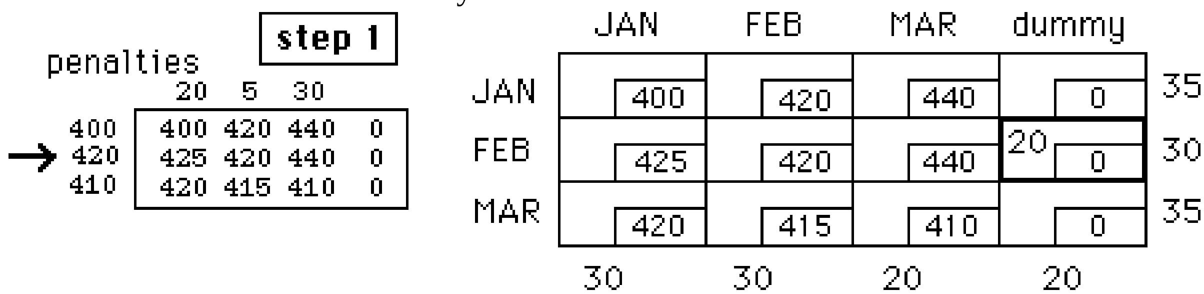
Note that a "dummy" destination is required, so that the total demand will equal the total supply (which is 100). The "shipping" cost actually consists of production and either backlogging or storage costs. For example, the cost of "shipping" from the JAN source to the MAR demand is $400 (production cost) + ($20/month)(2 months), the second term being the holding cost for inventory.

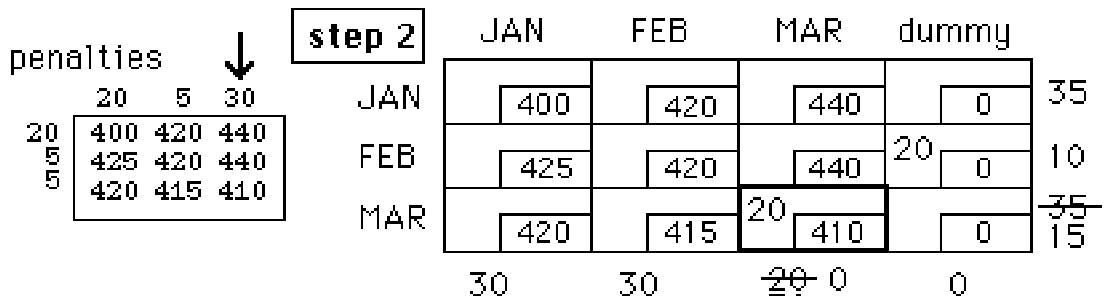
b. Use *either* Vogel's method *or* the Northwest-corner method to find a basic feasible solution of the transportation problem.
Solution: If we use the "Northwest-corner" method, we obtain the initial basic feasible solution:

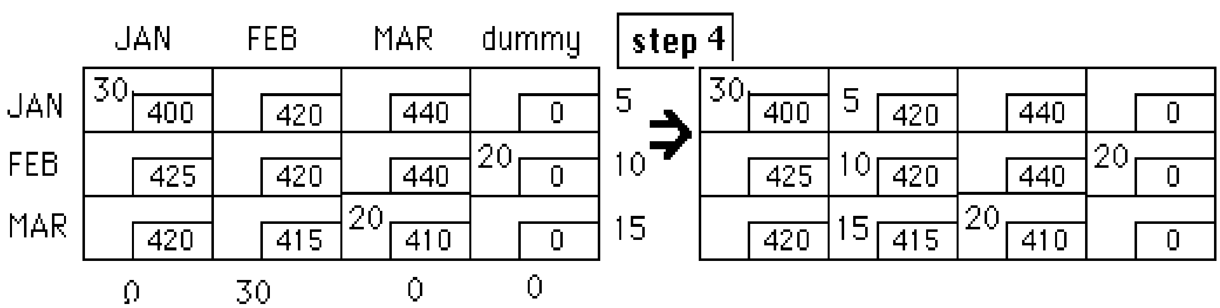
	JAN	FEB	MAR	dummy	
JAN	30 / 400	5 / 420	440	0	35
FEB	425	25 / 420	5 / 440	0	30
MAR	420	415	15 / 410	20 / 0	35
	30	30	20	20	

The cost of this "shipping" plan (actually a production plan) is 30($400) +5($420) +25($425) + 5($440) + 15($410) + 20($0) = $32950.

If we use Vogel's Approximation Method (VAM), we obtain the initial basic feasible solution below. Note that the feasible solution found by VAM has a lower cost than that found by NW-corner method.







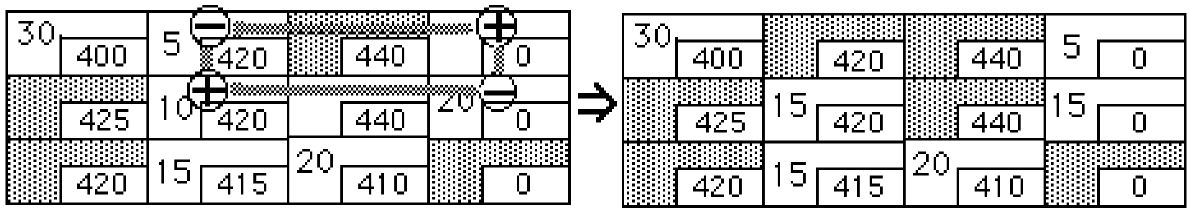
At step four of VAM, we obtain a matrix with nonzero demand in only one column, so the remaining shipments (to FEB demand) are determined. The total cost of this shipping plan is 30($400) +5($420) +10($420) +15($415) +20($410) +20($0) = $32725.

c. Use the transportation simplex method to determine how to meet each month's demand. Make sure to give an interpretation of your optimal solution. (For example, 20 units of month 2 demand is met from month 1 production, etc.)

Solution I: Let's use the starting basic feasible solution found by **VAM.**

- First, we calculate the dual variables. If we let $U_1=0$, then we obtain $U_1=0$, $U_2= 0$, $U_3=-5$, $V_1= 400$, $V_2=420$, $V_3= 415$, and $V_4= 0$.
- We then use these dual variables to compute the reduced costs $\underline{C}_{ij}$ of the nonbasic variables: $\underline{C}_{21} = 425-(0+400) = 25$; $\underline{C}_{31} = 420-(-5+400) = 25$; $\underline{C}_{13} = 440 - (0+415) = 25$; $\underline{C}_{23} = 440 - (0+400) = 40$; $\underline{C}_{14} = 0 - (0+0) = 0$; $\underline{C}_{34} = 0 - (-5+0) = 5$.
- Since $\underline{C}_{ij}$•0 for all i and j, the solution is optimal!

- However, since $\underline{C}_{14}$=0, this is not the only optimal solution. If we enter X_{14} into the basis, we obtain the "shipping" plan on the right below:

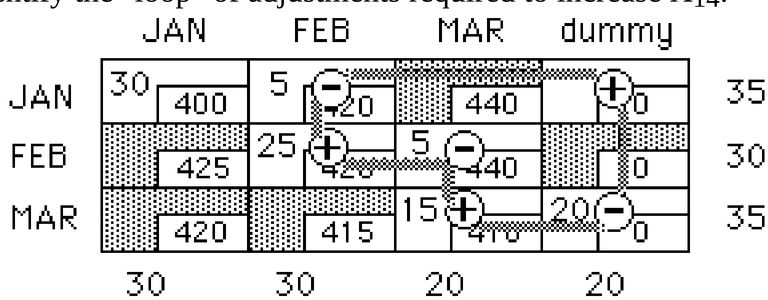


Solution II: Suppose that we use the initial basic feasible solution found by the **"Northwest-Corner" method:**

In order to price the nonbasic variables, we first compute the dual variables as shown (after arbitrarily setting U_1=0):

		JAN	FEB	MAR	dummy	
		400	**420**	**440**	**30**	
JAN	**0**	30 / 400	5 / 420	440	0	35
FEB	**0**	425	25 / 420	5 / 440	0	30
MAR	**-30**	420	415	15 / 410	20 / 0	35
		30	30	20	20	

- The reduced costs are now: $\underline{C}_{21}$=425-400>0, $\underline{C}_{31}$=420-370>0, $\underline{C}_{32}$=415-390>0, $\underline{C}_{13}$=440-440=0, $\underline{C}_{14}$=0-30<0, and $\underline{C}_{24}$=0-30<0. The solution can be improved by entering either X_{14} or X_{24} into the basis. Suppose that we choose X_{14}. We then identify the "loop" of adjustments required to increase X_{14}:



- For each unit increase in X_{14}, the variables X_{12}, X_{23}, and X_{34} are all decreased by one unit. When X_{14}=5, the basic variables X_{12} and X_{23} have both decreased (simultaneously) to zero. Only one of these should leave the basis, however (since the number of basic variables must remain constant at 6). Suppose we arbitrarily select X12 to leave the basis. Then the next basic tableau is

		JAN	FEB	MAR	dummy	
		400	450	410	0	
JAN	0	30 / 400	420	440	5 / 0	35
FEB	-30	425	30 / 420	0 / 440	0	30
MAR	0	420	415	20 / 410	15 / 0	35
		30	30	20	20	

with the new values of the dual variables as show. The reduced costs must now be re-computed: $\underline{C}_{12}$=420-450<**0**, $\underline{C}_{13}$=440-410>0, $\underline{C}_{21}$=425-370>0, $\underline{C}_{24}$=0-(-30)>0, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{32}$=415-450<**0**. Increaseing either X_{12} or X_{32} could improve the solution. Let's arbitrarily choose to increase X_{23}. We identify the loop of adjustments which must be made:

	JAN	FEB	MAR	dummy	
JAN	30 / 400	420	440	5 / 0	35
FEB	425	30 − / 420	0 + / 440	0	30
MAR	420	+ / 415	20 − / 410	15 / 0	35
	30	30	20	20	

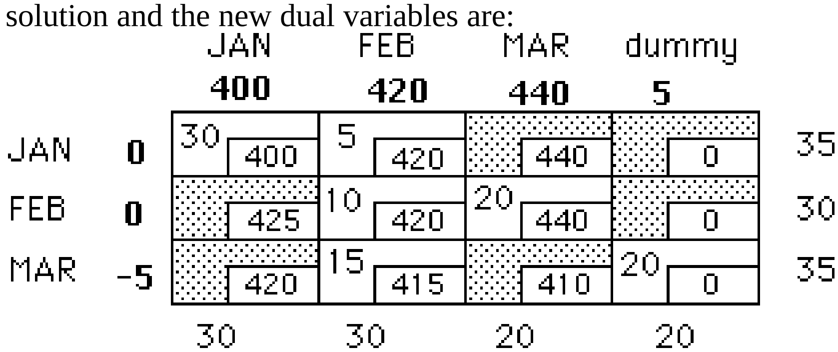
When X_{23} enters the basis, then, X_{33} leaves the basis, X_{22} decreases to 10, and X_{23} increases to 20. The new basic feasible solution and new values of the dual variables are:

		JAN	FEB	MAR	dummy	
		400	415	435	0	
JAN	0	30 / 400	420	440	5 / 0	35
FEB	5	425	10 / 420	20 / 440	0	30
MAR	0	420	20 / 415	410	15 / 0	35
		30	30	20	20	

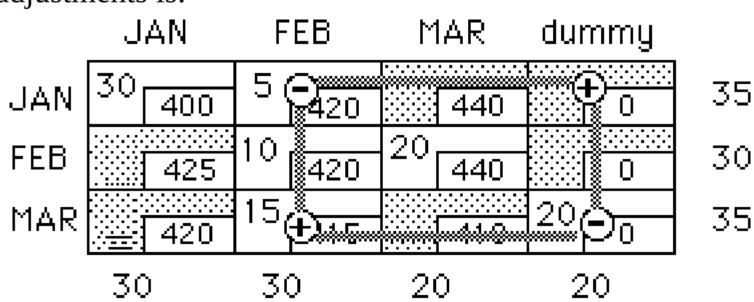
and the reduced costs are: $\underline{C}_{12}$=420-415<**0**, $\underline{C}_{13}$=440-435>0, $\underline{C}_{21}$=425-420>0, $\underline{C}_{24}$=0 - 5 <0, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{33}$=410-435<**0**. So Increasing either X_{12}, X_{24}, or X_{33} will improve the solution. Let's arbitrarily choose X_{12}, and obtain the following loop of adjustments:

	JAN	FEB	MAR	dummy	
JAN	30 / 400	+ / 420	440	5 − / 0	35
FEB	425	10 / 420	20 / 440	0	30
MAR	420	20 − / 415	410	15 + / 0	35
	30	30	20	20	

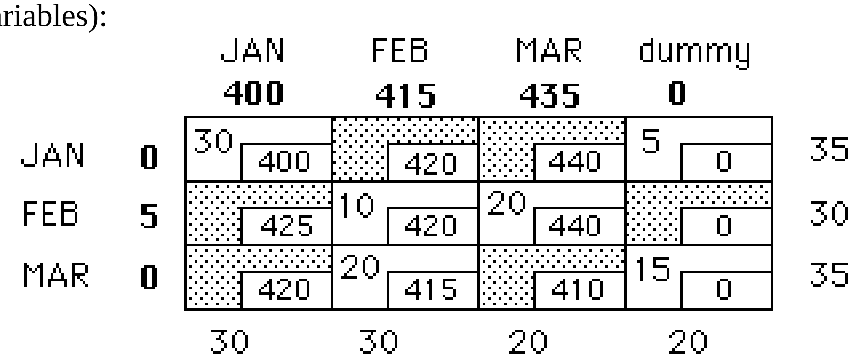
We see that X12 will replace X14 in the basis when X12=5. The new basic solution and the new dual variables are:



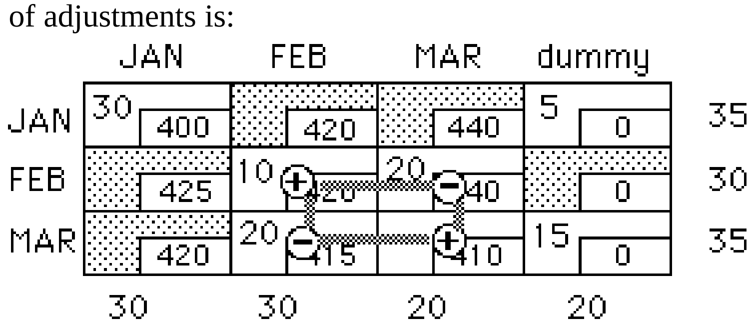
- The new reduced costs are: $\underline{C}_{13}$=440-440=0, $\underline{C}_{14}$=0-5<**0**, $\underline{C}_{21}$=425-400>0, $\underline{C}_{24}$=0-5<**0**, $\underline{C}_{31}$=420-395>0, and $\underline{C}_{33}$=410-435<**0**. Let's enter X_{14} into the basis. The loop of adjustments is:



and we see that X12 leaves the basis, giving us the new basic solution (& new dual variables):



- The new reduced costs are: $\underline{C}_{12}$=420-415>0, $\underline{C}_{13}$=440-435>0, $\underline{C}_{21}$=425-400>0, $\underline{C}_{24}$=0-5<**0**, $\underline{C}_{31}$=420-400>0, and $\underline{C}_{33}$=410-440<**0**. Let's enter X_{33} into the basis. Tbe loop of adjustments is:



Either X_{32} or X_{23} will become nonbasic. Let's arbitrarily keep X_{32} in the basis to obtain the degenerate feasible solution:

		JAN 400	FEB 415	MAR 410	dummy 0	
JAN	0	30 [400]	[420]	[440]	5 [0]	35
FEB	5	[425]	30 [420]	[440]	[0]	30
MAR	0	[420]	0 [415]	20 [410]	15 [0]	35
		30	30	20	20	

The new reduced costs are: $\underline{C}_{12}$=420-415>0, $\underline{C}_{13}$=440-410>0, $\underline{C}_{21}$=425-405>0, $\underline{C}_{23}$=440-415>0, $\underline{C}_{24}$=0-5**<0**, and $\underline{C}_{31}$=420-400>0. The solution is still not optimal, and so we enter X_{24} into the basis, obtaining the loop of adjustments:

	JAN	FEB	MAR	dummy	
JAN	30 [400]	[420]	[440]	5 [0]	35
FEB	[425]	30 − [420]	[440]	+ [0]	30
MAR	[420]	0 + [415]	20 [410]	15 − [0]	35
	30	30	20	20	

and, when X34 leaves the basis, obtain the solution

		JAN 400	FEB 420	MAR 415	dummy 0	
JAN	0	30 [400]	[420]	[440]	5 [0]	35
FEB	0	[425]	15 [420]	[440]	15 [0]	30
MAR	-5	[420]	15 [415]	20 [410]	[0]	35
		30	30	20	20	

- The reduced costs are now: $\underline{C}_{12}$=420-420=0, $\underline{C}_{13}$=440-415>0, $\underline{C}_{21}$=425-400>0, $\underline{C}_{23}$=440-415>0, $\underline{C}_{31}$=420-395>0, and $\underline{C}_{34}$=0-(-5)>0. Since the reduced costs are nonnegative, this is an optimal solution. As explained before, since $\underline{C}_{12}$=0, another optimal solution can be obtained by entering X_{12} into the basis.

• The optimal production plan #1 is: produce 35 units in January to satisfy the January demand of 30, and store 5; 10 units of February's demand is satisfied by February production, and 5 units of demand is satisfied by the inventory stored at the end of January, leaving 15 units of February demand unsatisfied; In March, produce 35 units, to satisfy the 20 units of demand in March and the 15 units of February's demand which was backlogged. This plan will result in 20 units of unused production capacity in February.
• Optimal solution #2 is: produce 30 units in January to satisfy the January demand of 30, with 5 units of production capacity unused; produce 15 units in February to partially satisfy the February demand, with 15 units of capacity unused and 15 units of demand backlogged; produce 35 units in March, 20 of which will satisfy the March

demand and the other 15 of which will be used for the February demand which was backlogged.

2. Assignment Problem: (Exercise 2 of Review Problems, page 371 of text by Winston)

Solution:

Define a "dummy" job (job #5) to get a square cost matrix:

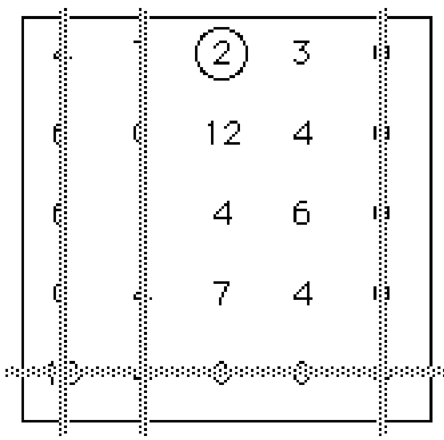
Column reduction (columns 1,2,3,4)

10	15	10	15	0
12	8	20	16	0
12	9	12	18	0
6	12	15	16	0
16	12	8	12	0

⇒

4	7	2	3	0
6	0	12	4	0
6	1	4	6	0
0	4	7	4	0
10	4	0	0	0

Now each row and each column has at least one zero, which we try to cover with the minimum number of lines. One of several which require only 4 lines is:



⇒

4	7	0	1	0
6	0	10	2	0
6	1	2	4	0
0	4	5	2	0
12	6	0	0	2

Since the number of lines is less than five, we perform another reduction, subtracting 2 from each number without a line, and adding 2 to the three intersections of lines. We then again cover the zeroes with the snallest number of lines, and discover that it cannot be done with less than five lines. It then follows that there is a zero-cost assignment using this cost matrix. Since columns 1, 2, and 4 have single zeroes, we make the corresponding assignments (4,1), (2,2), and (5,4). Row 3 also has a single zero, so we make the corresponding assignment (3,5). All workers & jobs are now assigned except worker #4 and job #3, so we make the assignment (4,3):

4	7	0	1	0
6	0	10	2	0
6	1	2	4	0
0	4	5	2	0
12	6	0	0	2

Assignments:

Worker #	to	Job #
1		3
2		2
3		5 ("dummy" job)
4		1
5		4

The cost of this assignment is 36 (hrs.) Note that worker #3 is actually assigned to do nothing!

Since you might cover the zeroes above with 4 lines in several ways, you might obtain different cost matrices.... however, the optimal solution will be the same. For example, suppose we cover the zeroes after the first step with lines in rows 2,4, and 5 and a line in column 5:

4	7	2	3	0
6	0	12	4	0
6	1	4	6	0
0	4	7	4	0
10	4	0	0	0

⇒

3	6	1	2	0
6	0	12	4	1
5	0	3	5	0
0	4	7	4	1
10	4	0	0	1

When we reduce the matrix (as indicated, by subtracting 1 from each number without a line and adding 1 to the three intersections), the resulting matrix has zeroes distributed differently than at this step in the earlier solution. Again, four lines are sufficient to cover the zeroes:

3	6	1	2	0
6	0	12	4	
5	0	3	5	0
0	4	7	4	
10	4	0	0	

⇒

3	6	0	1	0
6	0	11	3	1
5	0	2	4	0
0	4	6	3	1
11	5	0	0	2

Attempting to cover the zeroes in the matrix on the right above will soon convince you that it cannot be done in fewer than five lines. Therefore, we should be able to make zero-cost assignments in this matrix to obtain a solution to the problem:

3	6	0	1	0
6	0	11	3	1
5	0	2	4	0
0	4	6	3	1
11	5	0	0	2

Although the non-zero costs are different from the final matrix obtained before, the same assignment can be made.

3. O.R. resources on the WWW. *From the course web page, go to the page of O.R. resources on the WWW (as in last week's homework assignment). Find the link to INFORMS*

a. What does the acronym INFORMS abbreviate?

Institute For Operations Research & Management Sciences

❍❍❍❍❍❍❍❍❍❍ Homework #7 ❍❍❍❍❍❍❍❍❍

1. Bayes' Rule

Elektra receives 75% of its electronic components from vendor A and the remaining 25% from vendor B. The percentage of defective components from vendors A and B are 1% and 2%, respectively. A carton of components is selected at random from the inventory, and a sample of five components from the carton is inspected. One defective component is found. We wish to determine the probability that the carton was received from vendor B.

a. What is the (conditional) probability that exactly one out of five components is defective,
...*given that* the lot was produced by vendor A? ____________
...*given that* the lot was produced by vendor B? ____________

b. According to Bayes' rule, what is the (conditional) probability that the carton was received from vendor B, given that one defective component was found?
____

2. Decision Analysis. One of four machines must be selected for manufacturing a product, before the required quantity Q (demand) is known. Each machine has a different setup cost K and variable cost/unit C, as shown in the table below:

Machine i	Setup cost K_i ($)	Variable cost C_i ($/unit)
1	10,000	5
2	4,000	12
3	15,000	3
4	9,000	8

The cost of manufacturing Q units using machine i is then $TC_i = K_i + C_iQ$.

The required quantity Q is unknown, but is a multiple of 1000, and 1000 Q   4000.

a. Prepare a total cost table:

Machine	Demand (Q)			
selected	1000	2000	3000	4000
1	________	________	________	________
2	________	________	________	________
3	________	________	________	________
4	________	________	________	________

b. What is the optimal choice,
... given that one wishes to minimize the maximum cost? _____
... given that one wishes to minimize the minimum cost? _____

c. Prepare a regret table (regret   0):

Machine	Demand (Q)			
selected	1000	2000	3000	4000
1	________	________	________	________
2	________	________	________	________

3	_______	_______	_______	_______
4	_______	_______	_______	_______

d. What is the optimal choice of a machine which minimizes the maximum regret? _____

e. Suppose that the four possible values of Q have equal probabilities. What is the optimal choice of a machine which
....minimizes the expected cost? _____
....minimizes the expected regret? _____

3. Decision Tree

You are the author of what promises to be a successful book. You have the option to either publish the book yourself or through a publisher. The publisher is offering you $20,000 for signing the contract. If the book is successful, it will sell 200,000 copies. If it isn't, however, it will sell only 10,000 copies. The publisher pays royalties at the rate of $1 per copy sold. A market survey by the publisher indicates that there is a 70% chance that the book will be successful. You also have an option of publishing the book yourself. If you publish the book yourself, you will incur an initial cost of $90,000 for printing and marketing, but each copy sold will net you $2 instead of $1.

Before you decide how to publish the book, you may contract a literary agent to conduct an independent survey concerning its potential success. From past experience, the agent claims that when a book will be successful, the survey will predict success 80% of the time. On the other hand, when the book will not be successful, the survey will give the correct prediction 85% of the time. The cost of contracting this survey is $1000.

a. Draw a tree for this decision problem, labelling all decisions and uncertain outcomes, including probabilities of outcomes.

b. Fold back the tree in order to find the strategy which will maximize the expected payoff.

c. What is the expected value of the independent survey? $________

d. What would be the expected value of *perfect information* ? $ ________

4. O.R. Resources on WWW

I want you to find a reference to a software package (commercial or public domain) for analysis of decision trees. Load the URL:

<http://www.yahoo.com/Science/Mathematics/Operations_Research>

On this web page, find the link to "Decision/Risk Analysis".
Under "Online Sources", link to "Software", where you will find a link to numerous sources of software for decision analysis (some of which include analysis of decision trees). What is one software package which will analyze decision trees?
Name: _______________
Company: ______________________________

❍❍❍❍❍❍❍❍❍❍ Homework #7 Solutions ❍❍❍❍❍❍❍❍❍

1. Bayes' Rule Solution:

a. What is the (conditional) probability that exactly one out of five components is defective,
...given that the lot was produced by vendor A? 4.803%
...given that the lot was produced by vendor B? 9.224%

Solution:

The number of defects ("successes") in 5 (=n) trials has a binomial distribution, i.e.,

$$P\{x \text{ successes in n trials}\} = \binom{n}{x} p^x (1-p)^{n-x}$$

where the probability of success is the probability that a single component is defective (either 1% or 2%, depending upon its origin.) Hence,

$$P\{1 \text{ defect among 5 components } \mid \text{carton from A}\} = \binom{5}{1}(0.01)^1(0.99)^4 = 0.04803$$

and

$$P\{1 \text{ defect among 5 components } \mid \text{carton from B}\} = \binom{5}{1}(0.02)^1(0.98)^4 = 0.09224$$

b. According to Bayes' rule, what is the (conditional) probability that the carton was received from vendor B, given that one defective component was found? 40.55%

Solution:

$$P\{\text{parts from B} \mid \text{1 defect among 5 parts}\}$$
$$= \frac{P\{\text{1 defect among 5 parts} \mid \text{parts from B}\} \times P\{\text{parts from B}\}}{P\{\text{1 defect among 5 parts}\}}$$
$$= \frac{(0.09224)(0.25)}{0.0590825} = \frac{0.02396}{0.0590825} = 0.4055346$$

where the denominator has been calculated as follows:

$$P\{\text{1 defect among 5 parts}\} =$$
$$P\{\text{1 defect among 5 parts} \mid \text{parts from B}\} \times P\{\text{parts from B}\}$$
$$+ P\{\text{1 defect among 5 parts} \mid \text{parts from B}\} \times P\{\text{parts from A}\}$$
$$= (0.09224)(0.25) + (0.04803)(0.75)$$
$$= 0.02306 + 0.0360225 = 0.0590825$$

2. Decision Analysis. Solution:

a. Prepare a total cost table:

Solution:

Using the revised setup costs shown above, we would obtain

Machine	Demand (Q)			
selected	1000	2000	3000	4000
1	15000	20000	25000	30000
2	16000	28000	40000	52000

3	18000	21000	24000	27000
4	17000	25000	33000	41000

b. What is the optimal choice,

... given that one wishes to minimize the maximum cost? **Solution**: Machine #3 ,

since the row maxima are \$30000, \$52000, \$27000, and \$41000.

... given that one wishes to minimize the minimum cost? **Solution**: Machine #1 ,since the row minima are \$15000, \$16000, \$18000, and \$17000.

c. Prepare a regret table (regret •0):

Solution: Subtracting the smallest cost for each Q from the remaining costs in the respective columns yields:

Machine selected	Demand (Q) 1000	2000	3000	4000
1	0	0	1000	3000
2	1000	8000	16000	25000
3	3000	1000	0	0
4	2000	5000	9000	14000

d. What is the optimal choice of a machine which minimizes the maximum regret?

Solution: Either of Machines # 1 or 3 , since the maximum regrets in each row are \$3000, \$25000, \$3000, and \$14000.

e. Suppose that the four possible values of Q have equal probabilities. What is the optimal choice of a machine which

....minimizes the expected cost? Either of Machines # 1 or 3

Solution: The expected costs of machines 1 through 4 would be \$22500, \$34000, \$22500, and \$29000, respectively.

....minimizes the expected regret? Either of Machines # 1 or 3

Solution: The expected regrets of machines 1 through 4 would be \$1000, \$12500, \$1000, and \$7500, respectively.

3. Decision Tree-- Solution:

a. *Draw a tree for this decision problem, labelling all decisions and uncertain outcomes, including probabilities of outcomes.*

Solution:

Given

$$P\{\text{Predict Hit} \mid \text{Hit}\} = 0.80$$
$$P\{\text{Predict Flop} \mid \text{Hit}\} = 0.20$$
$$P\{\text{Predict Hit} \mid \text{Flop}\} = 0.15$$
$$P\{\text{Predict Flop} \mid \text{Flop}\} = 0.85$$

and

$$\left.\begin{array}{l} P\{\text{Hit}\} = 0.70 \\ P\{\text{Flop}\} = 0.30 \end{array}\right\} \textit{prior probabilitie}$$

First we compute

$$P\{\text{Predict Hit}\} = P\{\text{Predict Hit}|\text{ Hit}\}P\{\text{Hit}\} + P\{\text{Predict Hit | Flow}\}P\{\text{Flop}\}$$
$$= (0.80)(0.70) + (0.15)(0.30) = 0.605$$

and $P\{\text{Predict Flop}\} = 1 - P\{\text{Predict Hit}\} = 0.395$.

Then we use Bayes' Rule to compute

$$P\{\text{Hit | Predict Hit}\} = \frac{P\{\text{Predict Hit | Hit}\} \times P\{\text{Hit}\}}{P\{\text{Predict Hit}\}}$$
$$= \frac{(0.80)(0.70)}{0.605} = 0.9256$$

so that $P\{\text{Flop | Predict Hit}\} = 1 - P\{\text{Hit | Predict Hit}\} = 0.0744$

Likewise,

$$P\{\text{Flop | Predict Flop}\} = \frac{P\{\text{Predict Flop | Flop}\} \times P\{\text{Flop}\}}{P\{\text{Predict Flop}\}}$$
$$= \frac{(0.85)(0.30)}{0.395} = 0.64564$$

so that $P\{\text{Hit | Predict Flop}\} = 1 - P\{\text{Flop | Predict Flop}\} = 0.3544$

These conditional probabilities are assigned to the appropriate branches of the decision tree below:

b. Fold back the tree in order to find the strategy which will maximize the expected payoff.

Solution: See above.

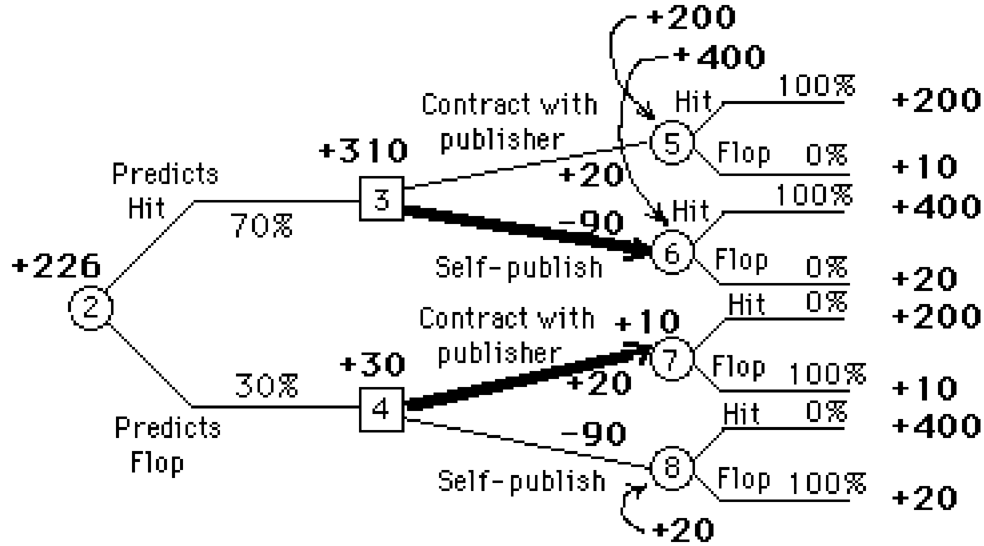
c. What is the expected value of the independent survey? $ 208.89 - 196 = $ 12.89 (x1000)

Solution: Evaluate the difference between the expected values of nodes #2 and #9 above.

d. What would be the expected value of *perfect information* ? $ 226 - 196 = $30 (x1000)

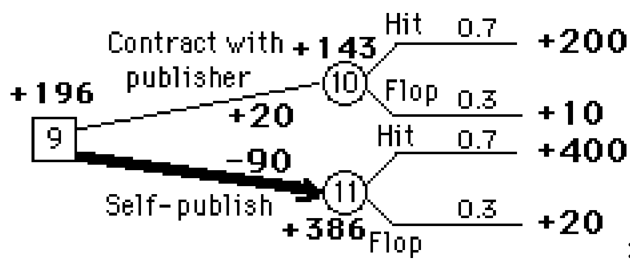
Solution: Evaluate the difference between the expected values of nodes #2 and #9 below, with the probabilities revised to reflect a survey giving perfect information. Note that the probabilities of the perfect predictions are the same as the *prior* probabilities (70% for a hit, 30% for a flop).

With Perfect Information



Compare this to the case in which we do not have any information provided by the consultant:

Without Perfect Information



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❍❍❍❍❍❍❍❍❍❍ Homework #8 ❍❍❍❍❍❍❍❍❍

1. **Integer Programming Model Formulation.** (Exercise 19, §9.2, page 494 of text by Winston) Comquat owns four production plants at which personal computers are produced. Comquat can sell up to 20,000 computers per year at a price of $3500 per computer. For each plant, the production capacity, the production cost per computer, and the fixed cost of operating a plant for a year are given in the table below:

Plant #	Production Capacity	Plant Fixed Cost	Cost per Computer
1	10,000	$9 million	$1000
2	8,000	$5 million	$1700
3	9,000	$3 million	$2300
4	6,000	$1 million	$2900

Determine how many computers Comquat should produce at each plant in order to maximize its yearly revenue. (Note that if no computers are produced by a plant during the year, Comquat need not pay the fixed cost of operating the plant that year.)

2. **Integer Programming Model Formulation.** (Problem 29, §9.2, page 497 of text by Winston) You have been assigned to arrange the songs on the cassette version of Madonna's latest album. A cassette tape has two sides (1 and 2). The songs on each side of the cassette must total between 14 and 16 minutes in length. The length and type of each song are given in the table below:

Song	Type	Length (minutes)
1	Ballad	4
2	Hit	5
3	Ballad	3
4	Hit	2
5	Ballad	4
6	Hit	3
7	neither ballad nor hit	5
8	Ballad & hit	4

The assignment of songs to the tape must satisfy the following conditions:

1. Each side must have exactly two ballads.
2. Side 1 must have at least three hit songs.
3. Either song 5 or song 6 must be on side 1.
4. If both songs 2 & 4 are on side 1, then song 5 must be on side 2.

Formulate an integer linear programming model to determine whether there is an arrangement of songs satisfying these restrictions, and use LINDO to solve the problem.

3. **O.R. on the WWW.** INFORMS (Institute for Operations Research & Management Sciences) has recently begun publication on the web of *Interactive Transactions of OR/MS*, an electronic journal, which can be found at:

http://orcs.bus.okstate.edu/itorms

Article No. 4 in Volume 1 is titled "Annotated Bibliography On Linear Programming Models", by Frederic H. Murphy.
Find a published paper listed in this bibliography which is in an application area of interest to you. (Most of them will be in a journal which is available in the University of Iowa Libraries, and copy the abstract of the paper.)

❍❍❍❍❍❍❍❍❍❍ Homework #8 Solutions ❍❍❍❍❍❍❍❍❍

1. Integer Programming Model Formulation. (Exercise 19, §9.2, page 494 of text by Winston)

Solution: We require two sets of decision variables:
Y_i = 1 if the computers are produced at plant i, 0 otherwise *(binary)*
and
X_i = quantity of computers produced at plant i *(continuous)*
The objective function consists of the variable production costs (which depend upon the number of computers produced in a plant) plus the fixed costs (which are independent of the number of computers produced, except that this fixed cost is avoided if no production takes place in the plant).
Objective:
Maximize $(3500 - 1000)X_1 + (3500 - 1700)X_2 + (3500 - 2300)X_3 + (3500 - 2900)X_4$
$- 9000000Y_1 - 5000000Y_2 - 3000000Y_3 - 1000000Y_4$ *(fixed costs)*
Contraints:
Total production should be limited to the potential market of 20,000 computers/year:
$X_1 + X_2 + X_3 + X_4$ • 20000
The production of a plant cannot exceed its capacity:
X_1 • $10000Y_1$
X_2 • $8000Y_2$
X_3 • $9000Y_3$
X_4 • $6000Y_4$
Note that the capacity (upper bound of X_i) is determined by whether the fixed cost of the plant has been paid-- if $Y_i = 0$, then the corresponding term in the objective function is zero and the capacity (right-hand-side of the capacity constraint) is zero, while if $Y_i=1$, then the fixed cost is subtracted in the objective function, and the capacity is that given in the data table.

LINDO Solution:
The input to LINDO was:

```
MAX   + 2500 X1 + 1800 X2 + 1200 X3 + 600 X4
- 9000000 Y1 - 5000000 Y2 - 3000000 Y3 - 1000000 Y4          ST
  X1 + X2 + X3 + X4 <   20000
  X1  - 10000 Y1  <   0
  X2 - 8000 Y2    <   0
  X3  - 9000 Y3    <   0
  X4  - 6000 Y4    <   0
 END
```

```
INTE    Y1
INTE    Y2
INTE    Y3
INTE    Y4
```

When echoing the input (see below), LINDO has re-arranged the terms in the objective so that the integer variables precede the continuous variables, and has added the SUB (simple upper bounds) lines, also. Note that the "INTE" (or "INTEGER") command specifies that the variable being specified is a *binary* (zero-one) variable.

The LINDO output is:

```
  : look all

  MAX   - 9000000 Y1 - 5000000 Y2 - 3000000 Y3 - 1000000 Y4 + 2500 X1
       + 1800 X2 + 1200 X3 + 600 X4
  SUBJECT TO
         2)   X1 + X2 + X3 + X4 <=   20000
         3) - 10000 Y1 + X1 <=   0
         4) - 8000 Y2 + X2 <=   0
         5) - 9000 Y3 + X3 <=   0
         6) - 6000 Y4 + X4 <=   0
  END
  SUB       Y1        1.00000
  INTE       Y1
  SUB       Y2        1.00000
  INTE       Y2
  SUB       Y3        1.00000
  INTE       Y3
  SUB       Y4        1.00000
  INTE       Y4

 : go
 LP OPTIMUM FOUND AT STEP      1
 OBJECTIVE VALUE =   27133334.0
 FIX ALL VARS.(    3)  WITH RC >   .000000
 SET       Y3 TO >=    1 AT    1, BND=  0.2480E+08 TWIN= 0.2540E+08
5

 NEW INTEGER SOLUTION OF    24800000.0     AT BRANCH      1 PIVOT
5

        OBJECTIVE FUNCTION VALUE

        1)     0.2480000E+08

  VARIABLE        VALUE          REDUCED COST
        Y1         1.000000   -4000000.000000
        Y2         1.000000     200000.000000
        Y3         1.000000    3000000.000000
        Y4          .000000    1000000.000000
        X1     10000.000000          .000000
        X2      8000.000000          .000000
        X3      2000.000000          .000000
        X4          .000000        600.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000       1200.000000
```

```
       3)           .000000       1300.000000
       4)           .000000        600.000000
       5)       7000.000000           .000000
       6)           .000000           .000000

 NO. ITERATIONS=       5
 BRANCHES=    1 DETERM.=  1.000E    0
 BOUND ON OPTIMUM: 0.2713333E+08
 FLIP       Y3 TO <=         0 AT    1 WITH BND=    25400000.

 NEW INTEGER SOLUTION OF    25400000.0     AT BRANCH      1 PIVOT
5

       OBJECTIVE FUNCTION VALUE

       1)     0.2540000E+08

  VARIABLE        VALUE          REDUCED COST
       Y1         1.000000  -16000000.000000
       Y2         1.000000   -9400000.000000
       Y3          .000000   -7800000.000000
       Y4          .000000   -2600000.000000
       X1     10000.000000           .000000
       X2      8000.000000           .000000
       X3          .000000           .000000
       X4          .000000           .000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)      2000.000000           .000000
       3)          .000000       2500.000000
       4)          .000000       1800.000000
       5)          .000000       1200.000000
       6)          .000000        600.000000

 NO. ITERATIONS=       5
 BRANCHES=    1 DETERM.=  1.000E    0
 BOUND ON OPTIMUM: 0.2713333E+08
 DELETE       Y3 AT LEVEL     1
 RELEASE FIXED VARIABLES
 FIX ALL VARS.(    2)  WITH RC >  0.146667E+07
 SET       Y3 TO <=    0 AT    1, BND=  0.2560E+08 TWIN= 0.2380E+08
14

 NEW INTEGER SOLUTION OF    25600000.0     AT BRANCH      2 PIVOT
14

       OBJECTIVE FUNCTION VALUE

       1)     0.2560000E+08

  VARIABLE        VALUE          REDUCED COST
       Y1         1.000000   -9000000.000000
       Y2         1.000000   -3600000.000000
       Y3          .000000   -2400000.000000
       Y4         1.000000           .000000
       X1     10000.000000           .000000
       X2      8000.000000           .000000
       X3          .000000           .000000
       X4      2000.000000           .000000
```

```
       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)          .000000        600.000000
        3)          .000000       1900.000000
        4)          .000000       1200.000000
        5)          .000000        600.000000
        6)      4000.000000           .000000
        7)          .000000    1000000.000000

 NO. ITERATIONS=      14
 BRANCHES=    2 DETERM.=  1.000E    0
 BOUND ON OPTIMUM: 0.2560000E+08
 DELETE       Y3 AT LEVEL     1
 ENUMERATION COMPLETE. BRANCHES=     2 PIVOTS=      14

 LAST INTEGER SOLUTION IS THE BEST FOUND
 RE-INSTALLING BEST SOLUTION...
```

The optimal solution is therefore to produce 10,000 units (max capacity) at plant #1, 8000 units (max capacity) at plant #2, and 2000 units at plant #4 (and none at plant #3).

2. **Integer Programming Model Formulation.** (Problem 29, §9.2, page 497 of text by Winston)

Solution: One might define the 16 decision variables

X_{ij} = 1 if song i is placed on side j; 0 otherwise

where i=1,2,3,...8 and j=1,2.

However, since we would require $X_{i2} = 1-X_{i1}$, it is sufficient to define only 8 variables:

Y_i = 1 if song i is placed on side 1; 0 if it is placed on side 2

where i=1,2,3,...8.

There is no stated *objective* other than to find a feasible solution to this problem, so that any arbitrary function may be optimized (either minimized or maximized), e.g.,

Maximize Y_1

Constraints:

The time constraints on side #1 are:

$$4Y_1 + 5Y_2 + 3Y_3 + 2Y_4 + 4Y_5 + 3Y_6 + 5Y_7 + 4Y_8 \quad 14$$
$$4Y_1 + 5Y_2 + 3Y_3 + 2Y_4 + 4Y_5 + 3Y_6 + 5Y_7 + 4Y_8 \quad 16$$

Since the total length of all the songs is 30 minutes, the time constraints on side #2 will automatically be satisfied if they are satisfied on side #1, and we need not include them here.

(If we did wish to include them, then we would have to repeat the above constraints with Y_i replaced by $1-Y_i$ [which is 1 if Y_i is zero, i.e., song i is not on side #1]. The resulting constraints would reduce to the same two constraints shown above!)

The other constraints are:

• *Each side must have exactly two ballads*, i.e., side #1 must contain 2 ballads:

$$Y_1 + Y_3 + Y_5 + Y_8 = 2$$

Again, since the total number of ballads is 2, if side #1 contains exactly 2 ballads, then so also must side #2 and we need not add the extra contraint, which would appear as

$(1-Y_1) + (1-Y_3) + (1-Y_5) + (1-Y_8) = 2$ or equivalently,

$4 - (Y_1 + Y_3 + Y_5 + Y_8) = 2$, which reduceds again to

$$Y_1 + Y_3 + Y_5 + Y_8 = 2$$

• *Side 1 must have at least three hit songs:* This constraint is similar in form to the previous constraint on the ballads, namely :

$$Y_2 + Y_4 + Y_6 + Y_8 \bullet 3$$

• *Either song 5 or song 6 must be on side 1:*

$$Y_5 + Y_6 \bullet 1$$

(Note that I have interpreted this NOT as an "exclusive OR", i.e., it is requiring the either song 5, or song 6, <u>or both</u>, be on side 1.)

• *If both songs 2 & 4 are on side 1, then song 5 must be on side 2:*

We need to restrict Y_5 to be 0 (i.e., song 5 is on side 2) if it is true that $Y_2 + Y_4 = 1$, while we wish to place no binding restriction on Y_5 otherwise. This can be accomplished by the constraint

$$Y_5 \bullet 2 - Y_2 - Y_4$$

Note that the right-hand-side of this constraint has three possible values:

0 if $Y_2 + Y_4 = 2$, i.e., both songs 2 & 4 are on side 1

1 if $Y_2 + Y_4 = 1$, i.e., exactly 1 of songs 2 & 4 is on side #1

2 if $Y_2 + Y_4 = 0$, i.e., neither song 2 nor song 4 are on side #1

Only in the first case will Y_5 be restricted to the value 0, i.e., song 5 will be forced onto side 2; in the other cases Y5 might be either 0 or 1. This constraint would be written as input to LINDO as

$$Y_2 + Y_4 + Y_5 \bullet 2$$

LINDO solution:

The LINDO input is

```
: MAX Y1
 ?st
 ?4Y1 + 5Y2 + 3Y3 + 2Y4 + 4Y5 + 3Y6 + 5Y7 + 4Y8 > 14
 ?4Y1 + 5Y2 + 3Y3 + 2Y4 + 4Y5 + 3Y6 + 5Y7 + 4Y8 < 16
 ?Y1 + Y3 + Y5 + Y8 = 2
 ?Y2 + Y4 + Y6 + Y8 > 3
 ?Y5 + Y6 > 1
 ?Y2 + Y4 + Y5 < 2
 ?END


: INTE 8
```

The LINDO output is:

```
: look all

  MAX     Y1
  SUBJECT TO
        2)   4 Y1 + 5 Y2 + 3 Y3 + 2 Y4 + 4 Y5 + 3 Y6 + 5 Y7 + 4 Y8 >=
14
        3)   4 Y1 + 5 Y2 + 3 Y3 + 2 Y4 + 4 Y5 + 3 Y6 + 5 Y7 + 4 Y8 <=
16
        4)   Y1 + Y3 + Y5 + Y8 =    2
        5)   Y2 + Y4 + Y6 + Y8 >=   3
        6)   Y5 + Y6 >=   1
        7)   Y2 + Y4 + Y5 <=   2
  END
  INTE     8

 : GO
 LP OPTIMUM FOUND AT STEP     11
 OBJECTIVE VALUE =   1.00000000
 FIX ALL VARS.(    5)  WITH RC >   .000000

 NEW INTEGER SOLUTION OF    1.00000000     AT BRANCH      0 PIVOT
13

        OBJECTIVE FUNCTION VALUE

        1)      1.000000

  VARIABLE        VALUE          REDUCED COST
        Y1         1.000000         -1.000000
        Y2         1.000000           .000000
        Y3          .000000           .000000
        Y4          .000000           .000000
        Y5          .000000           .000000
        Y6         1.000000           .000000
        Y7          .000000           .000000
        Y8         1.000000           .000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         2.000000           .000000
        3)          .000000           .000000
        4)          .000000           .000000
        5)         -.000000           .000000
        6)         -.000000           .000000
        7)         1.000000           .000000
        8)          .000000           .000000

 NO. ITERATIONS=      13
 BRANCHES=    0 DETERM.=  1.000E    0
 BOUND ON OPTIMUM:  1.000000
 ENUMERATION COMPLETE. BRANCHES=     0 PIVOTS=      13

 LAST INTEGER SOLUTION IS THE BEST FOUND
 RE-INSTALLING BEST SOLUTION...
```

That is, the feasible solution found above is to place songs #1, 2, 6, & 8 on side #1 (and therefore the remaining songs, namely 3, 4, 5, & 7, on side #2. Note that a different objective might result in LINDO finding a different solution, but it would be feasible as well!

❍❍❍❍❍❍❍❍❍❍ Homework #9 ❍❍❍❍❍❍❍❍❍

1. Consider an inventory system in which the number of items on the shelf is checked at the end of each day. The maximum number on the shelf is 8. If 3 or fewer units are on the shelf, the shelf is refilled overnight. The demand distribution is as follows:

x	0	1	2	3	4	5	6
P{D=x}	.1	.15	.25	.25	.15	.05	.05

The system is modeled as a Markov chain, with the state defined as the number of units on the shelf at the end of each day. The probability transition matrix is:

Transition Probabilities

from \ to	1	2	3	4	5	6	7	8	9
1	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
2	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
3	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
4	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1
5	0.25	0.25	0.25	0.15	0.1	0	0	0	0
6	0.1	0.15	0.25	0.25	0.15	0.1	0	0	0
7	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0	0
8	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1	0
9	0	0	0.05	0.05	0.15	0.25	0.25	0.15	0.1

a. Explain the derivation of the values P_{19}, P_{35}, P_{51}, P_{83} above. (Note that state 1=inventory level 0, etc.)

The steady-state distribution of the above Markov chain is:

Steady State Distribution

i	Pi
1	0.06471513457
2	0.07698357218
3	0.1304613771
4	0.1355295351
5	0.16322964
6	0.1698706746
7	0.1384131423
8	0.0754980776
9	0.04529884656

b. Write two of the equations which define this steady-state distribution. How many equations must be solved to yield the solution above?

c. What is the average number on the shelf at the end of each day?

The mean first passage matrix is:

Mean First Passage Times

from \ to	1	2	3	4	5	6	7	8	9
1	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
2	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
3	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
4	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756
5	12.2711	10.4926	6.64702	7.25983	6.12634	6.35574	7.50281	13.7757	23.1867
6	14.3163	11.5197	6.47734	6.42023	5.85772	5.88683	7.68799	13.9609	23.3719
7	14.632	12.4406	7.01794	6.24734	5.2518	5.79028	7.22475	14.3004	23.7114
8	15.2275	12.1857	7.62978	6.77218	5.19576	5.29848	7.10625	13.2454	24.1281
9	15.4523	12.9898	7.6651	7.37847	5.69593	5.24463	6.39169	12.6646	22.0756

d. If the shelf is full Monday morning, what is the expected number of days until the shelf is first emptied ("stockout")? ______________

e. What is the expected time between stockouts? ______________

f. How frequently will the shelf be restocked? (i.e. what is the average number of days between restocking?) ______________

2. Consider a manufacturing process in which raw parts (blanks) are machined on three machines, and inspected after each machining operation. The relevant data is as follows:

Manufacturing System Parameters: 56:171 O.R. HW

Station	Machine Operation			Inspection			
i	T	C	S	T	C	S	R
1	0.5	20	10	0.1	15	10	5
2	0.75	20	5	0.2	15	10	3
3	0.25	20	2	0.25	15	5	2

Pack & Ship: 0.1 hrs at 10 $/hr
Cost per blank: $50
Scrap Value: $10

T = time (hrs) per operation
C = cost ($/hr) of operation
S = scrap rate (%)
R = rework rate (%)

For example, machine #1 requires 0.5 hrs, at $20/hr., and has a 10% scrap rate. Those parts completing this operation are inspected, requiring 0.1 hr. at $15/hr. The inspector scraps 10%, and sends 5% back to machine #1 for rework (after which it is again inspected, etc.)

The Markov chain model of a part moving through this system has transition probability matrix:

Transition Probabilities

from \ to	1	2	3	4	5	6	7	8
1	0	0.9	0	0	0	0	0	0.1
2	0.05	0	0.85	0	0	0	0	0.1
3	0	0	0	0.95	0	0	0	0.05
4	0	0	0.03	0	0.87	0	0	0.1
5	0	0	0	0	0	0.98	0	0.02
6	0	0	0	0	0.02	0	0.93	0.05
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

a. Draw the diagram for this Markov chain and describe each state.

b. Which states are transient?____________ which are absorbing? ____________

The absorption probabilities are:

A = Absorption Probability Matrix

OK	Scrap
0.6335	0.3665
0.7039	0.2961
0.7909	0.2091
0.8325	0.1675
0.9296	0.07038
0.9486	0.05141

The matrix E is as follows:

E = Expected No. Visits

1	2	3	4	5	6
1.047	0.9424	0.8245	0.7833	0.6951	0.6812
0.05236	1.047	0.9162	0.8704	0.7723	0.7569
0	0	1.029	0.9779	0.8678	0.8504
0	0	0.03088	1.029	0.9134	0.8952
0	0	0	0	1.02	0.9996
0	0	0	0	0.0204	1.02

c. Explain how E was computed.

Explain how A was computed, given E.

d. What percent of the parts which are started are successfully completed?
____

e. What is the expected number of blanks which should be required to fill an order for 100 completed parts? ____________

f. What percent of the parts arriving at machine #2 will be successfully completed? ____________

g. What is the expected total number of inspections which entering parts will undergo? ____________

h. Explain the meaning of the number appearing in row 3, column 2 of the A matrix.

i. Explain the meaning of the number appearing in row 3, column 3 of the E matrix.

j. To fill the order for 100 completed parts, what is the expected man-hour requirement for each machine? for each inspection station? ____________

k. What are the expected direct costs (row materials + operating costs - scrap value of rejected parts) per completed part? ____________

❍❍❍❍❍❍❍❍❍❍ Homework #9 Solutions ❍❍❍❍❍❍❍❍❍

1. Solution*:*

a. Explain the derivation of the values P_{19}, P_{35}, P_{51}, P_{83} above. (Note that state 1=inventory level 0, etc.)

Solution:

- A transition from today's state 1 (0 on shelf at end of day, below the reorder point **s**) to tomorrow's state 9 (8 on shelf at end of day) occurs only when the shelf is restocked to its maximum level of 8 overnight, but no units are sold the following day. Hence
  P_{19} = P{tomorrow's demand D = 0} = 0.1

- A transition from today's state 3 (2 on shelf at end of day, at the reorder point **s**) to tomorrow's state 5 (4 on shelf at end of day) occurs only when the shelf is restocked to its capacity (8) overnight, and the store sells 4 of these 8 units. Hence,
  P_{35} = P{tomorrow's demand D = 4} = 0.15

- A transition from today's state 5 (4 on shelf at end of day, so the shelf will <u>not</u> be restocked) to tomorrow's state 1 (none on shelf at end of day) occurs only when tomorrow the store sells all four of its stock, i.e., the demand was <u>at least</u> 4. This happens with probability
  P{D 4} = P{D=4} + P{D=5} + P{D=6}
  = 0.15 + 0.05 + 0.05 = 0.25

- Similarly, P_{83} is the probability that tomorrow's demand is 5, i.e., 0.05.

b. Write two of the equations which define this steady-state distribution. How many equations must be solved to yield the solution above?

Solution: The equations which define the steadystate probability distribution ð are of two types. First is the constraint that says that the probabilities must sum to 1.00, i.e.,

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 + \pi_9 = 1$$

The other constraints are obtained from the system of equations written (in matrix form) as

$$\pi\Lambda = 0$$

where is the matrix of transition <u>rates</u> (not probabilities!) That is, the inner product of vector ð and *each* of the nine columns of Λ must equal 0, i.e.,

$$\begin{cases} 0.25\pi_5 + 0.1\pi_6 + 0.05\pi_7 = 0 \\ 0.25\pi_5 + 0.15\pi_6 + 0.05\pi_7 + 0.05\pi_8 = 0 \\ 0.05\pi_1 + 0.05\pi_2 + 0.05\pi_3 + 0.05\pi_4 + 0.25\pi_5 + 0.25\pi_6 + 0.15\pi_7 + 0.05\pi_8 + 0.05\pi_9 = 0 \\ 0.05\pi_1 + 0.05\pi_2 + 0.05\pi_3 + 0.05\pi_4 + 0.15\pi_5 + 0.25\pi_6 + 0.25\pi_7 + 0.15\pi_8 + 0.05\pi_9 = 0 \\ 0.15\pi_1 + 0.15\pi_2 + 0.15\pi_3 + 0.15\pi_4 + 0.1\pi_5 + 0.15\pi_6 + 0.25\pi_7 + 0.25\pi_8 + 0.15\pi_9 = 0 \\ 0.25\pi_1 + 0.25\pi_2 + 0.25\pi_3 + 0.25\pi_4 + 0.1\pi_6 + 0.15\pi_7 + 0.25\pi_8 + 0.25\pi_9 = 0 \\ 0.25\pi_1 + 0.25\pi_2 + 0.25\pi_3 + 0.25\pi_4 + 0.1\pi_7 + 0.15\pi_8 + 0.25\pi_9 = 0 \\ 0.15\pi_1 + 0.15\pi_2 + 0.15\pi_3 + 0.15\pi_4 + 0.1\pi_8 + 0.15\pi_9 = 0 \\ 0.1\pi_1 + 0.1\pi_2 + 0.1\pi_3 + 0.1\pi_4 + 0.1\pi_9 = 0 \end{cases}$$

There are nine variables to compute, and so we need nine equations, including the equation which restricts the sum of the variables to the value 1.00. (The nine equations obtained from $\pi\Lambda = 0$ include one redundant equation.)

c. What is the average number on the shelf at the end of each day?

Solution: $0\pi_1 + \pi_2 + 2\pi_3 + 3\pi_4 + 4\pi_5 + 5\pi_6 + 6\pi_7 + 7\pi_8 + 8\pi_9 \approx 3.968$

d. If the shelf is full Monday morning, what is the expected number of days until the shelf is first emptied ("stockout")?

Solution: If the shelf is full Monday morning, then the state of the system when last observed was either 1,2,3, or 9 (i.e., either the shelf was full Sunday evening or else it was restocked that night.) The rows of M (the mean first passage time matrix) corresponding to these four states are identical, so it does not matter which row is used. The quantity sought here is the mean first passage time from one of these four states to state 1 (stockout), e.g., m_{91} = <u>15.4523 (days)</u>

e. What is the expected time between stockouts?

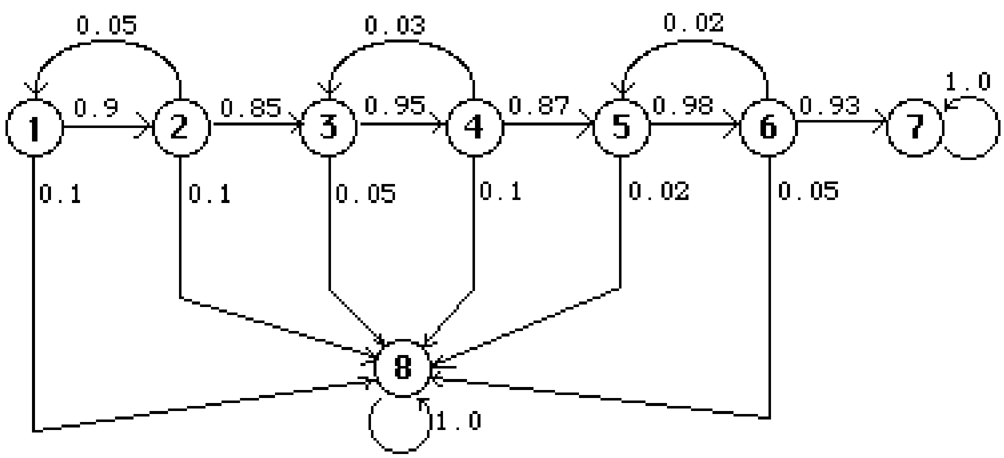
Solution: $m_{11} = 1/\pi_1$ = <u>15.4523 (days)</u>

f. How frequently will the shelf be restocked? (i.e. what is the average number of days between restocking?)

Solution: The probability that the shelf is restocked on any given day in steady state is the probability that the inventory level at the end of the day is s=3, i.e., sum of the probabilities π_1, π_2, π_3 and π_4, namely 0.4076. The reciprocal of this probability of restocking, namely $1/0.4076$ = 2.45 (days), gives us the frequency, i.e., the expected time between visits to this set of states {1,2,3,4}.

2. Solution

a. Draw the diagram for this Markov chain and describe each state.
Solution:



b. Which states are transient? **Solution**: 1,2,3,4,5, & 6
which are absorbing? **Solution**: 7 & 8

c. Explain how E was computed.

Solution: $E = (I - Q)^{-1}$, where Q is the submatrix of P whose rows and columns both correspond to transient states, i.e.,

from \ to	1	2	3	4	5	6	7	8
1	0	0.9	0	0	0	0	0	0.1
2	0.05	0	0.85	0	0	0	0	0.1
3	0	0	0	0.95	0	0	0	0.05
4	0	0	0.03	0	0.87	0	0	0.1
5	0	0	0	0	0	0.98	0	0.02
6	0	0	0	0	0.02	0	0.93	0.05
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

Explain how A was computed, given E.

Solution: A = E R, where the matrix E was computed as in (c) and R is a submatrix of P, as shown above.

d. What percent of the parts which are started are successfully completed?
Solution: The value a_{17} in the A matrix above is the probability that the system, beginning in state 1 (i.e., the part is at Machine #1), is eventually absorbed by state 7 (i.e., the part has reached the packing & shipping area). This should give us the desired probability, 63.35%

e. What is the expected number of blanks which should be required to fill an order for 100 completed parts?
Solution: We expect to need $^1/_{0.6335} = 1.579$ entering blanks per successfully completed parts, or 157.9 blanks total, i.e.,

$$\frac{\text{one entering blank}}{\text{0.6335 finished part}} = \frac{\text{157.9 entering blanks}}{\text{100 finished parts}}$$

f. What percent of the parts arriving at machine #2 will be successfully completed?
Solution: This value is given in the absorption probability matrix A: a_{37} = 79.09%

g. What is the expected total number of inspections which entering parts will undergo?
Solution: The expected number of visits to states 2, 4, and 6 (where the part is in an inspection station), given that a part begins in state 1, is $e_{12} + e_{14} + e_{16}$ = 2.4069, where the elements e_{ij} are found in the E matrix above.

h. Explain the meaning of the number appearing in row 3, column 2 of the A matrix.
Solution: Row 3 of the A matrix corresponds to the transient state 3 (part on machine #2), while column 2 of the matrix corresponds to the second absorbing state (#8, "scrap"). Thus, 0.2091 is the probability that a part which has arrived at the second machine will eventually be scrapped.

i. Explain the meaning of the number appearing in row 3, column 3 of the E matrix.

Solution: Row 3 and column 3 of the E matrix both correspond to the 3rd transient state, i.e., the machine has arrived at Machine #2. Therefore, e_{33} = 1.029 is the expected number of times (including the initial visit) that the part visits this machine, given that it begins at this machine.

j. To fill the order for 100 completed parts, what is the expected man-hour requirement for each machine?

Solution: Since e_{1j}, for j=1,3, and 5, are the expected number of visits to the three machines (starting at machine 1), and T_j is the time requirement per visit, $T_1 e_{11} + T_3 e_{13} + T_5 e_{15} = 1.31575$ (hrs) is the total expected machine time requirements per entering blank. Multiplying this by 157.9 entering blank per 100 successfully completed part yields 207.757 machine hours to complete the order for 100 parts.

... for each inspection station?

Solution: Since e_{1j}, for j=2,4, and 6, are the expected number of visits to the three inspection stations (starting at machine 1), and T_j is the time requirement per visit, $T_2 e_{12} + T_4 e_{14} + T_6 e_{16} = 0.421207$ (hrs) is the total inspection time requirements per entering blank. Multiplying this by 157.9 entering blank per 100 successfully completed part yields 66.5086 inspection hours to complete the order for 100 parts.

k. What are the expected direct costs (raw materials + operating costs - scrap value of rejected parts) per completed part?

Solution:

Operation	Cost per hour	x	Hours per visit	x	Expected # visits	=	Cost
Machine 1	$20	x	0.5	x	1.0471	=	$10.47
Inspection 1	$15	x	0.1	x	0.9424	=	1.41
Machine 2	$20	x	0.75	x	0.8246	=	12.36
Inspection 2	$15	x	0.2	x	0.7833	=	2.35
Machine 3	$20	x	0.25	x	0.6951	=	3.48
Inspection 3	$15	x	0.25	x	0.6812	=	2.55
Pack & Ship	$10	x	0.1	x	0.6335	=	0.63
			Total expected labor cost/entering part			=	$33.27

Raw material cost per entering part = $50
Scrap value retrieved per entering part = $10 x 0.3665 parts scrapped/entering part
= $3.66
Total expected cost per entering part: $50 + $33.27 - $3.66 = $79.61

To successfully complete 1 part, the expected number of entering parts required is 1.579, and so
Total expected cost per completed part is $79.61 x 1.579 = $125.70

❍❍❍❍❍❍❍❍❍❍❍ **Homework #10** ❍❍❍❍❍❍❍❍❍❍

1. Model the following situation using a **Discrete-time Markov chain**, and use the MARKOV workspace of APL code to analyze the model and answer the questions below. *(The MARKOV workspace is available on ICAEN's Courseware fileserver, in the I.E. program's folder. To use it, however, you must have the APL 68000 Level II interpreter. See the "Software" link on the course web page for more instructions.)*

The Green Valley Christmas Tree Farm owns a plot of land with 5000 evergreen trees. Each year they allow individuals to select and cut their own Christmas trees. However, they protect small trees (usually less than 4 feet tall) so that they will grow and be available for sale in future years. Currently 1500 trees are classified as protected trees, while the remaining 3500 are available for cutting. However, even though a tree is available for cutting in a given year, it possibly might not be selected for cutting until future years. While most trees not cut in a given year live until the next year(protected or unprotected), approximately 15% are lost to disease. Each year, approximately 60% of the unprotected trees are cut, and 30% of the protected trees surviving from the previous year have matured sufficiently to be made available for cutting.

(a.) Define a (discrete-time) Markov chain model of the system consisting of a *single* tree.

- List the states, with a definition of each state.

- Sketch the transition diagram and write down the probability matrix.

(b.) Which are the absorbing states of this model?

(c.) What is the probability that a tree which is now protected is eventually sold? ________

...that it eventually dies of disease? ________

(d.) How many of the farm's 5000 trees are expected to be sold eventually? ________

How many will be lost to disease? ________

(e.) If a tree is now protected, what is the expected number of years until it is either sold or dies? ________

2. **Continuous-time Markov chain**. A certain machine has occasional breakdowns. After the first breakdown, it can be repaired at a cost of \$1000. However, such a repair can only be done once, and consequently, the machine has to be replaced by a new one after the second breakdown. The replacement cost of the machine is \$5000. The time until the first breakdown follows an exponential distribution with an expected value of 5 years, and the time between the first breakdown (repair) and the second breakdown (replacement) is also exponentially distributed, but with an expected value of 4 years. The time to perform the repairs is negligible.

(a.) Formulate the problem as a continuous-time Markov chain, and draw its transition diagram. (Only 2 states are necessary. Let state 1 represent the condition before the first repair is made, and state 2 the condition after the first repair but before the second breakdown.)

(b.) Write the balance equations and find the steady-state distribution of the state of the system,
ð$_1$ = ________ and ð$_2$= ________.

(c.) We want to find the *average cost per year of repairs and replacements.* To do this, first find:

(i.) (Conditional) rate at which repairs are made <u>when in state 1</u>

(ii.) Rate at which repairs are made (value from (i) times ð$_1$) ________

(iii.) (Conditional) rate at which replacements are made *when in state 2.*
____

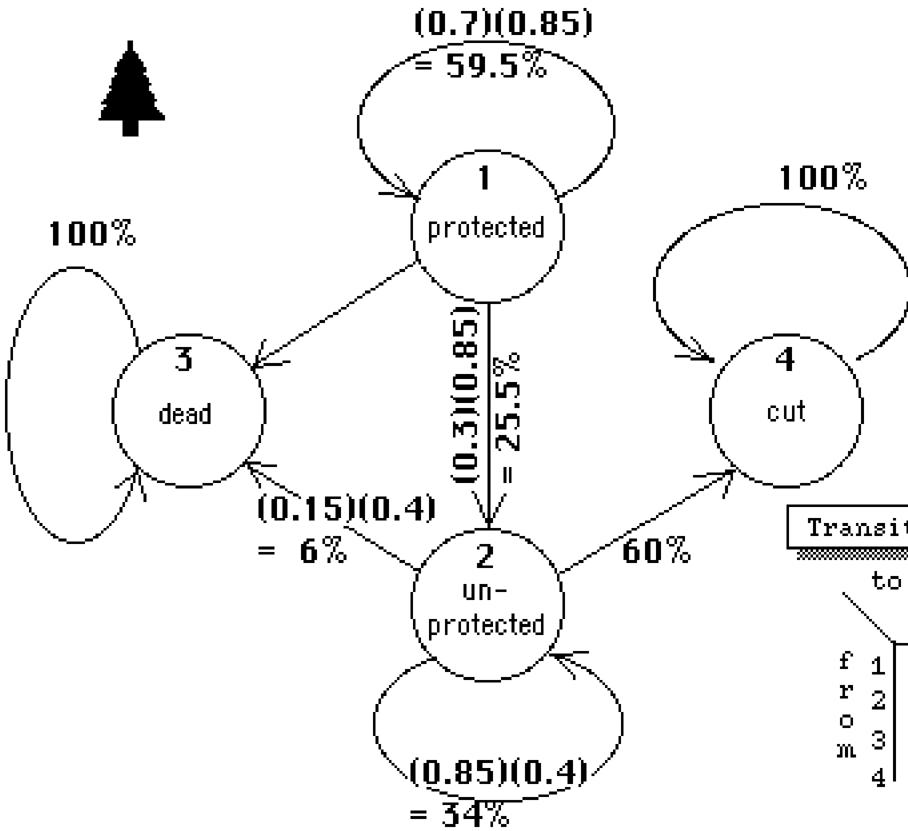
(iv.) Rate at which replacements are made (value from (iii) times ð$_2$)
____

(v.) Average cost per year (sum of the rates of repairs (ii) and replacements (iv) times the
appropriate costs). ________

❍❍❍❍❍❍❍❍❍❍ Homework #10 Solutions ❍❍❍❍❍❍❍❍❍

1. Discrete-Time Markov Chains. Define a Markov chain in which the state of the tree is observed each year immediately ***before*** the Christmas season begins.

a. The four states of the system, and transition probabilities, are indicated below:



Transition Probability Matrix

from \ to	1	2	3	4
1	.595	.255	.15	0
2	0	.34	.06	.6
3	0	0	1	0
4	0	0	0	1

b. The absorbing states are #3: "dead" and #4: "cut ".

Using the MARKOV workspace, and selecting "Absorption Analysis" on the menu produces the following output:

A = Absorption Probabilities

from	3	4
1	0.45454545	0.54545455
2	0.090909091	0.90909091

This computation could also be done manually as follows:

$$Q = \begin{bmatrix} .595 & .255 \\ 0 & .34 \end{bmatrix},\ E=(I-Q)^{-1} = \begin{bmatrix} .405 & -.255 \\ 0 & .66 \end{bmatrix}^{-1} = \begin{bmatrix} 2.66667 & .90909 \\ 0 & 1.51515 \end{bmatrix}$$

$$A = ER = \begin{bmatrix} 2.66667 & .90909 \\ 0 & 1.51515 \end{bmatrix} \begin{bmatrix} .15 & 0 \\ .06 & .6 \end{bmatrix} = \begin{bmatrix} 4/9 & 5/9 \\ 1/9 & 8/9 \end{bmatrix}$$

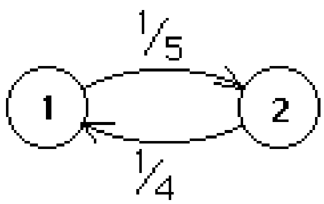
c. According to the A matrix above, a protected tree has probability 54.54% (=a_{14}) of eventually being cut & sold, and probability 45.45% (=a_{13}) of eventually being lost to disease.

d. The number which is expected to be eventually sold is $1500\times a_{14} + 3500\times a_{24}$ =1500(0.5454) + 3500(0.90909) = 818.1 + 3181.815 = 3999.9. That is, 54.54% of the protected trees (namely, 818.1) and 90.9% of the unprotected trees (namely, 3181.8) will be cut & sold, a total of about 4000 trees. The remaining 1000 (=3500+1500-4000) are expected to be lost to disease.

e. If the tree is initially protected, the expected number of visits to the transient state #1 is e_{11}=2.66667 and to transient state #3 is e_{12}=0.90909. Note, however, that e_{11} counts the initial visit to state #1 (i.e., the initial harvest season), so that the number of *additional* visits to this state is 1.66667 (the expected number of years that it will be protected), i.e., the expected lifetime of the tree will be 1.6667 + 0.90909 = 2.5757 years.

2. Continuous-Time Markov Chains. Solution:

(a.) *Formulate the problem as a continuous-time Markov process, and draw its transition diagram. (Only 2 states are necessary. Let state 1 represent the condition before the first repair is made, and state 2 the condition after the first repair but before the second breakdown.)*



(b.) *Write the balance equations and find the steady-state distribution of the state of the system, ð*$_1$ *and ð*$_2$*.* The balance equation for this C-T Markov Chain is

$$\frac{1}{5}\pi_1 = \frac{1}{4}\pi_2$$

That is, the rate of transitions from state 1 in steady state should equal the rate of transitions into state 1, where the left side above is the rate of transitions from state 1 and the right side is the rate of transitions into state 1. (In this case, we get the identical balance equation for state 2.)

As an alternative, we could write down the transition rate matrix Λ:

$$\Lambda = \begin{bmatrix} -1/5 & 1/5 \\ 1/4 & -1/4 \end{bmatrix}$$

and then write the system of equations represented by $\pi\Lambda = 0$, namely

$$\begin{cases} -1/5\ \pi_1 + 1/4\ \pi_2 = 0 \\ 1/5\ \pi_1 - 1/4\ \pi_2 = 0 \end{cases}$$

of which one may be discarded as redundant.

The remaining equation, together with the restriction on the sum of the probabilities, gives us a system of 2 equations with 2 unknowns:

$$\begin{cases} \frac{1}{5}\pi_1 = \frac{1}{4}\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

The optimal solution is

$$\begin{cases} \pi_1 = 5/9 \\ \pi_2 = 4/9 \end{cases}$$

(c.) We want to find the *average cost per year of repairs and replacements.* To do this, first find:

(i.) (Conditional) rate at which repairs are made <u>when in state 1</u> = $^1/_5$

(ii.) Rate at which repairs are made (value from (i) times ð$_1$) = ($^1/_5$) π_1 = $^1/_9$

(iii.) (Conditional) rate at which replacements are made <u>when in state 2</u> = $^1/_4$

(iv.) Rate at which replacements are made (value from (iii) times ð$_2$) = ($^1/_4$)π_2= $^1/_9$

(v.) Average cost per year (sum of the rates of repairs (ii) and replacements (iv) times the appropriate costs). ($^1/_9$)(\$1000)+($^1/_9$)(\$5000) = \$666.67/year

❍❍❍❍❍❍❍❍❍❍ Homework #11 ❍❍❍❍❍❍❍❍❍

1. The college is trying to decide whether to rent a slow or fast copy machine. It is believed that an employee's time is worth $15/hour. The slow copier rents for $4 per hour, and it takes an employee an average of 10 minutes to complete a copy job (exponentially distributed). The fast copier rents for $15 per hour, and it takes an employee an average of 6 minutes to complete a copy job (also exponentially distributed). An average of 4 employees per hour need to use the copying machines (interarrival times are exponentially distributed).

a. For each choice of machine, sketch the birth/death process model, showing the "birth" and "death" rates.

b. Which standard queueing model applies to this situation? (e.g., M/M/1, M/M/2, M/M/1/N, etc.)

c. For each machine, what is:

	Slow	Fast
Utilization of the machine	______%	______%
Average number of employees at the copy center:	______	______
Total cost (rental + employee time)	______	______

d. Which machine should the college rent?

2. *(Exercise 6, page 1083-1084 of text by Winston)* Bectol, Inc. is building a dam. A total of 10,000,000 cu ft of dirt is needed to construct the dam. A bulldozer is used to collect dirt for the dam. Then the dirt is moved via dumpers to the dam site. Only one bulldozer is available, and it rents for $100 per hour. Bectol can rent, at $40 per hour, as many dumpers as desired. Each dumper can hold 1000 cu ft of dirt. It takes an average of 12 minutes for the bulldozer to load a dumper with dirt, and it takes each dumper an average of five minutes to deliver the dirt to the dam and return to the bulldozer. Making appropriate assumptions about exponentiality so as to obtain a birth/death model, determine the optimal number of dumpers and the minimum total expected cost of moving the dirt needed to build the dam.

a. What is the optimal number of dumpers? ______

b. What is the minimum expected cost of moving the dirt to build the dam? ______

3. A neighborhood grocery store has only one check-out counter. Customers arrive at the check-out at a rate of one per 2 minutes. The grocery store clerk requires an average of one minute and 30 seconds to serve each customer. However, as soon as the waiting line exceeds 2 customers, including the customer being served, the manager helps by packing the groceries, which reduces the average service time to one minute. Assume a Poisson arrival process and exponentially-distributed service times.

a. Draw the flow diagram for a birth-death model of this system.

(b.) Either manually or using the **Birth/Death** workspace, compute the steady-state distribution of the number of customers at the check-out.

(c.) What fraction of the time will the check-out clerk be idle? _________%

(d.) What is the expected number of customers in the check-out area? __________

(e.) What is the expected length of time that a customer spends in the check-out area? ___________ minutes

(f.) Suppose that the store is being remodeled, and space is being planned so that the waiting line does not overflow the space allocated to it more than 1 percent of the time, and that 4 feet must be allocated per customer (with cart). How much space should be allocated for the waiting line? ___________ feet

(g.) What fraction of the time will the manager spend at the check-out area? _________%

❍❍❍❍❍❍❍❍❍❍ Homework #11 Solutions ❍❍❍❍❍❍❍❍❍❍

1. Solution:

Slow copy machine : (M/M/1 model)

$\lambda=2$, $\mu=4$

$W=1/(\mu-\lambda)=0.5$ (hr/job) = total time of employee per visit to copy machine

Average cost per hour=(# jobs/hr)($15/hr)(hr/job)+($4/hr)=4(15)(0.5)+4= $34.

Fast copy machine : (M/M/1 model)

$\lambda=4$, $\mu=10$

$W=1/(\mu-\lambda)=1/6$ hr/job) = total time of employee per visit to copy machine

Average cost per hour=(# jobs/hr)($15/hr)(hr/job)+($15/hr)=4(15)(1/6)+15= $25.

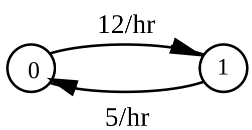
Therefore, we should choose the fast copy machine.

2. Solution:

We have to use 10,000,000/1000=10,000 loads to deliver all the dirt. If the loader were to have *100% utilization* , it would require 10000 loads / 5 loads/hr. = 2000 hours to complete the job. As the number of dumpers is increased, the utilization of the loader will *increase* (and the time required to complete the job will *decrease*), so that the cost will *decrease*. We use "trial & error" below to find the optimal number of dumpers.

Case 1 : One dumper :

Define state 0 : no dumper in the system,
state 1 : one dumper in the system.



Steady-state Distribution

i	Pi	CDF
0	0.294118	0.294118
1	0.705882	1.000000

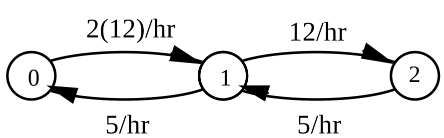
Utilization of loader is $1-\pi_0$ = 70.5882%, so that the total time to complete the job will be

$$\frac{10000 \text{ loads}}{.705882 \times 5 \text{ loads/hr.}} = 2833.33 \text{ hours.}$$

The hourly cost of renting the bulldozer and 1 dumper is \$100/hr + \$40/hr = \$140/hr, and so the total cost of completing the job will be
2833.33 hrs. (100\$/hr. + 40 \$/hr.) = \$396,666

Case 2 : Two dumpers.

Define state 0 : no dumper in the system,
state 1 : one dumper in the system,
state 2 : two dumpers in the system, one is being served and another is waiting.



Steady-state Distribution

i	Pi	CDF
0	0.057737	0.057737
1	0.277136	0.334873
2	0.665127	1.000000

Utilization of loader is $1-\pi_0$ = 94.2263%, so that the total time to complete the job will be

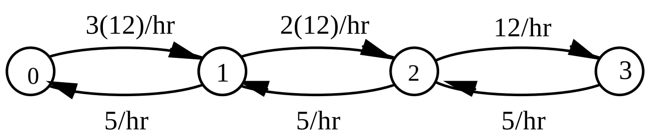
$$\frac{10000 \text{ loads}}{.942263 \times 5 \text{ loads/hr.}} = 2122.55 \text{ hours.}$$

The hourly cost of renting the bulldozer and 2 dumpers is \$100/hr + 2(\$40/hr) = \$180/hr, and so the total cost of completing the job will be
2122.55 hrs. × 180 \$/hr. = \$382,059

Case 3 : Three dumpers :

Define state 0 : no dumper in the system,
state 1 : one dumper in the system,
state 2 : two dumpers in the system, one is being served and another is waiting.

state 3 : three dumpers in the system, one is being served, and the other two are waiting.



Steady-state Distribution

i	Pi	CDF
0	0.007955	0.007955
1	0.057277	0.065233
2	0.659836	0.340164
3	0.659836	1.000000

Utilization of loader is $1-\pi_0 = 99.2045\%$, so that the total time to complete the job will be

$$\frac{10000 \text{ loads}}{.992045 \times 5 \text{ loads/hr.}} = 2016.04 \text{ hours.}$$

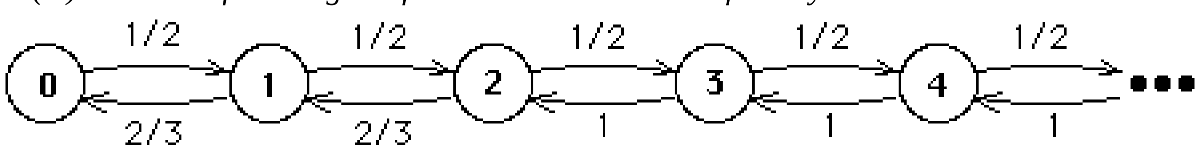
The hourly cost of renting the bulldozer and 3 dumpers is \$100/hr + 3(\$40/hr) = \$220/hr, and so the total cost of completing the job will be

2016.04 hrs. $\times$ 220 \$/hr. = \$443,529

The system cost when there are two dumpers is less than that obtained when the number of dumpers is either increased to 3 or decreased to 1. Therefore, it is reasonable to conclude that the optimal number of dumpers is 2.

3. Solution:

(a.) Draw the flow diagram for a birth-death model of this system.



The "birth" rate is 1/2 per minute for all states, while the "death" rate is 2/3 per minute in states 1 and 2, and 1/minute for higher-numbered states.

*(b.) Either manually or using the **Birth/Death** workspace, compute the steady-state distribution of the number of customers at the check-out.*

Manual computation:

$$\frac{1}{\pi_0} = 1 + \frac{1/2}{2/3} + \left(\frac{1/2}{2/3}\right)^2 + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right) + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right)^2 + \left(\frac{1/2}{2/3}\right)^2\left(\frac{1/2}{1}\right)^3 + \cdots$$

$$\frac{1}{\pi_0} = 1 + \frac{1/2}{2/3} + \left(\frac{1/2}{2/3}\right)^2\left\{1 + \left(\frac{1/2}{1}\right) + \left(\frac{1/2}{1}\right)^2 + \left(\frac{1/2}{1}\right)^3 + \cdots\right\}$$

The infinite series within the braces is a geometric series with sum

$$\frac{1}{1-\frac{1}{2}} = 2$$

and so

$$\frac{1}{\pi_0} = 1 + \frac{\frac{1}{2}}{\frac{2}{3}} + \left(\frac{\frac{1}{2}}{\frac{2}{3}}\right)^2 (2) = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 \times 2 = 1 + 0.75 + 2(0.5625) = 2.875$$

Therefore, $\pi_0 = 1/2.875 = 0.347826$ and $\pi_2 = 0.75\pi_0$, $\pi_2=0.5625\pi_0$, etc.

Use of Birth/Death workspace:

Entering the birth & death rates, and asking for computation of the steadystate probabilities up to π_{10}, we get:

Steady-State Distribution

i	Lambda	Mu	ρ	Pi	CDF
0	0.5000	0.0000	1.000000	0.347826	0.347826
1	0.5000	0.6667	0.750000	0.260870	0.608696
2	0.5000	0.6667	0.750000	0.195652	0.804348
3	0.5000	1.0000	0.500000	0.097826	0.902174
4	0.5000	1.0000	0.500000	0.048913	0.951087
5	0.5000	1.0000	0.500000	0.024457	0.975543
6	0.5000	1.0000	0.500000	0.012228	0.987772
7	0.5000	1.0000	0.500000	0.006114	0.993886
8	0.5000	1.0000	0.500000	0.003057	0.996943
9	0.5000	1.0000	0.500000	0.001529	0.998471
10	0.5000	1.0000	0.500000	0.000764	0.999236

The computation of π_0 is done by truncating the series after the 11th term, so that the probabilities above are approximations. However, from the CDF (cumulative distribution function), we see that p{# of customers   10} > 99.9%, so that the approximated probabilities should be very near to the actual probabilities.

(c.) What fraction of the time will the check-out clerk be idle?

π_0 = 34.78% of the time, the check-out clerk will be idle.

(d.) What is the expected number of customers in the check-out area?

$$L = \sum_{i=0}^{\infty} i\pi_i = 0 + 0.608696 + 2\times 0.804348 + 3\times 0.902174 + \cdots$$

According to the Birth/Death workspace, we get L=1.4267 customers (including the one being served, if any).

The Mean Population Size is 1.4267

(e.) What is the expected length of time that a customer spends in the check-out area?

To compute the average time in the system, W, we use Little's Law: $L = \underline{\lambda}W$, where $\underline{\lambda}$ is the average arrival rate, in this case 1/2. Therefore, $W = L/\underline{\lambda} =$ 1.4267/(0.5/minute) = 2.8534 minutes.

(f.) Suppose that the store is being remodeled, and space is being planned so that the waiting line does not overflow the space allocated to it more than 1 percent

of the time, and that 4 feet must be allocated per customer (with cart). How much space should be allocated for the waiting line?

By examining the cumulative probabilities (CDF) in the table above, we see that P{# in system 7} = 0.993886, i.e., P{# in system exceeds 7} = 0.6114%. Therefore, if we allocate enough space for 7 customers (6 waiting plus one being served), i.e., 24 feet plus space for the customer being served, the customers will overflow the space less than 1% of the time.

(g.) What fraction of the time will the manager spend at the check-out area?

$1-\pi_0 - \pi_1 - \pi_2$ = 1 - 0.804348 = 19.5652%.

❍❍❍❍❍❍❍❍❍❍ Homework #12 ❍❍❍❍❍❍❍❍❍

1. A system consists of 4 devices, each subject to possible failure, all of which must function in order for the system to function. In order to increase the reliability of the system, redundant units may be included, so that the system continues to function if at least one of the redundant units remains functional. The data are:

Device	Reliability (%)	Weight (kg.)
1	80	1
2	90	3
3	75	1
4	85	2

If we include a single unit of each device, then the system reliability will be only 45.9%.

a. Explain how it is determined that the reliability is only 45.9%.

However, by including redundant units of one or more devices, we can substantially increase the reliability.

Suppose that the system may weigh no more than 12 kg. (Since at least one of each device *must* be included, a total of 7 kg, this leaves 5 kg available for redundant units.) Assume that no more than 3 units of any type need be considered. We wish to compute the number of units of each device type to be installed in order to maximize the system reliability, subject to the maximum weight restriction.

The dynamic programming model arbitrarily assumes that the devices are considered in the order: #4, #3, #2, and finally, #1. The optimal value function is defined to be:

$F_n(S)$ = maximum reliability which can be achieved for devices #n, n-1, ... 1, given that the weight used by these devices cannot exceed S *(the state variable)*

Note that the computation is done in the backward order, i.e., first the optimal value function $F_1(S)$ is computed for each value of the available weight S, then $F_2(S)$, etc., until finally $F_4(S)$ has been computed.

Data

Weight

i	1	2	3	4
Wt[i]	1	3	1	2

Reliability (%) vs # redundant units:

i	1	2	3
1	80	96	99.2
2	90	99	99.9
3	75	93.75	98.4375
4	85	97.75	99.6625

b. Explain the computation of the 97.75% reliability for 2 units of device #4 above.

Details of Computations at each Stage:

Stage 1

s \ x:	1	2	3
1	0.80	¯99999999.99	¯99999999.99
2	0.80	0.96	¯99999999.99
3	0.80	0.96	0.99
4	0.80	0.96	0.99
5	0.80	0.96	0.99
6	0.80	0.96	0.99
7	0.80	0.96	0.99
8	0.80	0.96	0.99
9	0.80	0.96	0.99
10	0.80	0.96	0.99
11	0.80	0.96	0.99
12	0.80	0.96	0.99

Stage 3

s \ x:	1	2	3
5	0.54	¯99999999.99	¯99999999.99
6	0.65	0.68	¯99999999.99
7	0.67	0.81	0.71
8	0.67	0.84	0.85
9	0.71	0.84	0.88
10	0.74	$\boxed{}$	0.88
11	0.74	0.92	0.94
12	0.74	0.92	0.97

Stage 2

s \ x:	1	2	3
4	0.72	¯99999999.99	¯99999999.99
5	0.86	¯99999999.99	¯99999999.99
6	0.89	¯99999999.99	¯99999999.99
7	0.89	0.79	¯99999999.99
8	0.89	0.95	¯99999999.99
9	0.89	0.98	¯99999999.99
10	0.89	0.98	0.80
11	0.89	0.98	0.96
12	0.89	0.98	0.99

Stage 4

s \ x:	1	2	3
7	0.46	¯99999999.99	¯99999999.99
8	0.57	¯99999999.99	¯99999999.99
9	0.69	0.53	¯99999999.99
10	0.72	0.66	¯99999999.99
11	0.75	0.79	0.54
12	0.76	0.83	0.67

Optimal values & decisions at each stage:

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
7	0.46	1	5
8	0.57	1	6
9	0.69	1	7
10	0.72	1	8
11	0.79	2	7
12	0.83	2	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resul Sta
4	0.72	1	[illegible]
5	0.86	1	[illegible]
6	0.89	1	[illegible]
7	0.89	1	[illegible]
8	0.95	2	[illegible]
9			
10	0.98	2	[illegible]
11	0.98	2	[illegible]
12	0.99	3	[illegible]

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
5	0.54	1	4
6	0.68	2	4
7	0.81	2	5
8	0.85	3	5
9	0.88	3	6
10	0.89	2	8
11	0.94	3	8
12	0.97	3	9

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
1	0.80	1	0
2	0.96	2	0
3	0.99	3	0
4	0.99	3	1
5	0.99	3	2
6	0.99	3	3
7	0.99	3	4
8	0.99	3	5
9	0.99	3	6
10	0.99	3	7
11	0.99	3	8
12	0.99	3	9

c. What is the maximum reliability that can be achieved allowing 12 kg. total weight?

d. How many units of each device should be included in the system?

e. Four values have been blanked out in the output. What are they?

i. the optimal value $f_2(9)$ _____________

ii. the optimal decision $x_2^*(9)$ _____________

iii. the state which results from the optimal decision $x_2^*(9)$ _____________

iv. the value associated with the decision to include 2 units of device #3, given that 10 kg. is still available _____________

f. Suppose that only 11 kg. of capacity were available. What is the achievable system reliability? How many units of each device should be included?

2. We wish to plan production of an expensive, low-demand item for the next three months (January, February, & March).

- the cost of production is $15 for setup, plus $5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is $2 per unit, based upon the level at the beginning of the month.
- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand each month is random, with the same probability distribution:

d	0	1	2
P{D=d}	0.3	0.4	0.3

- there is a penalty of $25 per unit for any demand which cannot be satisfied. Backorders are not allowed.
- the inventory at the end of December is 1.
- a salvage value of $4 per unit is received for any inventory remaining at the end of the last month (March)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 3 = January, stage 2 = February, etc.** (i.e., n = # months remaining in planning period.)

a. What is the optimal production quantity for January? _______

b. What is the total expected cost for the three months? __________

c. If, during January, the demand is 1 unit, what should be produced in February?

______

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ ___________

ii. the optimal decision $x_2^*(1)$ ___________

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ____________

The table of costs for each combination of state & decision at stage 2 is:

---STAGE 2---

s \ x:	0	1	2	3	4
0	46.00	$\boxed{}$	34.62	31.89	33.60
1	26.69	31.62	28.89	30.60	35.00
2	13.62	25.89	27.60	32.00	37.00
3	7.89	24.60	29.00	34.00	39.00

The tables of the optimal value function $f_n(S_n)$ at each stage are:

Stage 3:

State	Optimal Values	Optimal Decisions
0	44.61	4
1	39.83	0
2	28.33	0
3	21.82	0

Stage 2:

State	Optimal Values	Optimal Decisions
0	31.89	3
1		
2	13.62	0
3	7.89	0

Stage 1:

State	Optimal Values	Optimal Decisions
0	21.00	2
1	8.30	0
2	0.00	0
3	¯2.00	0

❍❍❍❍❍❍❍❍❍❍ Homework #12 Solutions ❍❍❍❍❍❍❍❍❍

1. Solution:

a. *Explain how it is determined that the reliability is only 45.9%.*

Solution.
(0.8)(0.9)(0.75)(0.85)=0.459.

b. *Explain the computation of the 97.75% reliability for 2 units of device #4 above.*
Solution. 1- (1- 0.85)(1- 0.85)=0.9775 or 97.75%.

Details of Computations at each Stage:

Stage 1

s \ x:	1	2	3
1	0.80	¯99999999.99	¯99999999.99
2	0.80	0.96	¯99999999.99
3	0.80	0.96	0.99
4	0.80	0.96	0.99
5	0.80	0.96	0.99
6	0.80	0.96	0.99
7	0.80	0.96	0.99
8	0.80	0.96	0.99
9	0.80	0.96	0.99
10	0.80	0.96	0.99
11	0.80	0.96	0.99
12	0.80	0.96	0.99

Stage 3

s \ x:	1	2	3
5	0.54	¯99999999.99	¯99999999.99
6	0.65	0.68	¯99999999.99
7	0.67	0.81	0.71
8	0.67	0.84	0.85
9	0.71	0.84	0.88
10	0.74	$\boxed{0.89}$	0.88
11	0.74	0.92	0.94
12	0.74	0.92	0.97

Stage 2

s \ x:	1	2	3
4	0.72	¯99999999.99	¯99999999.99
5	0.86	¯99999999.99	¯99999999.99
6	0.89	¯99999999.99	¯99999999.99
7	0.89	0.79	¯99999999.99
8	0.89	0.95	¯99999999.99
9	0.89	0.98	¯99999999.99
10	0.89	0.98	0.80
11	0.89	0.98	0.96
12	0.89	0.98	0.99

Stage 4

s \ x:	1	2	3
7	0.46	¯99999999.99	¯99999999.99
8	0.57	¯99999999.99	¯99999999.99
9	0.69	0.53	¯99999999.99
10	0.72	0.66	¯99999999.99
11	0.75	0.79	0.54
12	0.76	0.83	0.67

Optimal values & decisions at each stage:

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
7	0.46	1	5
8	0.57	1	6
9	0.69	1	7
10	0.72	1	8
11	0.79	2	7
12	0.83	2	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resu St.
4	0.72	1	
5	0.86	1	
6	0.89	1	
7	0.89	1	
8	0.95	2	
9	$\boxed{0.98}$	$\boxed{2}$	
10	0.98	2	
11	0.98	2	
12	0.99	3	

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
5	0.54	1	4
6	0.68	2	4
7	0.81	2	5
8	0.85	3	5
9	0.88	3	6
10	0.89	2	8
11	0.94	3	8
12	0.97	3	9

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
1	0.80	1	0
2	0.96	2	0
3	0.99	3	0
4	0.99	3	1
5	0.99	3	2
6	0.99	3	3
7	0.99	3	4
8	0.99	3	5
9	0.99	3	6
10	0.99	3	7
11	0.99	3	8
12	0.99	3	9

c. *What is the maximum reliability that can be achieved allowing 12 kg. total weight?*

Solution. 0.83.

d. *How many units of each device should be included in the system?*

Solution.

Device #	# of units
1	1
2	1
3	3
4	2

e. *Four values have been blanked out in the output. What are they?*

i. the optimal value $f_2(9)$ ___0.98_____

ii. the optimal decision $x_2^*(9)$ ____2_____

iii. the state which results from the optimal decision $x_2^*(9)$ _____3_____

iv. the value associated with the decision to include 2 units of device #3, given that 10 kg. is still available ______0.98____

f. *Suppose that only 11 kg. of capacity were available. What is the achievable system reliability? How many units of each device should be included?*

Solution. Reliability=0.79.

Device #	# of units
1	1
2	1
3	2
4	2

2. Solution

a. What is the optimal production quantity for January? ___0____

b. What is the total expected cost for the three months? ___39.83____

c. If, during January, the demand is 1 unit, what should be produced in February? ___3__

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ ____26.69____

ii. the optimal decision $x_2^*(1)$ ___0_____

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ____44.69____

Solution.

P(d)	d	inventory cost	production cost	f_1^*+penalty
0.3	0	0	15+5	8.3
0.4	1	0	15+5	21
0.3	2	0	15+5	21+25

(0.3)(15+5+8.3)+(0.4)(15+5+21)+(0.3)(15+5+25+21)=44.69.

The table of costs for each combination of state & decision at stage 2 is:

---STAGE 2---

s \ x:	0	1	2	3	4
0	46.00	$\boxed{44.69}$	34.62	31.89	33.60
1	26.69	31.62	28.89	30.60	35.00
2	13.62	25.89	27.60	32.00	37.00
3	7.89	24.60	29.00	34.00	39.00

The tables of the optimal value function $f_n(S_n)$ at each stage are:

Stage 3:

State	Optimal Values	Optimal Decisions
0	44.61	4
1	39.83	0
2	28.33	0
3	21.82	0

Stage 2:

State	Optimal Values	Optimal Decisions
0	31.89	3
1	26.69	0
2	13.62	0
3	7.89	0

Stage 1:

State	Optimal Values	Optimal Decisions
0	21.00	2
1	8.30	0
2	0.00	0
3	¯2.00	0