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Homework Exercises -- Fall 1993

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Homework # 1

Matrix Algebra Review: The following problems are to be found in Chapter 2 of the text, *Operations Research (2nd edition)* by W. Winston:

(1.) Exercise # 4&6, p. 32
Use the Gauss-Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solutions (if any exist).

$$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 6 \\ x_1 + x_2 + x_3 = 4 \end{cases}$$

$$\begin{cases} 2x_2 + 2x_3 = 4 \\ x_1 + 2x_2 + x_3 = 4 \\ x_2 - x_3 = 0 \end{cases}$$

(2.) Exercise #2, p. 36
Determine if the following set of vectors is independent or linearly dependent:

$$V = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(3.) Exercise # 2, p. 42: *Find A^{-1} (if it exists) for the following matrix:*

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

Linear Programming Model Formulation: Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available both on the Apollo workstations and Macintoshes of ICAEN.) Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State verbally the optimal solution. *(All exercises are from Chapter 3 of the text. For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

(4.) Exercise #2, page 98 *(Furniture manufacturing)*
"Furnco manufactures tables and chairs. A table requires 40 board feet of wood, and a chair requires 30 board feet of wood. Wood may be purchased at a cost of $1 per board foot, and 40,000 board feet of wood are available for purchase. It takes 2 hours of skilled labor to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labor will turn an unfinished table into a finished table, and 2 more hours of skilled labor will turn an unfinished chair into a finished chair. A total of 6000 hours of skilled labor are available (and have already been paid for). All furniture produced can be sold at the following unit

prices: unfinished table, \$70; finished table, \$140; unfinished chair, \$60; finished chair, \$110. Formulate an LP that will maximize the contribution to profit from manufacturing tables and chairs."

◊◊◊◊◊◊◊◊ Homework #1 Solutions ◊◊◊◊◊◊◊◊

Matrix Algebra Review: The following problems are to be found in Chapter 2 of the text, Operations Research (2nd edition) by W. Winston:

(1.) Exercise # 4&6, p. 32
Use the Gauss-Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solutions (if any exist).

(A) $$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 6 \\ x_1 + x_2 + x_3 = 4 \end{cases}$$

(B) $$\begin{cases} 2x_2 + 2x_3 = 4 \\ x_1 + 2x_2 + x_3 = 4 \\ x_2 - x_3 = 0 \end{cases}$$

Solution: System (A) has infinite number of solutions, since

$$X_1 = a,\ X_2 = b,\ X_3 = 4 - a - b,\ X_4 = 2 - a + 2b,$$

$$where\quad a \in R, b \in R.$$

System (B) has unique solution $X_1 = X_2 = X_3 = 1.$

(2.) Exercise #2, p. 36
Determine if the following set of vectors is independent or linearly dependent:

$$V = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Solution: Let a,b,c $\in R$, we consider the following system

$$a\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = 0 \text{ ,i.e.,} \begin{matrix} 2a + b + 3c = 0 \\ a + 2b + 3c = 0 \\ c = 0 \end{matrix}$$

The only solution is a=b=c=0. Hence by the definition, we have that the set of vectors is linearly independent.

(3.) Exercise # 2, p. 42: *Find A^{-1} (if it exists) for the following matrix:*

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 0.5 & -0.5 & 0.5 \end{array}\right]$$

Hence

$$A^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ -1 & -2 & 3 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$$

Linear Programming Model Formulation: Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available both on the Apollo workstations and Macintoshes of ICAEN.) Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State verbally the optimal solution. (All exercises are from Chapter 3 of the text. For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)

(4.) Exercise #2, page 98 (Furniture manufacturing)
"Furnco manufactures tables and chairs. A table requires 40 board feet of wood, and a chair requires 30 board feet of wood. Wood may be purchased at a cost of $1 per board foot, and 40,000 board feet of wood are available for purchase. It takes 2 hours of skilled labor to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labor will turn an unfinished table into a finished table, and 2 more hours of skilled labor will turn an unfinished chair into a finished chair. A total of 6000 hours of skilled labor are available (and have already been paid for). All furniture produced can be sold at the following unit prices: unfinished table, $70; finished table, $140; unfinished chair, $60; finished chair, $110. Formulate an LP that will maximize the contribution to profit from manufacturing tables and chairs."

Solution: Suppose we produce

T1: number of unfinished tables,
T2: number of finished tables,
C1: number of unfinished chairs,
C2: number of finished chairs.

LP model :

Max	70T1+140T2	(profit for tables)	
	+60C1+110C2	(profit for chairs)	
	-40(T1+T2)	(cost for tables)	
	-30(C1+C2)	(cost for chairs)	
s.t.			
	2T1+5T2+2C1+4T2<=6000		(available board feet of wood)
	40T1+40T2+30C1+30C2<=40000		(available working hours)
	T1,T2,C1,C2>=0		

LINDO output :

```
 MAX     30 T1 + 100 T2 + 30 C1 + 80 C2
       SUBJECT TO
          2)    2 T1 + 5 T2 + 2 C1 + 4 C2 <=   6000
          3)    40 T1 + 40 T2 + 30 C1 + 30 C2 <=   40000
       END


      : go
         LP OPTIMUM FOUND  AT STEP      2

              OBJECTIVE FUNCTION VALUE

        1)         106666.664

       VARIABLE        VALUE          REDUCED COST
             T1           0.000000          76.666672
             T2           0.000000           6.666672
             C1           0.000000          50.000000
             C2       1333.333374            0.000000


       ROW         SLACK OR SURPLUS     DUAL PRICES
          2)          666.666687                   0.000000
          3)              0.000000                   2.666667

       NO. ITERATIONS=        2

       DO RANGE(SENSITIVITY) ANALYSIS?
      ? y


          RANGES IN WHICH THE BASIS IS UNCHANGED

                        OBJ COEFFICIENT RANGES
       VARIABLE         CURRENT         ALLOWABLE    ALLOWABLE
                          COEF            INCREASE        DECREASE
             T1            30.000000           76.666672         INFINITY
             T2          100.000000             6.666672         INFINITY
             C1            30.000000           50.000000         INFINITY
             C2            80.000000           INFINITY          5.000004

                             RIGHTHAND SIDE RANGES
          ROW         CURRENT        ALLOWABLE       ALLOWABLE
                          RHS            INCREASE          DECREASE
             2           6000.000000      INFINITY              666.666687
             3         40000.000000     5000.000000         40000.000000
```

Homework # 2

(1.) Consider the following LP:

$$\text{Minimize } z = 3x_1 + x_2$$
$$\text{s.t. } x_1 - 2x_2 \geq 2$$
$$x_1 + x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

a. Plot the feasible region of the LP, and identify the extreme points.
b. Convert the constraints (excluding the nonnegativity constraints) into equations, by the use of "surplus" variables.
c. How many basic variables will the set of two equality constraints have?
d. How many basic solutions (both feasible and infeasible) will the two equality constraint have, i.e., how many ways may these basic variables be chosen from the set of four variables in the equations?
e. Use the Big-M method to find the optimal solution. Plot the path which is taken on your graph in (a), and indicate which variables are basic at each basic solution along the path.
f. Use the two-phase method to find the optimal solution. If the path followed to the optimum is different from that in (e), plot it, and again indicate which variables are basic at each basic solution along this path.

(2.) Exercise #19, p. 116 ***Linear Programming Model Formulation:*** Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available both on the Apollo workstations and Macintoshes of ICAEN.) Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State verbally the optimal solution. *(For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

Brady Corporation produces cabinets. Each week, they require 90,000 cu ft of processed lumber. They may obtain processed lumber in two ways. First, they may purchase lumber from an outside supplier and then dry it at their kiln. Second, they may chop down logs on their land, cut them into lumber at their sawmill, and finally dry the lumber at their kiln.

Brady can purchase grade 1 or grade 2 lumber. Grade 1 lumber costs $3 per cu ft and when dried yields 0.7 cu ft of useful lumber. Grade 2 lumber costs $7 per cu ft and when dried yields 0.9 cu ft of useful lumber. It costs the company $3 per cu ft to chop down a log. After being cut and dried, one cubic foot of log yields 0.8 cu ft of lumber. Brady incurs costs of $4 per cu ft of lumber dried.

It costs $2.50 per cu ft of logs sent through their sawmill. Each week, the sawmill can process up to 35,000 cu ft of lumber. Each week, up to 40,000 cu ft of grade 1 lumber and up to 60,000 cu ft of grade 2 lumber can be purchased.

Each week, 40 hours of time are available for drying lumber. the time it takes to dry 1 cu ft of grade 1 lumber, grade 2 lumber, or logs is as follows:

grade 1: 2 seconds
grade 2: 0.8 seconds
log: 1.3 seconds

Formulate an LP to help Brady minimize the weekly cost of meeting the demand for processed lumber.

◊◊◊◊◊◊◊◊ Homework #2 Solutions ◊◊◊◊◊◊◊◊

(1.) Consider the following LP:

$$\text{Minimize } z = 3x_1 + x_2$$
$$\text{s.t. } x_1 - 2x_2 \geq 2$$
$$x_1 + x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

a. Plot the feasible region of the LP, and identify the extreme points.
b. Convert the constraints (excluding the nonnegativity constraints) into equations, by the use of "surplus" variables.
c. How many basic variables will the set of two equality constraints have?

d. How many basic solutions (both feasible and infeasible) will the two equality constraint have, i.e., how many ways may these basic variables be chosen from the set of four variables in the equations?

e. Use the Big-M method to find the optimal solution. Plot the path which is taken on your graph in (a), and indicate which variables are basic at each basic solution along the path.

f. Use the two-phase method to find the optimal solution. If the path followed to the optimum is different from that in (e), plot it, and again indicate which variables are basic at each basic solution along this path.

Solution:

(a).

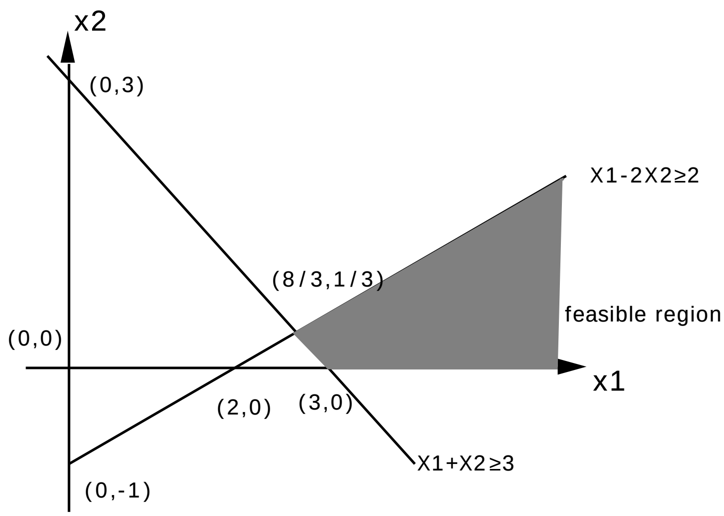

Fig 1. Feasible region for the problem.

(b) Min 3X1+X2

s.t. X1-2X2-s1=2

X1+X2-s2=3

X1,X2,s1,s2≥0

Here s1 and s2 are slack variables.

(c). Two basic variables.(Since we have two constraints)

(d). Because we have 4 variables and 2 constraints, hence we have $\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$ basic solutions.(the extreme points in Figure 1). Two of them are feasible and four of them are infeasible.

(e) Big-M Method.

Min (-Z)+3X1+X2+Ma1+Ma2

s.t. X1-2X2-s1+a1=2

X1+X2-s2+a2=3

X1,X2,s1,s2,a1,a2≥0,

where a1and a2 are artificial varibles, and M is a very large scalar.

The system can be rewritten as

Min (-Z)+3X1+X2+M(2-X1+2X2+s1)+M(3-X1-X2+s2)

s.t. X1-2X2-s1+a1=2

X1+X2-s2+a2=3

X1,X2,s1,s2,a1,a2≥0

Simplex tableau:

-Z	X1	X2	s1	s2	a1	a2	RHS	basic vars
1	3-2M	1+M	M	M	0	0	-5M	-Z=-5M
0	1	-2	-1	0	1	0	2	a1=2
0	1	1	0	-1	0	1	3	a2=3

-Z	X1	X2	s1	s2	a1	a2	RHS	basic vars
1	0	-3M+7	-M+3	M	2M-3	0	-M-6	-Z=-M-6
0	1	-2	-1	0	1	0	2	X1=2
0	0	3	1	-1	-1	1	1	a2=1

-Z	X1	X2	s1	s2	a1	a2	RHS	basic vars
1	0	0	2/3	7/3	M-(2/3)	0	-25/3	-Z=-25/3
0	1	0	-1/3	-2/3	*	*	8/3	X1=8/3
0	0	1	1/3	-1/3	*	*	1/3	X2=1/3

Because all reduced costs are nonnegative, the solution is optimal.
The path is (0,0) ---> (2,0) ---> (8/3,1/3) (see Fig 1).

(f). Two Phase Method.

Min w=a1+a2
s.t. (-Z)+3X1+X2=0
X1-2X2-s1+a1=2
X1+X2-s2+a2=3
X1,X2,s1,s2,a1,a2≥0

-W	-Z	X1	X2	s1	s2	a1	a2	RHS
1	0	0	0	0	0	1	1	0
0	1	3	1	0	0	0	0	0
0	0	1	-2	-1	0	1	0	2
0	0	1	1	0	-1	0	1	3

Phase (I) we want to minimize W.

-W	-Z	X1	X2	s1	s2	a1	a2	RHS
1	0	-2	1	1	1	0	0	-5
0	1	3	1	0	0	0	0	0
0	0	1	-2	-1	0	1	0	2
0	0	1	1	0	-1	0	1	3

-W	-Z	X1	X2	s1	s2	a1	a2	RHS
1	0	-2	1	1	1	0	0	-5
0	1	3	1	0	0	0	0	0
0	0	1	-2	-1	0	1	0	2
0	0	1	1	0	-1	0	1	3

-W	-Z	X1	X2	s1	s2	a1	a2	RHS
1	0	0	-3	-1	1	2	0	-1
0	1	0	7	3	0	-3	0	-6
0	0	1	-2	-1	0	1	0	2
0	0	0	3	1	-1	-1	1	1

-W	-Z	X1	X2	s1	s2	a1	a2	RHS
1	0	0	0	0	0	1	1	0
0	1	0	0	2/3	7/3	*	*	-25/3
0	0	1	0	-1/3	-2/3	*	*	8/3
0	0	0	1	1/3	-1/3	*	*	1/3

All artifical variables are now non-basic, we can drop the phase one objective row and the artifical variables from the tableau.

-Z	X1	X2	s1	s2	RHS
1	0	0	2/3	7/3	-25/3
0	1	0	-1/3	-2/3	8/3
0	0	1	1/3	-1/3	1/3

Note that this is optimal. The path is (0,0) -->(2,0) -->(8/3,1/3).(see Fig.1).

(2.) Exercise #19, p. 116 ***Linear Programming Model Formulation:*** Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available both on the Apollo workstations and Macintoshes of ICAEN.) Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State verbally the optimal solution. *(For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

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Brady can purchase grade 1 or grade 2 lumber. Grade 1 lumber costs \$3 per cu ft and when dried yields 0.7 cu ft of useful lumber. Grade 2 lumber costs \$7 per cu ft and when dried yields 0.9 cu ft of useful lumber. It costs the company \$3 per cu ft to chop down a log. After being cut and dried, one cubic foot of log yields 0.8 cu ft of lumber. Brady incurs costs of \$4 per cu ft of lumber dried.

It costs \$2.50 per cu ft of logs sent through their sawmill. Each week, the sawmill can process up to 35,000 cu ft of lumber. Each week, up to 40,000 cu ft of grade 1 lumber and up to 60,000 cu ft of grade 2 lumber can be purchased.

Each week, 40 hours of time are available for drying lumber. the time it takes to dry 1 cu ft of grade 1 lumber, grade 2 lumber, or logs is as follows:

grade 1: 2 seconds
grade 2: 0.8 seconds
log: 1.3 seconds

Formulate an LP to help Brady minimize the weekly cost of meeting the demand for processed lumber.

Solution:

Let
G1 be the # of cu ft of grade 1 lumber by purchasing,
G2 be the # of cu ft of grade 2 lumber by purchasing,
LOG be the # of cu ft of the coperation's own lumber.

LP Model :

Min	3G1+7G2	(purchase cost)
	+4(G1+G2+LOG)	(dry cost for lumber)
	+3LOG	(cost for chopping)
	+2.5LOG	(cost for sawmill)
s.t.		
	G1≤40000	(available cu ft of grade 1 per week)
	G2≤60000	(available cu ft of grade 2 per week)
	LOG≤35000	(available cu ft of own lumber per week)
	0.7G1+0.9G2+0.8LOG≥90000	(constraint for demand)
	2G1+0.8G2+1.3LOG≤144000	(available hours for drying)
	G1,G2,LOG≥0	

LINDO output :

```
 MIN     7 G1 + 11 G2 + 9.5 LOG
 SUBJECT TO
        G1 <=   40000
        G2 <=   60000
        LOG<=   35000
        0.7 G1 + 0.9 G2 + 0.8 LOG >=   90000
        2 G1 + 0.8 G2 + 1.3 LOG <=   144000
 END

 : go
   LP OPTIMUM FOUND  AT STEP     4

        OBJECTIVE FUNCTION VALUE

  1)       1033584.94

 VARIABLE        VALUE          REDUCED COST
       G1     40000.000000         0.000000
       G2     55471.703125         0.000000
      LOG    15094.337891          0.000000


 ROW        SLACK OR SURPLUS     DUAL PRICES
   2)            0.000000          0.905661
   3)         4528.298340          0.000000
   4)        19905.662109          0.000000
   5)            0.000000        -12.641510
```

6) 0.000000 0.471698

NO. ITERATIONS= 4

DO RANGE(SENSITIVITY) ANALYSIS?
? y

RANGES IN WHICH THE BASIS IS UNCHANGED

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
G1	7.000000	0.905661	INFINITY
G2	11.000000	0.695652	0.312500
LOG	9.500000	0.277778	0.387097

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	40000.000000	3478.257080	8508.064453
3	60000.000000	INFINITY	4528.298340
4	35000.000000	INFINITY	19905.662109
5	90000.000000	1846.152100	13187.499023
6	144000.000000	11722.221680	2999.997070

Homework # 3

1. For each of the tableaus below, the objective is to be *maximized*, and the variable in the first column is (-z).
   - Is the tableau optimal?
   - If optimal, is the optimum unique?
   - If not optimal, circle every possible pivot location which might be selected by the simplex method.
   - Indicate if an unbounded objective is detected.

(a)

1	-3	0	2	-1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	15
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

(b)

1	-3	0	-2	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	4
0	5	0	6	-2	0	1	2	-5	10

(c)

1	3	0	-2	1	0	0	1	-4	-30
0	-2	0	3	4	1	0	-2	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

(d)

1	2	0	4	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	0	1	4	0	0	0	1	2	4
0	-1	0	6	-2	0	1	2	-5	10

(e)

1	3	0	-2	-1	0	0	1	4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	10
0	5	0	6	-2	0	1	2	-5	0

(f)

1	-3	0	2	1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

2. For each of the tableaus below, the objective is to be *minimized*, and the variable in the first column is (-z).
    - Is the tableau optimal?
    - If optimal, is the optimum unique?
    - If not optimal, circle every possible pivot location which might be selected by the simplex method.
    - Indicate if an unbounded objective is detected.

(g)

1	-3	0	2	-1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	15
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

(h)

1	-3	0	-2	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	4
0	5	0	6	-2	0	1	2	-5	10

(i)

1	3	0	-2	1	0	0	1	-4	-30
0	-2	0	3	4	1	0	-2	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

(j)

1	2	0	4	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	0	1	4	0	0	0	1	2	4
0	-1	0	6	-2	0	1	2	-5	10

(k)

1	3	0	-2	-1	0	0	1	4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	10
0	5	0	6	-2	0	1	2	-5	0

(l)

1	-3	0	2	1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

3. Consider the following LP in its initial tableau:

1	1	4	5	0	0	0	0
0	1	2	3	1	0	0	14
0	3	1	2	0	1	0	8
0	2	3	1	0	0	1	15

Without pivoting in the tableau as in the ordinary simplex method, find the missing values in the following tableau which occurs after several iterations, as in the revised simplex metod:

1	4	0	0	-3	2	0	?
?	-7	1	?	2	-3	0	4
?	5	0	?	-1	2	0	?
?	18	0	?	-5	7	1	?

(Explain briefly how you determine the missing values.)

4. Consider the following problem:

"A manufacturer produces two types of plastic cladding. These have the trade names Ankalor and Beslite. One yard of Ankalor requires 8 lb of polyamine, 2.5 lb of diurethane and 2 lb of monomer. A yard of Beslite needs 10 lb of polyamine, 1 lb of diurethane, and 4 lb of monomer. The company has in stock 80,000 lb of polyamine, 20,000 lb of diurethane, and 30,000 lb of monomer. Both plastics can be produced by alternate parameter settings of the production plant, which is able to produce sheeting at the rate of 12 yards per hour. A total of 750 production plant hours are available for the next planning period. The contribution to profit on Ankalor is \$10/yard and \$20/yard on Beslite.

The company has a contract to deliver at least 3,000 yards of Ankalor. What production plan should be implemented in order to maximize the contribution to the firm's profit from this product division."

Decision variables:

A = Number of yards of Ankalor produced

B = Number of yards of Beslite produced

LP model:

$$
\begin{array}{lll}
\text{Maximize } 10A + 20B & & \\
\text{subject to} & 8A + 10B & \leq 80{,}000 \text{ (Polyamine)} \\
& 2.5A + 1B & \leq 20{,}000 \text{ (Diurethane)} \\
& 2A + 4B & \leq 30{,}000 \text{ (Monomer)} \\
& A + B & \leq 9{,}000 \text{ (Plant capacity)} \\
& A & \geq 3{,}000 \text{ (Contract)} \\
& A \geq 0,\ B \geq 0 &
\end{array}
$$

The LINDO solution is:

```
? GO
LP OPTIMUM FOUND AT STEP 2
                         OBJECTIVE FUNCTION VALUE
1)   142000.000
       VARIABLE        VALUE     REDUCED COST
           A          3000.000      0.000
           B          5600.000      0.000

       ROW        SLACK OR SURPLUS  DUAL PRICES
        2)               0.000        2.000
        3)            6900.000        0.000
        4)            1600.000        0.000
```

```
        5)               400.000         0.000
        6)                 0.000        -6.000
NO. ITERATIONS =2
RANGES IN WHCH THE BASIS IS UNCHANGED
                          OBJ COEFFICIENT RANGES
     VARIABLE    CURRENT    ALLOWABLE   ALLOWABLE
                   COEF      INCREASE   DECREASE
        A         10.000       6.000    INFINITY
        B         20.000     INFINITY     7.500

                          RIGHTHAND SIDE RANGES
     ROW          CURRENT       ALLOWABLE    ALLOWABLE
                    RHS         INCREASE     DECREASE
       2         80000.000      4000.000     56000.000
       3         20000.000      INFINITY      6900.000
       4         30000.000      INFINITY      1600.000
       5          9000.000      INFINITY       400.000
       6          3000.000      2000.000      1333.333
```

a. How many yards of each product should be manufactured? _____________
b. How much of each raw material will be used, and how much will be unused? _____________
c. Suppose that the company can purchase 2000 pounds of additional polyamine for $1.50 per pound. Should they make the purchase? _____________ Why or why not? _____________
d. If the profit contribution from Ankalor were to decrease to $8/yard, will the optimal solution change? _____________ Why or why not? _____________
e. If the profit contribution from Beslite were to increase to $25/yard, will the optimal solution change? _____________ Why or why not? _____________
f. If the company could deliver less than the contracted amount of Ankalor by forfeiting a penalty of $8/yard, should they do so? ____________ If so, how much should they deliver? _____________

◊◊◊◊◊◊◊◊ Homework #3 Solutions ◊◊◊◊◊◊◊◊

1. For each of the tableaus below, the objective is to be *maximized*, and the variable in the first column is (-z).
   - Is the tableau optimal?
   - If optimal, is the optimum unique?
   - If not optimal, circle every possible pivot location which might be selected by the simplex method.
   - Indicate if an unbounded objective is detected.

(a)

1	-3	0	2	-1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	15
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

(b)

1	-3	0	-2	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	4
0	5	0	6	-2	0	1	2	-5	10

Optimal but not unique. Since the variable of column 8 can enter to the basic without changing the objective value.

(c)

1	3	0	-2	1	0	0	1	-4	-30
0	-2	0	3	4	1	0	-2	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

(d)

1	2	0	4	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	0	1	4	0	0	0	1	2	4
0	-1	0	6	-2	0	1	2	-5	10

Not optimal.
Unbounded.

(e)

1	3	0	-2	-1	0	0	1	4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	10
0	5	0	6	-2	0	1	2	-5	0

Not optimal.

(f)

1	-3	0	2	1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

2. For each of the tableaus below, the objective is to be *minimized*, and the variable in the first column is (-z).
   - Is the tableau optimal?
   - If optimal, is the optimum unique?
   - If not optimal, circle every possible pivot location which might be selected by the simplex method.
   - Indicate if an unbounded objective is detected.

(g)

1	-3	0	2	-1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	15
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

(h)

1	-3	0	-2	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	4
0	5	0	6	-2	0	1	2	-5	10

Not optimal.

(i)

1	3	0	-2	1	0	0	1	-4	-30
0	-2	0	3	4	1	0	-2	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

(j)

1	2	0	4	-1	0	0	0	-4	-30
0	-2	0	3	4	1	0	-2	3	5
0	0	1	4	0	0	0	1	2	4
0	-1	0	6	-2	0	1	2	-5	10

Not optimal.

(k)

1	3	0	-2	-1	0	0	1	4	-30
0	-2	0	3	4	1	0	-2	3	5
0	2	1	4	0	0	0	1	2	10
0	5	0	6	-2	0	1	2	-5	0

Not optimal.

(l)

1	-3	0	2	1	0	0	1	4	-30
0	0	0	-3	4	1	0	5	3	0
0	2	1	4	0	0	0	-1	2	10
0	5	0	6	-2	0	1	2	0	20

Not optimal.

3. Consider the following LP in its initial tableau:

1	1	4	5	0	0	0	0
0	1	2	3	1	0	0	14
0	3	1	2	0	1	0	8
0	2	3	1	0	0	1	15

Without pivoting in the tableau as in the ordinary simplex method, find the missing values in the following tableau which occurs after several iterations, as in the revised simplex method:

1	4	0	0	-3	2	0	?
?	-7	1	?	2	-3	0	4
?	5	0	?	-1	2	0	?
?	18	0	?	-5	7	1	?

(Explain briefly how you determine the missing values.)

Solution:

B={2,3,6}, $A^B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ $(A^B)^{-1} = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ -5 & 7 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

b=[14,8 15]^T

Thus, $(A^B)^{-1}A = \begin{bmatrix} -7 & 1 & 0 & 2 & -3 & 0 \\ 5 & 0 & 1 & -1 & 2 & 0 \\ 18 & 0 & 0 & -5 & 7 & 1 \end{bmatrix}$,

$(A^B)^{-1}b = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$, and

$$C_B(A^B)^{-1}b = \begin{bmatrix}-4 & -5 & 0\end{bmatrix}\begin{bmatrix}2 & -3 & 0\\ -1 & 2 & 0\\ -5 & 7 & 1\end{bmatrix}\begin{bmatrix}14\\ 8\\ 15\end{bmatrix} = -26$$

Therefore the table becomes:

1	4	0	0	-3	2	0	-26
0	-7	1	0	2	-3	0	4
0	5	0	1	-1	2	0	2
0	18	0	0	5	7	1	1

4. Consider the following problem:
"A manufacturer produces two types of plastic cladding. These have the trade names Ankalor and Beslite. One yard of Ankalor requires 8 lb of polyamine, 2.5 lb of diurethane and 2 lb of monomer. A yard of Beslite needs 10 lb of polyamine, 1 lb of diurethane, and 4 lb of monomer. The company has in stock 80,000 lb of polyamine, 20,000 lb of diurethane, and 30,000 lb of monomer. Both plastics can be produced by alternate parameter settings of the production plant, which is able to produce sheeting at the rate of 12 yards per hour. A total of 750 production plant hours are available for the next planning period. The contribution to profit on Ankalor is $10/yard and $20/yard on Beslite.
The company has a contract to deliver at least 3,000 yards of Ankalor. What production plan should be implemented in order to maximize the contribution to the firm's profit from this product division."

Decision variables:
A = Number of yards of Ankalor produced
B = Number of yards of Beslite produced

LP model:

Maximize 10 A + 20 B

subject to

8 A + 10 B	≤ 80,000 (Polyamine)
2.5 A + 1 B	≤ 20,000 (Diurethane)
2 A + 4 B	≤ 30,000 (Monomer)
A + B	≤ 9,000 (Plant capacity)
A	≥ 3,000 (Contract)
A ≥ 0, B ≥ 0	

The LINDO solution is:

```
? GO
LP OPTIMUM FOUND AT STEP 2
                              OBJECTIVE FUNCTION VALUE
1)    142000.000
       VARIABLE        VALUE     REDUCED COST
           A         3000.000       0.000
           B         5600.000       0.000

       ROW        SLACK OR SURPLUS  DUAL PRICES
        2)               0.000         2.000
        3)            6900.000         0.000
        4)            1600.000         0.000
        5)             400.000         0.000
        6)               0.000        -6.000
NO. ITERATIONS =2
RANGES IN WHCH THE BASIS IS UNCHANGED
                               OBJ COEFFICIENT RANGES
     VARIABLE    CURRENT    ALLOWABLE   ALLOWABLE
                  COEF      INCREASE    DECREASE
        A        10.000       6.000     INFINITY
```

B	20.000	INFINITY	7.500

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	80000.000	4000.000	56000.000
3	20000.000	INFINITY	6900.000
4	30000.000	INFINITY	1600.000
5	9000.000	INFINITY	400.000
6	3000.000	2000.000	1333.333

a. How many yards of each product should be manufactured?
Produce Ankalor 3000 yards and Beslite 5600 yards.

b. How much of each raw material will be used, and how much will be unused?
Polyamine will be used 80000 lb, diurethane will be used 13100 lb. and 6900 lb. unused, and monomer will used 28400 lb. and 1600 lb. unused.

c. Suppose that the company can purchase 2000 pounds of additional polyamine for $1.50 per pound. Should they make the purchase? **Yes** Why or why not? **Since (1) the allowable increase for polyamine is 4000, and (2) $1.5<$2(dual price).**

d. If the profit contribution from Ankalor were to decrease to $8/yard, will the optimal solution change? **NO.** Why or why not? **Since the allowable decrease of objective coefficient for Ankalor is infinity.**

e. If the profit contribution from Beslite were to increase to $25/yard, will the optimal solution change? **NO.** Why or why not? **Because the allowable increase of objective coefficient for Beslite is infinity.**

f. If the company could deliver less than the contracted amount of Ankalor by forfeiting a penalty of $8/yard, should they do so? **NO** If so, how much should they deliver? **Not deliver. Because $8(penalty)>$6(see dual price)**

Homework # 4

1. Consider the following LP:

$$\text{Max } 3X_1 - 11X_2$$
$$\text{subject to } 2X_1 + 13X_2 \geq 5$$
$$3X_1 - 4X_2 \leq 16$$
$$X_1 \geq 0,\ X_2 \geq 0$$

a. Plot the feasible region of the LP.
b. Identify all of the basic solutions of the problem. Which are feasible and which are infeasible?
c. Write the dual of this LP.
d. Plot the feasible region of the dual LP.
e. Identify all of the basic solutions of the dual LP. Which are feasible and which are infeasible?
f. Check that the optimal solutions of the primal and dual LP satisfy the complementary slackness conditions.

2. Consider the following four primal LPs:

P_A: Max $2X_1 + X_2$	P_B: Maximize $2X_1 + X_2$
subject to $X_1 + X_2 \geq 4$	subject to $X_1 - X_2 \geq 4$
$X_1 - X_2 \leq 2$	$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$	$X_1 \geq 0, X_2 \geq 0$

P_C: Max $2X_1 + X_2$	P_D: Maximize $2X_1 + X_2$
subject to $X_1 + X_2 \leq 4$	subject to $X_1 - X_2 \leq 4$
$X_1 - X_2 \leq 2$	$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$	$X_1 \geq 0, X_2 \geq 0$

a. Write the dual LP for each of P_A, P_B, P_C, and P_D. (Label them D_A, etc.)
b. Sketch the feasible region (if it exists) for each LP.
c. Classify each primal-dual pair according to the categories:
   i. Primal & Dual both feasible
   ii. Primal infeasible, Dual feasible & unbounded
   iii. Primal feasible and unbounded, Dual infeasible
   iv. Both Primal & Dual infeasible.

3. Find the dual LP for the LP:

$$\begin{aligned} \text{Minimize } & 3X_1 + 2X_2 - 3X_3 + 4X_4 \\ \text{subject to } & X_1 - 2X_2 + 3X_3 + 4X_4 \leq 3 \\ & X_2 + 3X_3 + 4X_4 \geq -5 \\ & 2X_1 - 3X_2 - 7X_3 - 4X_4 = 2 \\ & X_1 \geq 0,\ X_4 \leq 0 \end{aligned}$$

4. The following questions refer to the LP for "Production Planning for a Tire Manufacturer" in your notes:
   a. A Wheeling machine breaks down in June, and 8 hours of production time are lost before the machine is repaired. How much will this increase the objective function? How should the production plan be modified, according to the "substitution rates"?
   b. How many Nylon tires will be in storage at the end of July? If the storage cost for Nylon tires in July were to drop from 10¢ per tire to 5¢ per tire, how many tires should be put into storage at the end of July?
   c. In the optimal solution, no fiberglass tires are manufactured on the Regal machine in June. Suppose that the production manager mistakenly has 10 such tires manufactured on the Regal machine. How much will this error increase the cost function? How should the production plan be modified to best compensate for this error?

◊◊◊◊◊◊◊◊ Homework #4 Solutions ◊◊◊◊◊◊◊◊

1. Consider the following LP:

$$\begin{aligned} \text{Max } & 3X_1 - 11X_2 \\ \text{subject to } & 2X_1 + 13X_2 \geq 5 \\ & 3X_1 - 4X_2 \leq 16 \\ & X_1 \geq 0,\ X_2 \geq 0 \end{aligned}$$

a. Plot the feasible region of the LP.
b. Identify all of the basic solutions of the problem. Which are feasible and which are infeasible?
c. Write the dual of this LP.
d. Plot the feasible region of the dual LP.

e. Identify all of the basic solutions of the dual LP. Which are feasible and which are infeasible?
f. Check that the optimal solutions of the primal and dual LP satisfy the complementary slackness conditions.

Solution: (a).

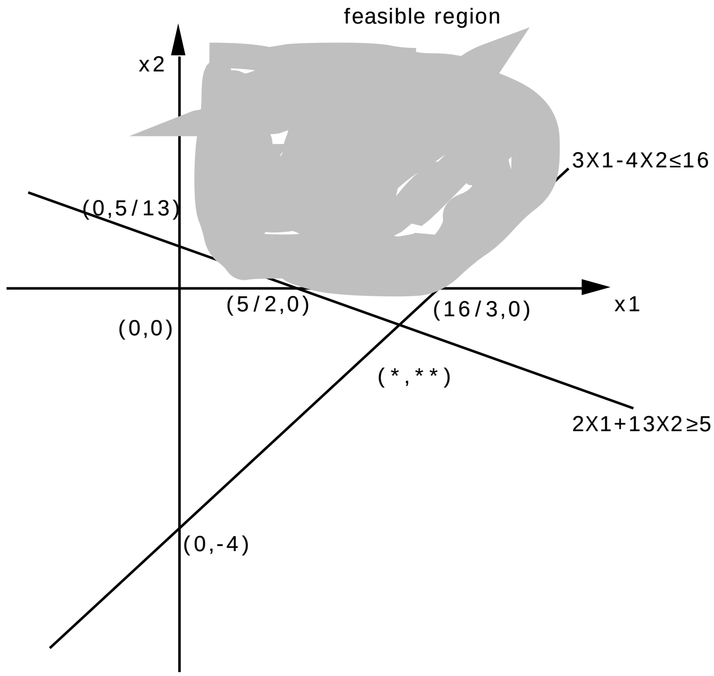


Fig 1.Feasible region for the problem.

(b). There are six basic variables (see Fig 1). (0,5/13), (5/2,0), and (16/3,0) are feasible, the other three ((0,-4), (0,0) and (*,**) above) are infeasible.

(c). (Dual) Min $5Y1+16Y2$

s.t.. $2Y1+3Y2 \geq 3$

$13Y1-4Y2 \geq -11$

$Y1 \leq 0,\ Y2 \geq 0$

(d).

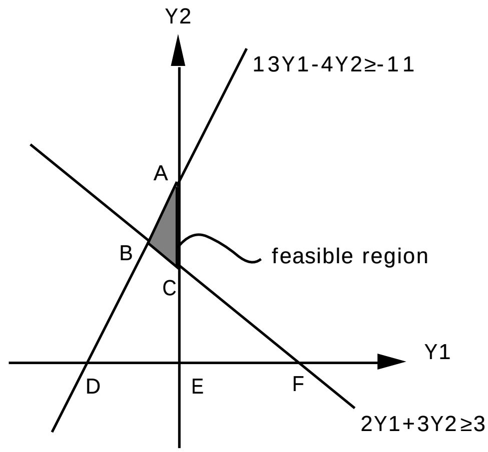


Fig 2 Feasible region for dual system.

There are six basic solutions for dual, points A, B,..., and F in Fig 2. But only three are feasible, i.e. points A, B, and C.

(e). The optimal solution for primal problem is (X1*,X2*)=(16/3,0), and the optimal solution for dual is (Y1*,Y2*)=(0,1), i.e., point C in Fig. 2.

(1) X1*>0. Constraint 1 in dual is tight, i.e., 2Y1+3Y2=2(0)+3(1)=3.

(2) Y2*>0. Constraint 2 in primal is tight, i.e., 3X1-4X2-16=3(16/3)-4(0)-16=0.

(3) Constraint 2 in dual is slack, i.e., 13Y1-4Y2=13(0)-4(1)=-4>-11. We have X2*=0.

(4) Constraint 1 in primal is slack, i.e., 2X1+13X2-5=2(16/3)+13(0)-5>0. We have Y1*=0.

2. Consider the following four primal LPs:

P_A: Max $2X_1 + X_2$
subject to $X_1 + X_2 \geq 4$
$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$

P_B: Maximize $2X_1 + X_2$
subject to $X_1 - X_2 \geq 4$
$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$

P_C: Max $2X_1 + X_2$
subject to $X_1 + X_2 \leq 4$
$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$

P_D: Maximize $2X_1 + X_2$
subject to $X_1 - X_2 \leq 4$
$X_1 - X_2 \leq 2$
$X_1 \geq 0, X_2 \geq 0$

a. Write the dual LP for each of P_A, P_B, P_C, and P_D. (Label them D_A, etc.)
b. Sketch the feasible region (if it exists) for each LP.
c. Classify each primal-dual pair according to the categories:
   i. Primal & Dual both feasible
   ii. Primal infeasible, Dual feasible & unbounded
   iii. Primal feasible and unbounded, Dual infeasible
   iv. Both Primal & Dual infeasible.

Solution:

(a). (D_A) Min 4Y1+2Y2
s.t. Y1+Y2≥2
Y1-Y2≥1
Y1≤0, Y2≥0.

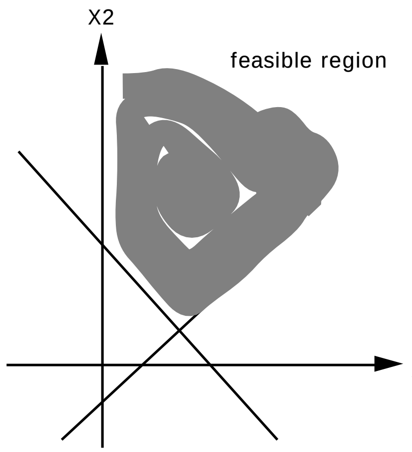
(D_B) Min 4Y1+2Y2
s.t. Y1+Y2≥2
-Y1-Y2≥1
Y1≤0, Y2≥0.

(D_C) Min $4Y1+2Y2$
s.t. $Y1+Y2\geq 2$
$Y1-Y2\geq 1$
$Y1\geq 0,\ Y2\geq 0.$

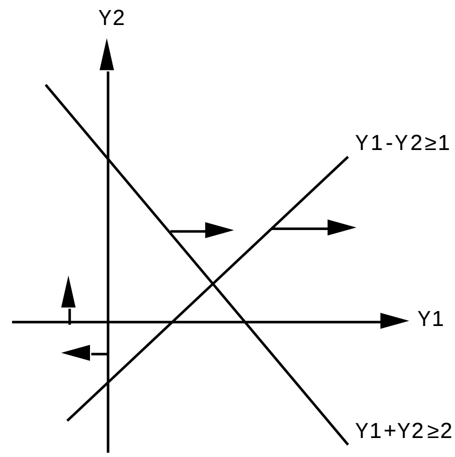
(D_D) Min $4Y1+2Y2$
s.t. $Y1+Y2\geq 2$
$-Y1-Y2\geq 1$
$Y1\geq 0,\ Y2\geq 0.$

(b & c) Feasible regions for primal and dual for each problem.

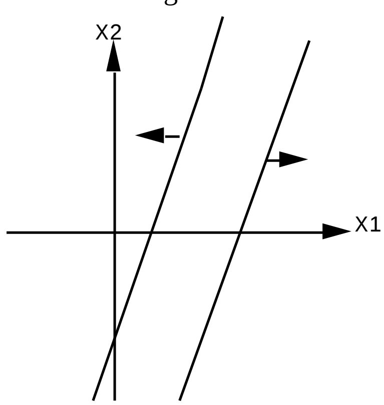
(P_A) Primal is feasble.



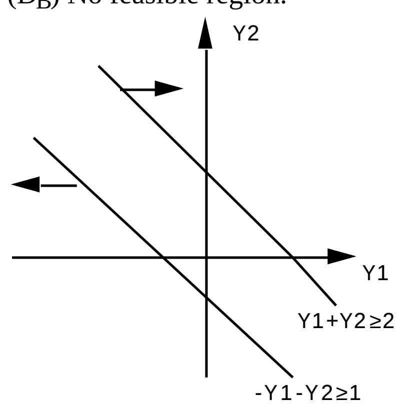
(D_A) No feasible region.



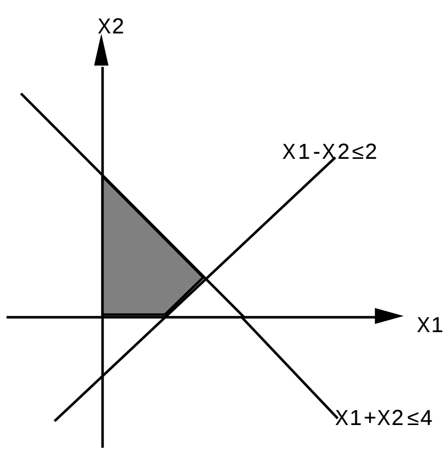
(P_B) No feasible region.



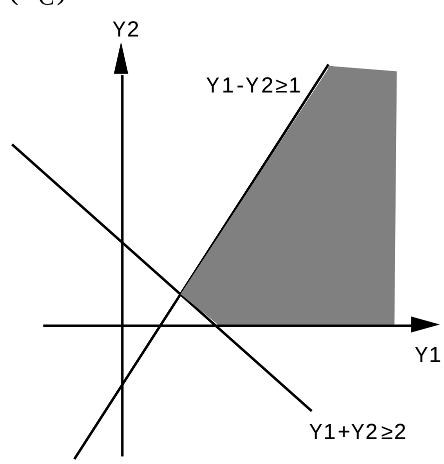
(D_B) No feasible region.



(P_C) Primal is feasible.

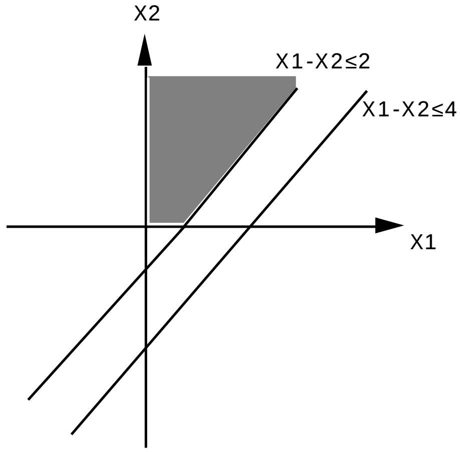


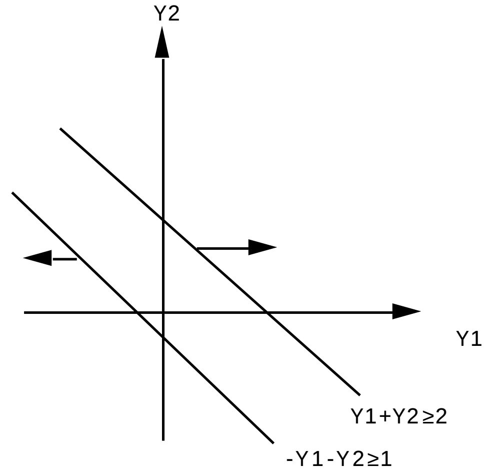
(D_C) Dual is feasible.



(P_D) Primal is feasible.

(D_D) No feasible region.





3. Find the dual LP for the LP:

$$\text{Minimize } 3X_1 + 2X_2 - 3X_3 + 4X_4$$
$$\text{subject to } X_1 - 2X_2 + 3X_3 + 4X_4 \le 3$$
$$X_2 + 3X_3 + 4X_4 \ge -5$$
$$2X_1 - 3X_2 - 7X_3 - 4X_4 = 2$$
$$X_1 \ge 0,\ X_4 \le 0$$

Solution: Max 3Y1-5Y2+2Y3
s.t. Y1+2Y3≤3
-2Y1+Y2-3Y3=2
3Y1+3Y2-7Y3=-3
4Y1+4Y2-4Y3≥4
Y1≤0, Y2≥0, Y3 no restriction in sign.

4. The following questions refer to the LP for "Production Planning for a Tire Manufacturer" in your notes:
   a. A Wheeling machine breaks down in June, and 8 hours of production time are lost before the machine is repaired. How much will this increase the objective function? How should the production plan be modified, according to the "substitution rates"?
   b. How many Nylon tires will be in storage at the end of July? If the storage cost for Nylon tires in July were to drop from 10¢ per tire to 5¢ per tire, how many tires should be put into storage at the end of July?
   c. In the optimal solution, no fiberglass tires are manufactured on the Regal machine in June. Suppose that the production manager mistakenly has 10 such tires manufactured on the Regal machine. How much will this error increase the cost function? How should the production plan be modified to best compensate for this error?

Solution:
(a).From the tableau, one can find the relation between SLK2 and RHS as :

$$\begin{bmatrix} ART \\ WN1 \\ SLK3 \\ WG2 \\ RN2 \\ WN3 \\ SLK7 \\ RN1 \\ WG1 \\ IN1 \\ IG1 \\ RN3 \\ WG3 \end{bmatrix} = \begin{bmatrix} -191.733 \\ 18.667 \\ 2.787 \\ 25.000 \\ 25.000 \\ 26.667 \\ 2.467 \\ 76.333 \\ 35.000 \\ 55.000 \\ 25.000 \\ 3.333 \\ 50.000 \end{bmatrix} - \begin{bmatrix} 0.333 \\ 6.667 \\ 1.067 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ -6.667 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} SLK2$$

SLK2 will increase 8 hours if we lose 8 hours capacity in Wheeling machine in June. Hence, the objective value will increase 8(.333)=2.667, WN1 will decrease 8(6.667), RN1 will increase 8(6.667), and SLK3 will decrease 8(1.067).

(b). (1). From the tableau we can see that there will be no tires stored at the end of July.

(2). Since allowable decrease of the objective cofficient range for IN2 is \$0.2, the optimal solution will not change, i.e., no tire will be stored at the end of July when cost drops from 10 cents per tire to 5 cents per tire.

(c).

$$\begin{bmatrix} ART \\ WN1 \\ SLK3 \\ WG2 \\ RN2 \\ WN3 \\ SLK7 \\ RN1 \\ WG1 \\ IN1 \\ IG1 \\ RN3 \\ WG3 \end{bmatrix} = \begin{bmatrix} -191.733 \\ 18.667 \\ 2.787 \\ 25.000 \\ 25.000 \\ 26.667 \\ 2.467 \\ 76.333 \\ 35.000 \\ 55.000 \\ 25.000 \\ 3.333 \\ 50.000 \end{bmatrix} - \begin{bmatrix} 0.060 \\ -0.800 \\ 0.012 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.800 \\ 1.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} RG1$$

It is obvious that increasing 10 units of RG1 will increase 10(0.060)=0.6 in objective value.
At this time, WN1 will increase 10(0.800)=8 units, RN1 will decrease 10(0.8)=8 units, and WG1 will decrease 10(1)=10 units.

Homework # 5

1. Consider the transportation problem having the following cost and requirements table:

<u>Destination</u>

		1	2	3	4	Supply
Source	1	3	7	6	4	5
	2	2	4	3	2	2
	3	4	3	8	5	3
Demand		3	4	1	1	

a. Write the regular LP formulation of this problem.
b. How many basic variables will the LP have (in addition to -z, the objective value)?
c. Use each of the following methods to obtain an initial basic feasible solution:
   i) Northwest Corner Method
   ii) Least-Cost Method
   iii) Vogel's Approximation Method
d. For each of the basic feasible solutions you found in (c), apply the transportation simplex method to find the optimal solution.
e. Compare the number of iterations required starting at each solution obtained in (c).
f. What are the values of the dual variables (simplex multipliers) for the optimal solution?

2. A manufacturer of electronic calculators is working on setting up his production plans for the next six months. One product is particularly puzzling to him. The orders on hand for the coming season are:

Month	Orders
January	100
February	150
March	200
April	100
May	200
June	150

The product will be discontinued after satisfying the June demand. Therefore, there is no need to keep any inventory after June. The production cost, using regular manpower, is $10 per unit. Producing the calculator on overtime costs an additional $2 per unit. The inventory-carrying cost is $0.50 per unit per month. The regular shift production is limited to 100 units per month, and overtime production is limited to an additional 75 units per month. In every month except June, backorders are allowed, i.e. demand can be shipped one month late, but there is an estimated $3 per unit loss, including the loss of the customer's "good will". (Orders cannot be satisfied more than 1 month late.) Set up, but do not solve, the transportation tableau which could be used to find the optimal production schedule.

◊◊◊◊◊◊◊◊ Homework #5 Solutions ◊◊◊◊◊◊◊◊

1. Consider the transportation problem having the following cost and requirements table:

		Destination 1	2	3	4	Supply
Supply	1	3	7	6	4	5
	2	2	4	3	2	2
	3	4	3	8	5	3
Demand		3	4	1	1	

a. Write the regular LP formulation of this problem.
b. How many basic variables will the LP have (in addition to -z, the objective value)?
c. Use each of the following methods to obtain an initial basic feasible solution:
   i) Northwest Corner Method
   ii) Least-Cost Method
   iii) Vogel's Approximation Method
d. For each of the basic feasible solutions you found in (c), apply the transportation simplex method to find the optimal solution.

e. Compare the number of iterations required starting at each solution obtained in (c).
f. What are the values of the dual variables (simplex multipliers) for the optimal solution?

Solutions:
(a) Since the total supply is greater than total demand, we add a dummy column with demand (5+2+3)-(3+4+1+1)=1.

Destination

Supply	1	2	3	4	dummy 5	
1	$\boxed{3}$	$\boxed{7}$	$\boxed{6}$	$\boxed{4}$	$\boxed{0}$	5
2	$\boxed{2}$	$\boxed{4}$	$\boxed{3}$	$\boxed{2}$	$\boxed{0}$	2
3	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$	$\boxed{5}$	$\boxed{0}$	3
	3	4	1	1	1	

Let Xij be the number of units transported from supply i to destination j.

Min 3X11+7X12+6X13+4X14+2X12+4X22+3X23+2X24+4X31+3X32+8X33+5X34

s.t.

X11+X12+X13+X14+X15=1
X21+X22+X23+X24+X25=1
X31+X32+X33+X34+X35=1
X11+X21+X31=3
X12+X22+X32=4
X13+X23+X33=1
X14+X24+X34=1
X15+X25+X35=1
All Xij≥0.

(b). m+n-1=3+5-1=7, i.e., seven basic variables.
(c). i) Northwest Corner Method:

Destination

Supply	1	2	3	4	dummy 5	
1	3 $\boxed{3}$	2 $\boxed{7}$	$\boxed{6}$	$\boxed{4}$	$\boxed{0}$	5
2	$\boxed{2}$	2 $\boxed{4}$	$\boxed{3}$	$\boxed{2}$	0 $\boxed{0}$	2
3	$\boxed{4}$	$\boxed{3}$	1 $\boxed{8}$	1 $\boxed{5}$	1 $\boxed{0}$	3
	3	4	1	1	1	

X11=3, X12=2, X22=2, X33=1, X34=1, X35=1, X25=0.
Total cost=44.

ii) Least-Cost Method:

Destination

Supply	1	2	3	4	5 (dummy)	
1	1 (3)	1 (7)	1 (6)	1 (4)	1 (0)	5
2	2 (2)	(4)	(3)	(2)	(0)	2
3	(4)	3 (3)	(8)	(5)	(0)	3
	3	4	1	1	1	

X11=X12=X13=X14=X15=1, X21=2, X32=3.
Total cost=33.

iii) Vogel's Approximation Method:

Destination

Supply	1	2	3	4	5 (dummy)	
1	3 (3)	2 (7)	(6)	(4)	(0)	5
2	(2)	(4)	1 (3)	1 (2)	0 (0)	2
3	(4)	2 (3)	(8)	(5)	1 (0)	3
	3	4	1	1	1	

X11=3, X12=2, X23=X24=1, X32=2, X35=1, X25=0.
Total cost=34.

(d).i) Northwest Corner Method:

u\v	3	7	11	8	3
0	3 (3)	2 (7)	(6)	(4)	(0)
-3	(2)	2 (4)	θ (3)	(2)	0−θ (0)
-3	(4)	(3)	1−θ (8)	1 (5)	1+θ (0)

reduced cost:
c13=-5
c14=-4
c15=-3
c21=2
c23=-5 <<----
c24=-3
c31=4
c32=-1

X25 out of basic.

u\v	3	7	6	3	-2
0	3 (3)	2 (7)	(6)	(4)	(0)
-3	(2)	2−θ (4)	0+θ (3)	(2)	(0)
2	(4)	θ (3)	1−θ (8)	1 (5)	1 (0)

reduced cost:
c13=0
c14=1
c15=2
c21=2
c24=2
c25=5
c31=-1
c32=-6<<----

X33 out of basic.

u\v	3	7	6	9	4
0	3 $\boxed{3}$	2−θ $\boxed{7}$	$\boxed{6}$	θ $\boxed{4}$	$\boxed{0}$
-3	$\boxed{2}$	1 $\boxed{4}$	1 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
-4	$\boxed{4}$	1+θ $\boxed{3}$	$\boxed{8}$	1−θ $\boxed{5}$	1 $\boxed{0}$

reduced cost:
c13=0
c14=-5<<----
c15=-4
c21=2
c24=-4
c25=-1
c31=5
c33=6

X34 out of basis.

u\v	3	7	6	4	4
0	3 $\boxed{3}$	1−θ $\boxed{7}$	$\boxed{6}$	1 $\boxed{4}$	θ $\boxed{0}$
-3	$\boxed{2}$	1 $\boxed{4}$	1 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
-4	$\boxed{4}$	2+θ $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	1−θ $\boxed{0}$

reduced cost:
c13=0
c15=-4<<----
c21=2
c24=1
c25=-1
c31=5
c33=6
c34=5

X35 out of basis.

u\v	3	7	6	4	0
0	3 $\boxed{3}$	0 $\boxed{7}$	$\boxed{6}$	1 $\boxed{4}$	1 $\boxed{0}$
-3	$\boxed{2}$	1 $\boxed{4}$	1 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
-4	$\boxed{4}$	3 $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	$\boxed{0}$

reduced cost:
c13=0
c21>0
c24>0
c25>0
c31>0
c33>0
c34>0
c35>0

Since all reduced costs for non-basic variables are nonnegative, it is an optimal solution.
Total cost=29.

ii) Least-Cost Method:

	3	7	6	4	0
0	1+θ $\boxed{3}$	1 $\boxed{7}$	1−θ $\boxed{6}$	1 $\boxed{4}$	1 $\boxed{0}$
−1	2−θ $\boxed{2}$	$\boxed{4}$	θ $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
−4	$\boxed{4}$	3 $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	$\boxed{0}$

reduced cost:
c22=-2
c23=-2<<----
c24=-1
c25=1
c31=5
c33=6
c34=5
c35=1

X13 out of basis.

	3	7	4	4	0
0	2+θ $\boxed{3}$	1−θ $\boxed{7}$	$\boxed{6}$	1 $\boxed{4}$	1 $\boxed{0}$
−1	1−θ $\boxed{2}$	θ $\boxed{4}$	1 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
−4	$\boxed{4}$	3 $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	$\boxed{0}$

reduced cost:
c13=2
c22=-2<<----
c24=-1
c25=1
c31=5
c33=8
c34=5
c35=4

X12 out of basis.

	3	5	4	4	0
0	3+θ 3	 7	 6	1−θ 4	1 0
−1	0−θ 2	1 4	1 3	θ 2	 0
−2	 4	3 3	 8	 5	 0

reduced cost:
c12=2
c13=2
c24=-1<<----
c25=1
c31=3
c33=6
c34=3
c35=2

X21 out of basis.

	3	6	5	4	0
0	3 3	 7	 6	1 4	1 0
−2	 2	1 4	1 3	0 2	 0
−3	 4	3 3	 8	 5	 0

reduced cost:
c12>0
c13>0
c21>0
c25>0
c31>0
c33>0
c34>0
c35>0

Since all reduced costs for non-basic variables are nonnegative, it is an optimal solution.
Total cost=29.

iii). Vogel's Approximation Method

	3	7	7	6	4
0	3 3	2−θ 7	 6	 4	θ 0
−4	 2	 4	1 3	1 2	0 0
−4	 4	2+θ 3	 8	 5	1−θ 0

reduced cost:
c13=-1
c14=-2
c15=-4<<----
c21=3
c22=1
c31=5
c33=5
c34=3

X35 out of basis.

	3	7	3	2	0
0	3 3	1−θ 7	 6	 4	1+θ 0
0	 2	θ 4	1 3	1 2	0−θ 0
−4	 4	3 3	 8	 5	 0

reduced cost:
c13=3
c14=2
c21=-1
c22=-3<<----
c31=5
c33=9
c34=7
c35=4

X25 out of basis.

	3	7	6	5	0
0	3 3	1−θ 7	 6	θ 4	1 0
−3	 2	0+θ 4	1 3	1−θ 2	 0
−4	 4	3 3	 8	 5	 0

reduced cost:
c13=0
c14=-1<<----
c21=2
c25=3
c31=5
c33=6
c34=4
c35=4

X24 out of basis.

	3	7	6	4	0
0	3 $\boxed{3}$	0 $\boxed{7}$	$\boxed{6}$	1 $\boxed{4}$	1 $\boxed{0}$
−3	$\boxed{2}$	1 $\boxed{4}$	1 $\boxed{3}$	$\boxed{2}$	$\boxed{0}$
−4	$\boxed{4}$	3 $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	$\boxed{0}$

reduced cost:
c13=0
c21>0
c24>0
c25>0
c31>0
c33>0
c34>0
c35>0

Since all reduced costs for non-basic variables are nonnegative, it is an optimal solution.
Total cost=29.

(e). Northwest Corner Method------------ 4 iterations.
Least-Cost Method--------------------- 3 iterations.
Vogel's Approximation Method------ 3 iterations.

(f) The values of dual variables are shown on the above optimal matrix (u and v).

2. A manufacturer of electronic calculators is working on setting up his production plans for the next six months. One product is particularly puzzling to him. The orders on hand for the coming season are:

Month	Orders
January	100
February	150
March	200
April	100
May	200
June	150

The product will be discontinued after satisfying the June demand. Therefore, there is no need to keep any inventory after June. The production cost, using regular manpower, is $10 per unit. Producing the calculator on overtime costs an additional $2 per unit. The inventory-carrying cost is $0.50 per unit per month. The regular shift production is limited to 100 units per month, and overtime production is limited to an additional 75 units per month. Backorders are allowed, in that demand can be shipped one month late, but there is an estimated $3 per unit loss, including the loss of the customer's "good will". Set up but do not solve the transportation tableau which could be used to find the optimal production schedule.

Solution:

Since the total supply is greater than the total demand, we add a dummy column with capacity=150 (total supply-total demand=150). The cost for the dummy column is zero.

		JAN	FEB	MAR	APRIL	MAY	JUNE	Dummy		Supply
JAN	RT	10	10.5	11	11.5	12	12.5	0		100
	OT	12	12.5	13	13.5	14	14.5	0		75
FEB	RT	13	10	10.5	11	11.5	12	0		100
	OT	15	12	12.5	13	13.5	14	0		75
MAR	RT	∞	13	10	10.5	11	11.5	0		100
	OT	∞	15	12	12.5	13	13.5	0		75
APRIL	RT	∞	∞	13	10	10.5	11	0		100
	OT	∞	∞	15	12	12.5	13	0		75
MAY	RT	∞	∞	∞	13	10	10.5	0		100
	OT	∞	∞	∞	15	12	12.5	0		75
JUNE	RT	∞	∞	∞	∞	13	10	0		100
	OT	∞	∞	∞	∞	15	12	0		75
Demand		100	150	200	100	200	150	150	Total	1050

Note: RT means regular time, OT means overtime.

Homework # 6

1. Consider the assignment problem having the following cost matrix:

\	1	2	3	4
A:	4	1	0	1
B:	1	3	4	0
C:	3	2	1	3
D:	2	2	3	0

   a. Consider this as a transportation problem. What should be the supplies and demands?
   b. Use the "Least Cost Rule" or Vogel's Approximation Method to get an initial basic feasible solution. Be sure to indicate which variables are basic. (This may require some arbitrary choices.)
   c. Starting with the solution indicated in (b), apply the transportation simplex method to find the optimal solution. How many iterations, i.e., changes of basis, were required? In how many iterations was the objective function improved?
   d. Apply the Hungarian method to the same problem. How many iterations were required?

2. Three new automatic feed devices have been made available for existing punch presses. Six presses in the plant can be fitted with this equipment. The plant superintendent estimates that the increased output, together with the labor saved, will result in the following dollar *increase in profits* per day:

		Punch Press					
	\	A	B	C	D	E	F
Device	1:	22	17	22	19	17	18
	2:	21	19	20	23	20	14
	3:	20	21	20	22	23	17

   a. Convert this assignment problem with an equal number of rows and columns.
   b. Convert the objective from maximization to minimization by minimizing the negative of the increased profits.

c. How can the cost matrix be modified so that the costs are nonnegative as required in the Hungarian method? (What happens if you perform row reduction on a row?)

d. Find the assignment of the feed devices to the punch presses which will yield the greatest increase in profits.

◊◊◊◊◊◊◊◊ Homework #6 Solutions ◊◊◊◊◊◊◊◊

1. Consider the assignment problem having the following cost matrix:

\	1	2	3	4
A:	4	1	0	1
B:	1	3	4	0
C:	3	2	1	3
D:	2	2	3	0

a. Consider this as a transportation problem. What should be the supplies and demands?

b. Use the "Least Cost Rule" or Vogel's Approximation Method to get an initial basic feasible solution. Be sure to indicate which variables are basic. (This may require some arbitrary choices.)

c. Starting with the solution indicated in (b), apply the transportation simplex method to find the optimal solution. How many iterations, i.e., changes of basis, were required? In how many iterations was the objective function improved?

d. Apply the Hungarian method to the same problem. How many iterations were required?

Solution :

a. The supplies and demands should all be one.

b. Using the Least Cost Rule,we obtain the following initial basic feasible solution.

	1	2	3	4	
A	$\boxed{4}$	$\boxed{1}$	1 $\boxed{0}$	$\boxed{1}$	1
B	1 $\boxed{1}$	0 $\boxed{3}$	$\boxed{4}$	$\boxed{0}$	1
C	$\boxed{3}$	1 $\boxed{2}$	0 $\boxed{1}$	$\boxed{3}$	1
D	0 $\boxed{2}$	$\boxed{2}$	$\boxed{3}$	1 $\boxed{0}$	1
	1	1	1	1	

Note that this is one of several feasible solutions which might be found by Least Cost Rule. The number of basic variables is 7 (=m+n-1), and so three zero shipments must be basic. Suppose that we select X_{B2}, X_{C3}, and X_{D1} (indicated by the 0 in the boxes above). Next we calculate the reduced costs for all non-basic variables.

		1	2	3	4	
	u\v	-1	1	0	-3	
A	0	4	1	1 0	1	1
B	2	1 1	0 3	4	0	1
C	1	3	1 2	0 1	3	1
D	3	0 2	2	3	1 0	1
		1	1	1	1	

reduced cost for
non-basic vars:
CA1=5
CA2=0
CA4=4
CB3=2
CB4=1
CC1=3
CC4=5
CD2=-2
CD3=0

Since reduced cost for D2(row D, column 2) is negative, it will be chosen to be basic variable.

		1	2	3	4	
	u\v	-1	1	0	-3	
A	0	4	1	1 0	1	1
B	2	1+θ 1	0−θ 3	4	0	1
C	1	3	1 2	0 1	3	1
D	3	0−θ 2	θ 2	3	1 0	1
		1	1	1	1	

It is easy to find that θ=0. We arbitrarily choose B2 to be removed from the basic set and then have the following. (Note we could instead choose D1 instead of B2 to be removed.)

		1	2	3	4	
	u\v	1	1	0	-1	
A	0	4	1	1 0	1	1
B	0	1 1	3	4	0	1
C	1	3	1 2	0 1	3	1
D	1	0 2	0 2	3	1 0	1
		1	1	1	1	

reduced cost for
non-basic vars:
CA1=3
CA2=0
CA4=2
CB2=2
CB3=4
CB4=1
CC1=1
CC4=3
CD3=2

Because reduced costs of all non-basics are nonnegative, so it is optimal. We used one iteration, but obtained no improvement in objective. Total cost=3.

d. First we use row reduction and obtain the following cost matrix.

	1	2	3	4
A	4	1	0	1
B	1	3	4	0
C	2	1	0	2
D	2	2	3	0

Then by using column reduction in above cost matrix, we obtain the following cost matrix.

	1	2	3	4
A	3	0	0	1
B	0	2	4	0
C	1	0	0	2
D	1	1	3	0

It is obvious that four lines are needed to cover all zeros in above cost matrix.
Hence we stop and can assign as A->2,B->1,C->3,D->4 or A->3,B->1,C->2,D->4.
Total cost=3.

2. Three new automatic feed devices have been made available for existing punch presses. Six presses in the plant can be fitted with this equipment. The plant superintendent estimates that the increased output, together with the labor saved, will result in the following dollar *increase in profits* per day:

		Punch Press					
	\	A	B	C	D	E	F
Device	1:	22	17	22	19	17	18
	2:	21	19	20	23	20	14
	3:	20	21	20	22	23	17

a. Convert this assignment problem with an equal number of rows and columns.
b. Convert the objective from maximization to minimization by minimizing the negative of the increased profits.
c. How can the cost matrix be modified so that the costs are nonnegative as required in the Hungarian method? (What happens if you perform row reduction on a row?)
d. Find the assignment of the feed devices to the punch presses which will yield the greatest increase in profits.

Solution :a.

	A	B	C	D	E	F	
1	22	17	22	19	17	18	1
2	21	19	20	23	20	14	1
3	20	21	20	22	23	17	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
	1	1	1	1	1	1	

b.

	A	B	C	D	E	F	
1	-22	-17	-22	-19	-17	-18	1
2	-21	-19	-20	-23	-20	-14	1
3	-20	-21	-20	-22	-23	-17	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
	1	1	1	1	1	1	

c. Subtract min{-22,-17,-22,.....,-17}=-23 from all the elements of above matrix, we obtain the following matrix, which has nonnegative costs:

	A	B	C	D	E	F	
1	1	6	1	4	6	5	1
2	2	4	3	0	3	9	1
3	3	2	3	1	0	6	1
dummy	23	23	23	23	23	23	1
dummy	23	23	23	23	23	23	1
dummy	23	23	23	23	23	23	1
	1	1	1	1	1	1	

d. Applying the row reduction of Hungarian method, we can obtain the following table.

	A	B	C	D	E	F	
1	0	5	0	3	5	4	1
2	2	4	3	0	3	9	1
3	3	2	3	1	0	6	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
dummy	0	0	0	0	0	0	1
	1	1	1	1	1	1	

Since we need four lines to cover all zeros in matrix, so we stop and can assign either

1 --->A, 2 --->D, 3 --->E, or
1 --->C, 2 --->D, 3 --->E.

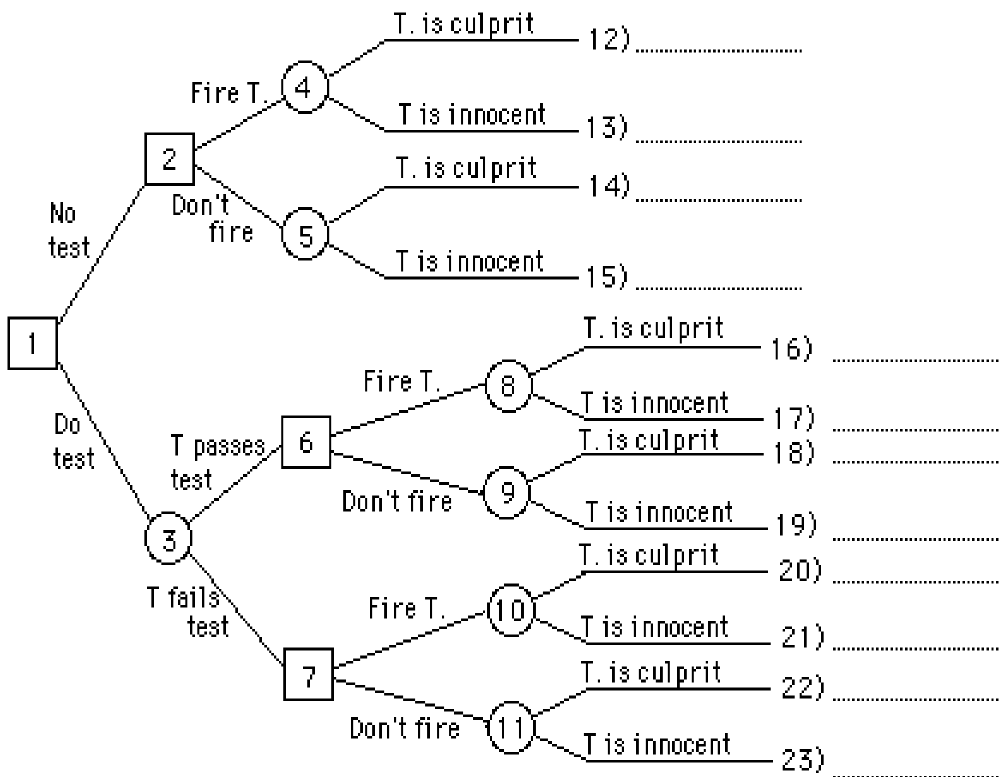
Total profit=68.

Homework # 7

1. **Decision Trees** The president of a firm in a highly competitive industry believes that an employee of the company is providing confidential information to a competing firm. She is 90% certain that this informer is the treasurer of the firm, whose contacts have been extremely valuable in obtaining financing for the company. If she fires the treasurer and he is indeed the informer, the company gains \$100K. If he is fired but is not the informer, the company loses his expertise and still has an informer on the staff, for a net loss to the company of \$500K. If the president does not fire the treasurer, the company loses \$300K whether or not he is the informer, since in either case the informer is still with the company.

   Before deciding the fate of the treasurer, the president could order lie detector tests. To avoid possible lawsuits, such tests would have to be administered to all company employees, at a total cost of \$30K. Another problem is that lie detector tests are not definitive. If a person is lying, the test will reveal it 90% of the time; but if a person is not lying, the test will incorrectly indicate that he is lying 30% of the time.

   a. Write the "pay-off" for each of the terminal nodes (12)-(23) of the decision tree below:



   b. Write the *prior probabilities* on the branches from nodes (4) and (5).

   c. Compute the probability that the treasurer fails the lie detector test, if it is administered:

$$P\{T\text{ fails}\} = P\{T\text{ fails}|T\text{ is culprit}\}P\{T\text{ is culprit}\} + P\{T\text{ fails}|T\text{ is innocent}\}P\{T\text{ is innocent}\}$$

$$= \_\_\_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_$$

   d. Using Bayes' rule, compute the *posterior probability* that the treasurer is the culprit, if he fails the lie detector test:

$$P\{T\text{ is culprit} \mid T\text{ fails test}\} = \frac{P\{T\text{ fails test} \mid T\text{ is culprit}\}P\{T\text{ is culprit}\}}{P\{T\text{ fails test}\}}$$

$$= \frac{(\_\_\_\_\_\_\_\_) \times (\_\_\_\_\_\_\_\_)}{(\_\_\_\_\_\_\_\_)}$$

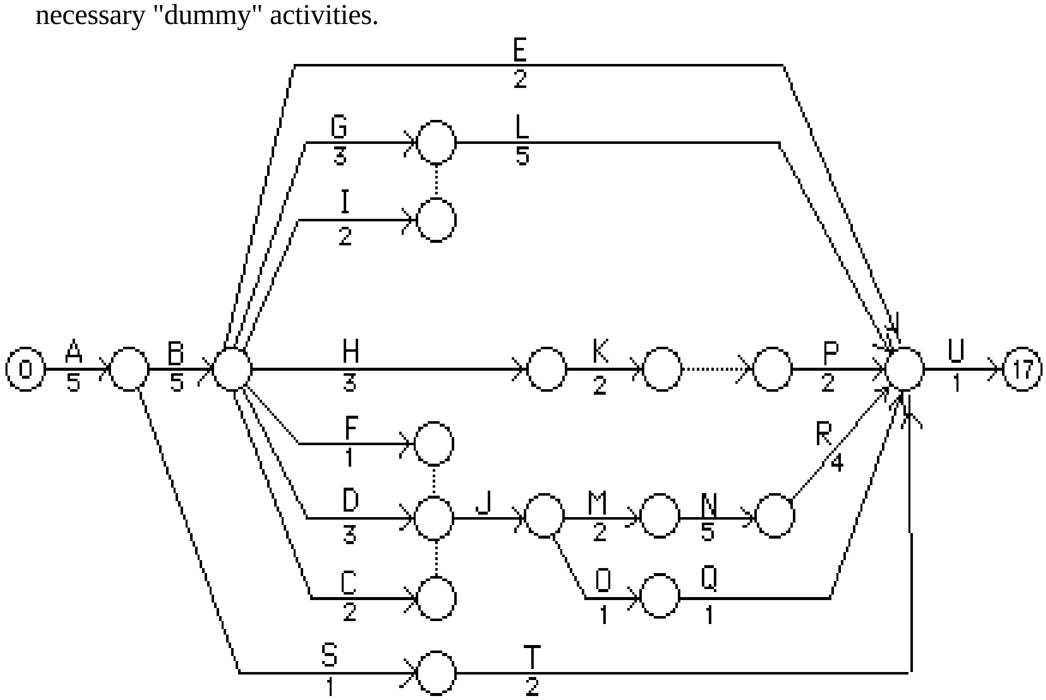
= _____________

e. Fold back the decision tree, computing the expected payoff at each node.
f. What is the optimal strategy?
g. What is the maximum expected payoff which is achieved with this strategy? __________
h. What is the EVSI, i.e., the expected value of the lie detector test? ___________
i. What is the EVPI, i.e., the expected value of perfect information? ___________

2. **Project Scheduling.** A building contractor is building a house. He has identified the component activities below, with their predecessors and durations as indicated:

ID	Description	Duration	Predecesors
A	Basement	5	-none-
B	Erect house frame	5	A
C	Rough electrical work	2	B
D	Rough plumbing	3	B
E	Roofing	2	B
F	Chimney	1	B
G	Siding	3	B
H	Hardwood floors	3	B
I	Windows & Doors	2	B
J	Wallboard	4	C, D, F
K	Trim wood	2	H, I, J
L	Paint exterior	5	G, I
M	Paint ceilings	2	J
N	Paint interior walls	5	K, M
O	Kitchen cabinets	1	J
P	Finished plumbing, heat	2	K, O
Q	Finished electrical	1	O
R	Finish floors	4	N
S	Fine grading of site	1	A
T	Landscape	2	S
U	Clean up site	1	E, L, P, Q, R, T

a. The building contractor has prepared the A-O-A Project Network shown below. There are, however, some "dummy" activities missing, and some "dummy" activities with no direction indicated. Draw the necessary "dummy" activities.



b. Label each of the nodes so that for each activity from node i to node j, i<j.

c. Compute the Early Time (ET) for each node (i.e., event):

Node	ET	LT
0	___	___
1	___	___
2	___	___
3	___	___
4	___	___
5	___	___
6	___	___
7	___	___
8	___	___
9	___	___
10	___	___
11	___	___
12	___	___
13	___	___
14	___	___
15	___	___
16	___	___
17	___	___

d. Compute, for each activity, the early start (ES) time, the early finish (EF) time, the late start (LS) time, and the late finish (LF) time, and the (total) float (TF).

ID	Description	Duration	ES	EF	LS	LF	TF
A	Basement	5	____	____	____	____	____
B	Erect house frame	5	____	____	____	____	____
C	Rough electrical work	2	____	____	____	____	____
D	Rough plumbing	3	____	____	____	____	____
E	Roofing	2	____	____	____	____	____
F	Chimney	1	____	____	____	____	____
G	Siding	3	____	____	____	____	____
H	Hardwood floors	3	____	____	____	____	____
I	Windows & Doors	2	____	____	____	____	____
J	Wallboard	4	____	____	____	____	____
K	Trim wood	2	____	____	____	____	____
L	Paint exterior	5	____	____	____	____	____
M	Paint ceilings	2	____	____	____	____	____
N	Paint interior walls	5	____	____	____	____	____
O	Kitchen cabinets	1	____	____	____	____	____
P	Finished plumbing, heat	2	____	____	____	____	____
Q	Finished electrical	1	____	____	____	____	____
R	Finish floors	4	____	____	____	____	____
S	Fine grading of site	1	____	____	____	____	____
T	Landscape	2	____	____	____	____	____
U	Clean up site	1	____	____	____	____	____

e. Circle the ID of each critical activity in the preceding table in (d), and indicate the critical path on the diagram in (a).

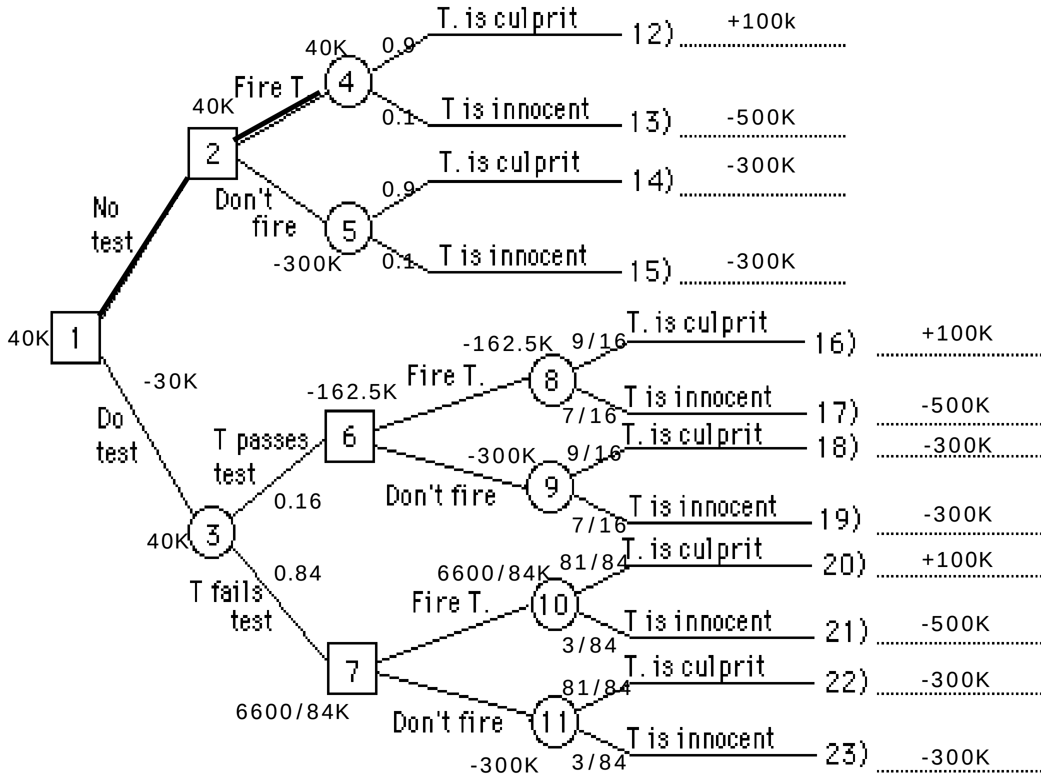
◊◊◊◊◊◊◊◊ Homework #7 Solutions ◊◊◊◊◊◊◊◊

1. **Decision Trees** The president of a firm in a highly competitive industry believes that an employee of the company is providing confidential information to a competing firm. She is 90% certain that this informer is the treasurer of the firm, whose contacts have been extremely valuable in obtaining financing for the company. If she fires the treasurer and he is indeed the informer, the company gains $100K. If he is fired but is not the informer, the company loses his expertise and still has an informer on the staff, for a net loss to the company of $500K. If the president does not fire the treasurer, the company loses $300K whether or not he is the informer, since in either case the informer is still with the company.

   Before deciding the fate of the treasurer, the president could order lie detector tests. To avoid possible lawsuits, such tests would have to be administered to all company employees, at a total cost of $30K. Another problem is that lie detector tests are not definitive. If a person is lying, the test will reveal it 90% of the time; but if a person is not lying, the test will incorrectly indicate that he is lying 30% of the time.

Questions and solutions.

a. Write the "pay-off" for each of the terminal nodes (12)-(23) of the decision tree below:



Here we calculate P(T culprit | T passes test) and P(T culprit | T fails test) by

$$P(T\text{ culprit}|\,T\text{ passes test}) = \frac{P(T\text{ passes test}|T\text{ culprit})P(T\text{ culprit})}{P(T\text{ passes test})} = \frac{(0.1)(0.9)}{0.16} = \frac{9}{16}$$

and

$$P(T\text{ culprit}|\,T\text{ fails test}) = \frac{P(T\text{ fails test}|T\text{ culprit})P(T\text{ culprit})}{P(T\text{ fails test})} = \frac{(0.9)(0.9)}{0.84} = \frac{81}{84}$$

b. Write the *prior probabilities* on the branches from nodes (4) and (5).

c. Compute the probability that the treasurer fails the lie detector test, if it is administered:

P{T fails} = P{T fails|T is culprit}P{T is culprit} + P{T fails|T is innocent}P{T is innocent}

= ____0.9___ x ___0.9___ + ____0.3___ x ___0.1___

= ____0.84__

d. Using Bayes' rule, compute the *posterior probability* that the treasurer is the culprit, if he fails the lie detector test:

$$P\{T\text{ is culprit} \mid T\text{ fails test}\} = \frac{P\{T\text{ fails test} \mid T\text{ is culprit}\}P\{T\text{ is culprit}\}}{P\{T\text{ fails test}\}}$$

$$= \frac{(\_\_\,0.9\,\_\_\_\_)\times(\_\_\_\,0.9\,\_\_\_)}{(\_\_\_\,0.84\,\_\_)}$$

= _____81/84_____

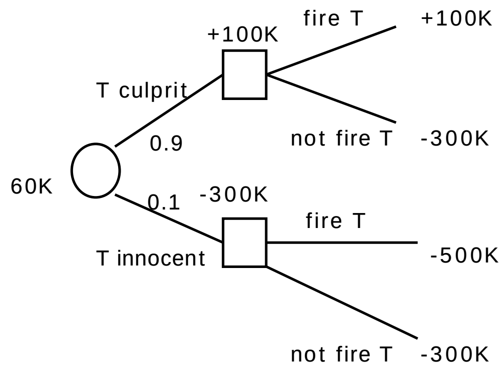
e. Fold back the decision tree, computing the expected payoff at each node.
See decision tree in (a).

f. What is the optimal strategy? Fire T without test.

g. What is the maximum expected payoff which is achieved with this strategy? 40K.

h. What is the EVSI, i.e., the expected value of the lie detector test? 0.
EVSI=EVWSI-EVWOI=40K-40K=0.

i. What is the EVPI, i.e., the expected value of perfect information? 20K.
EVPI=EVWPI-EVWOI=60K-40K=20K.
Here EVWPI is obtained by

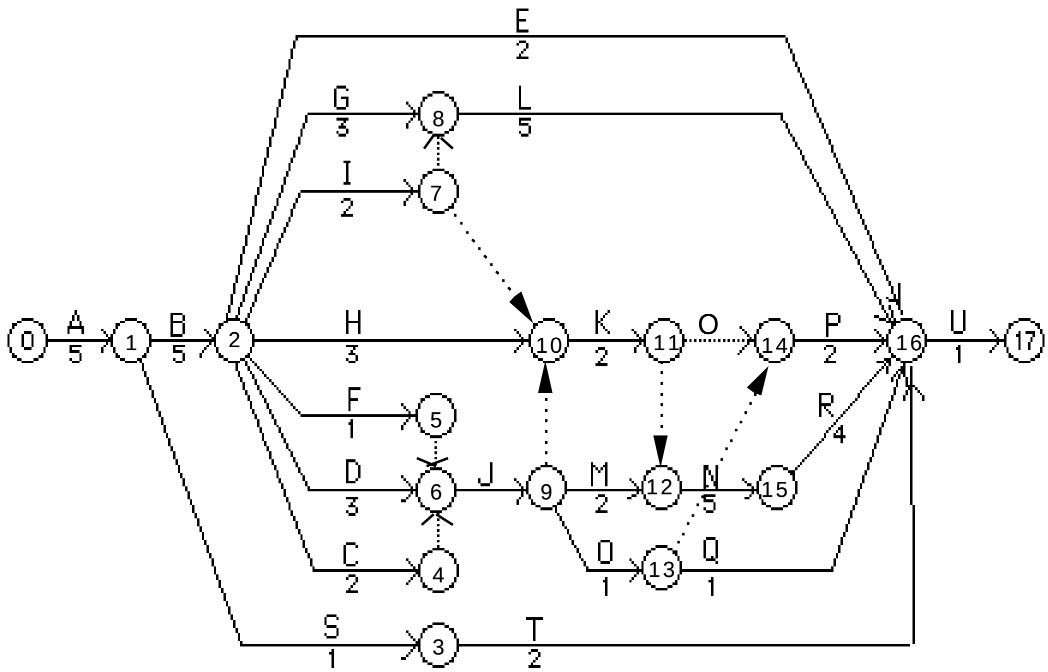


2. **Project Scheduling.** A building contractor is building a house. He has identified the component activities below, with their predecessors and durations as indicated:

ID	Description	Duration	Predecesors
A	Basement	5	-none-
B	Erect house frame	5	A
C	Rough electrical work	2	B
D	Rough plumbing	3	B
E	Roofing	2	B
F	Chimney	1	B
G	Siding	3	B
H	Hardwood floors	3	B
I	Windows & Doors	2	B
J	Wallboard	4	C, D, F
K	Trim wood	2	H, I, J
L	Paint exterior	5	G, I
M	Paint ceilings	2	J
N	Paint interior walls	5	K, M
O	Kitchen cabinets	1	J
P	Finished plumbing, heat	2	K, O
Q	Finished electrical	1	O
R	Finish floors	4	N
S	Fine grading of site	1	A
T	Landscape	2	S
U	Clean up site	1	E, L, P, Q, R, T

Questions and solutions.

a. The building contractor has prepared the A-O-A Project Network shown below. There are, however, some "dummy" activities missing, and some "dummy" activities with no direction indicated. Draw the necessary "dummy" activities.



b. Label each of the nodes so that for each activity from node i to node j, i<j.

c. Compute the Early Time (ET) for each node (i.e., event):

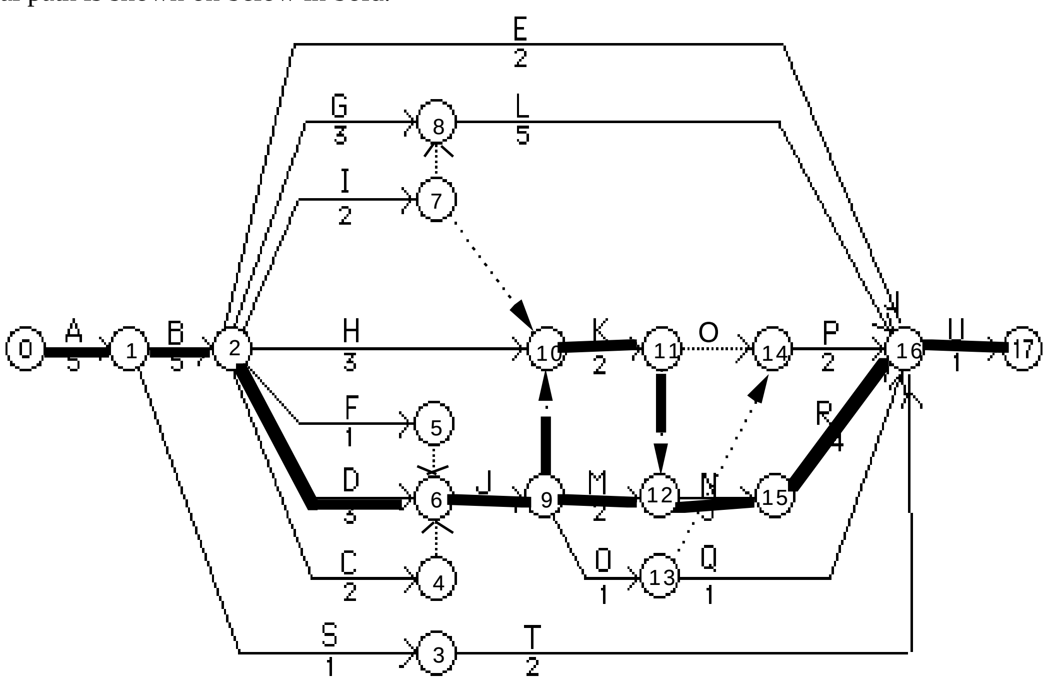
Node	ET	LT
0	0	0
1	5	5
2	10	10
3	6	26
4	12	13
5	11	13
6	13	13
7	12	17
8	13	23
9	17	17
10	17	17
11	19	19
12	19	19
13	18	26
14	19	26
15	24	24
16	28	28
17	29	29

d. Compute, for each activity, the early start (ES) time, the early finish (EF) time, the late start (LS) time, and the late finish (LF) time, and the (total) float (TF).

ID	Description	Duration	ES	EF	LS	LF	TF
*A	Basement	5	0	5	0	5	0
*B	Erect house frame	5	5	10	5	10	0
C	Rough electrical work	2	10	12	11	13	1
*D	Rough plumbing	3	10	13	10	13	0
E	Roofing	2	10	12	26	28	16

F	Chimney	1	10	11	12	13	2
G	Siding	3	10	13	20	23	10
H	Hardwood floors	3	10	13	14	17	4
I	Windows & Doors	2	10	12	15	17	5
*J	Wallboard	4	13	17	13	17	0
*K	Trim wood	2	17	19	17	19	0
L	Paint exterior	5	13	18	23	28	10
*M	Paint ceilings	2	17	19	17	19	0
*N	Paint interior walls	5	19	24	19	24	0
O	Kitchen cabinets	1	17	18	25	26	8
P	Finished plumbing, heat	2	19	21	26	28	7
Q	Finished electrical	1	18	19	27	28	9
*R	Finish floors	4	24	28	24	28	0
S	Fine grading of site	1	5	6	25	26	20
T	Landscape	2	6	8	26	28	20
*U	Clean up site	1	28	29	28	29	0

e. Circle the ID of each critical activity in the preceding table in (d), and indicate the critical path on the diagram in (a).
See above table, notation "*" is put before all critical activities.
Critical path is shown on below in bold.



Homework # 8

1. A company is considering three different products with the following characteristics:

Product	Fixed Cost	Unit Profit	Man-hrs Dept. 1	Man-hrs Dept. 2
A	$1000	$50	4	2
B	$1500	$75	6	2
C	$750	$40	2	3

The fixed cost is incurred only if the product is manufactured. Production is limited by the availability of manpower in the two departments of the manufacturing facility. The available man-hours during the next month are 2000 for department 1 and 1500 for department 2.

Define decision variables X_i = the quantity of product i to be manufactured (assumed to be a continuous variable) and Y_i = 1 if product i is manufactured, and 0 otherwise.

a. In order to formulate the integer programming model, it is useful to have an upper bound on the quantity of each product which could possibly be manufactured. Based on the man-hour requirements, compute these upper bounds.
b. Formulate an integer programming model to determine the optimal quantities to be manufactured in order to maximize profits.
c. Use LINDO (or similar software) to solve your model.

2. The Smalltown Fire Department currently has seven conventional ladder companies and seven alarm boxes. The two closest ladder companies to each alarm box are given in the table below. The city fathers want to maximize the number of conventional ladder companies that can be replaced with tower ladder companies. Unfortunately, political considerations dictate that a conventional company can be replaced only if, after replacement, at least one of the two closest companies to each alarm box is still a conventional company.

Alarm Box	Two closest Ladder Companies
1	2,3
2	3,4
3	1,5
4	2,6
5	3,6
6	4,7
7	5,7

a. Formulate an integer programming problem that can be used to maximize the number of conventional companies that can be replaced by tower companies.
b. Solve your model using LINDO (or similar software).

◊◊◊◊◊◊◊◊ Homework #8 Solutions ◊◊◊◊◊◊◊◊

1. A company is considering three different products with the following characteristics:

Product	Fixed Cost	Unit Profit	Man-hrs Dept. 1	Man-hrs Dept. 2
A	$1000	$50	4	2
B	$1500	$75	6	2
C	$750	$40	2	3

The fixed cost is incurred only if the product is manufactured. Production is limited by the availability of manpower in the two departments of the manufacturing facility. The available man-hours during the next month are 2000 for department 1 and 1500 for department 2.

Define decision variables X_i = the quantity of product i to be manufactured (assumed to be a continuous variable) and Y_i = 1 if product i is manufactured, and 0 otherwise.

a. In order to formulate the integer programming model, it is useful to have an upper bound on the quantity of each product which could possibly be manufactured. Based on the man-hour requirements, compute these upper bounds.
b. Formulate an integer programming model to determine the optimal quantities to be manufactured in order to maximize profits.
c. Use LINDO (or similar software) to solve your model.

Solutions.

(a). Upper bound for product A=min{2000/4, 1500/2}=500
Upper bound for product B=min{2000/6, 1500/2}=333.3
Upper bound for product C=min{2000/2, 1500/3}=500.

(b). ILP formation.

```
Max    -1000YA-1500YB-750YC        (fixed cost)
       +50XA+75XB+40XC             (profit)
```

s.t.

$4XA+6XB+2XC \le 2000$	(available hours for Dept.1)
$2XA+2XB+3XC \le 1500$	(available hours for Dept.2)
$XA \le 500YA$	(upper bound for product A)
$XB \le 333.3YB$	(upper bound for product B)
$XC \le 500YC$	(upper bound for product C)

(c). From the below LINDO outputs, one can find the optimal solution is : (XA,XB,XC)=(0,214.3,357.1), and the max profit is 28107.1426.

LINDO outputs :

```
 MAX   - 1000 YA - 1500 YB - 750 YC + 50 XA + 75 XB + 40 XC
 SUBJECT TO
    2)    4 XA + 6 XB + 2 XC <=   2000
    3)    2 XA + 2 XB + 3 XC <=   1500
    4)  - 500 YA +  XA <=   0
    5)  - 333.3 YB +  XB <=   0
    6)  - 500 YC +  XC <=   0
 END
 INTEGER-VARIABLES=      3

: INTE 3
: INTE YA
: INTE YB
: INTE YC

: go
     LP OPTIMUM FOUND  AT STEP     8

         OBJECTIVE FUNCTION VALUE

  1)        28857.1328

 VARIABLE        VALUE          REDUCED COST
       YA          0.000000          0.000000
       YB          0.642864          0.000000
       YC          0.714286          0.000000
       XA          0.000000          3.285687
       XB        214.285706          0.000000
       XC        357.142883          0.000000


 ROW        SLACK OR SURPLUS     DUAL PRICES
    2)           0.000000             9.607133
    3)           0.000000             6.428578
    4)           0.000000             2.000000
    5)           0.000000             4.500045
    6)           0.000000             1.500000

 NO. ITERATIONS=       8
  BRANCHES=    0 DETERM.=-11.667E  8
 SET       YC TO    1 AT    1 BND=   28642.848     TWIN=   28321.428
 SET       YB TO    1 AT    2 BND=   28107.143     TWIN=   28107.143

 NEW INTEGER SOLUTION OF   28107.1     AT BRANCH      2 PIVOT     10

         OBJECTIVE FUNCTION VALUE
```

1) 28107.1426

VARIABLE	VALUE	REDUCED COST
YA	0.000000	0.000000
YB	1.000000	1500.000000
YC	1.000000	750.000000
XA	0.000000	6.285716
XB	214.285721	0.000000
XC	357.142853	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	10.357143
3)	0.000000	6.428571
4)	0.000000	2.000000
5)	119.044266	0.000000
6)	142.857147	0.000000

```
NO. ITERATIONS=      10
 BRANCHES=    2 DETERM.=-70.000E  2
 BOUND ON OPTIMUM:  28321.43
 DELETE       YB AT LEVEL     2
   FLIP       YC TO     0 WITH BOUND   28321.43
 DELETE       YC AT LEVEL     1
    ENUMERATION COMPLETE. BRANCHES=     2  PIVOTS=    15

 LAST INTEGER SOLUTION IS THE BEST FOUND
```

2. The Smalltown Fire Department currently has seven conventional ladder companies and seven alarm boxes. The two closest ladder companies to each alarm box are given in the table below. The city fathers want to maximize the number of conventional ladder companies that can be replaced with tower ladder companies. Unfortunately, political considerations dictate that a conventional company can be replaced only if, after replacement, at least one of the two closest companies to each alarm box is still a conventional company.

Alarm Box	Two closest Ladder Companies
1	2,3
2	3,4
3	1,5
4	2,6
5	3,6
6	4,7
7	5,7

a. Formulate an integer programming problem that can be used to maximize the number of conventional companies that can be replaced by tower companies.

b. Solve your model using LINDO (or similar software).

Solutions.

(a). Let Yi be the decision variables, i=1 2,...,7.

Yi=1 if conventional ladder company i can be replaced with tower ladder company, and 0 otherwise.

ILP formulation.

Max $Y1+Y2+Y3+Y4+Y5+Y6+Y7$

s.t. $Y2+Y3\leq 1$ (at least one of two closest compamies is conventional)

$Y3+Y4\leq 1$

$Y1+Y5\leq 1$

$Y2+Y6 \leq 1$
$Y3+Y6 \leq 1$
$Y4+Y7 \leq 1$
$Y5+Y7 \leq 1$
Yi, i=1,2,...,7, are zeros or ones.

(b). The optimal solution can be obtained from the below LINDO outputs as : (Y1,Y2,Y3,Y4,Y5,Y6,Y7)=(1,1,0,0,0,0,1). Note that the optimal solution is degenerate, and one can find the other solutions with the same objective value, e.g. (0,1,0,1,1,0,0).

LINDO outputs.

```
MAX     Y1 +  Y2 +  Y3 +  Y4 +  Y5 +  Y6 +  Y7
SUBJECT TO
   2)    Y2 +  Y3 <=   1
   3)    Y3 +  Y4 <=   1
   4)    Y1 +  Y5 <=   1
   5)    Y2 +  Y6 <=   1
   6)    Y3 +  Y6 <=   1
   7)    Y4 +  Y7 <=   1
   8)    Y5 +  Y7 <=   1
END
INTEGER-VARIABLES=      7


: go
    LP OPTIMUM FOUND  AT STEP     8

         OBJECTIVE FUNCTION VALUE

  1)        3.50000000

VARIABLE        VALUE          REDUCED COST
      Y1          1.000000          0.000000
      Y2          0.500000          0.000000
      Y3          0.500000          0.000000
      Y4          0.000000          0.000000
      Y5          0.000000          0.000000
      Y6          0.500000          0.000000
      Y7          1.000000          0.000000


ROW        SLACK OR SURPLUS     DUAL PRICES
   2)          0.000000                0.500000
   3)          0.500000                0.000000
   4)          0.000000                1.000000
   5)          0.000000                0.500000
   6)          0.000000                0.500000
   7)          0.000000                1.000000
   8)          0.000000                0.000000

NO. ITERATIONS=       8
 BRANCHES=    0 DETERM.=  2.000E  0
SET       Y2 TO    1 AT    1 BND=   3.0000000     TWIN=   3.0000000

NEW INTEGER SOLUTION OF   3.00000     AT BRANCH      1 PIVOT      9
```

OBJECTIVE FUNCTION VALUE

1) 3.00000000

VARIABL	VALUE	REDUCED COST
Y1	1.000000	0.000000
Y2	1.000000	1.000000
Y3	0.000000	0.000000
Y4	0.000000	0.000000
Y5	0.000000	0.000000
Y6	0.000000	0.000000
Y7	1.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	1.000000
3)	1.000000	0.000000
4)	0.000000	1.000000
5)	0.000000	1.000000
6)	1.000000	0.000000
7)	0.000000	1.000000
8)	0.000000	0.000000

NO. ITERATIONS= 9
BRANCHES= 1 DETERM.= 1.000E 0
BOUND ON OPTIMUM: 3.000000
DELETE Y2 AT LEVEL 1
ENUMERATION COMPLETE. BRANCHES= 1 PIVOTS= 9

LAST INTEGER SOLUTION IS THE BEST FOUND

Homework # 9

1. A city's water supply comes from a reservoir. Careful study of this reservoir over the past 20 years has shown that, if the reservoir was full at the beginning of the summer, then the probability it would be full at the beginning of the following summer is 80%, independent of its status in previous years. On the other hand, if the reservoir was not full at the beginning of one summer, the probability it would be full at the beginning of the following summer is only 40%.

   a. Defining the states to be "full" and "not full", draw a transition diagram for a Markov chain model of this reservoir.
   b. Write the transition probability matrix.
   c. If the reservoir was full at the beginning of summer 1993, what is the probability that it will be full at the beginning of summer 1995?
   d. Write equations which determine the steadystate probability distribution for this Markov chain, and solve them.
   e. In what percent of the years will the reservoir be full at the beginning of the summer, according to this model?

2. A distributing company owns a fleet of ten cars for use by its salesmen. A car's "birthday" is always considered to be July 1, when decisions are made concerning replacement of used cars in the fleet. Thus, a new car purchased in the spring of 1993 (for example), becomes one year old on its first birthday (July 1, 1993). On July 1 they replace any car which is four years old with a new car costing $10,000. Cars which break down and cannot be easily repaired are replaced immediately. (Assume that the $10,000 is in addition to

the trade-in value of the car being replaced.) The probabilities of such a breakdown and the costs of operation & maintenance if <u>not</u> replaced are:

Age	P{breakdown}	Operating cost
Less than 1 year old	0%	$ 800
One year old	10%	$1500
Two years old	30%	$1800
Three years old	60%	$2200

The company has modeled each car as a Markov chain, with the state of the system being the "age" of the car on July 1. Consult the output of the APL workspace "Markov" below to answer the following questions:

a. Draw a diagram of the Markov chain model.
b. Explain the computation of the probabilities p_{11}, p_{31}, p_{34}, and p_{41}.
c. Find the average annual cost of the entire fleet of ten cars (purchasing, operating, and maintenance).
d. Find the average age of the fleet on July 1.
e. Find the average lifetime of a car in the fleet.
f. Find the probability that, if a car is 2 years old on July 1, 1993, its replacement will be a one-year-old car on July 1, 1996.
g. Find the probability that, if a car is 2 years old on July 1, 1993, its <u>next</u> replacement will be a one-year-old car on July 1,
1996.

$$P=\begin{bmatrix} 0.1 & 0.9 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad P^2=\begin{bmatrix} 0.28 & 0.09 & 0.63 & 0 \\ 0.45 & 0.27 & 0 & 0.28 \\ 0.46 & 0.54 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \end{bmatrix} \quad P^3=\begin{bmatrix} 0.433 & 0.252 & 0.063 & 0.252 \\ 0.406 & 0.405 & 0.189 & 0 \\ 0.208 & 0.414 & 0.378 & 0 \\ 0.28 & 0.09 & 0.63 & 0 \end{bmatrix}$$

$$P^4=\begin{bmatrix} 0.4087 & 0.3897 & 0.1764 & 0.0252 \\ 0.2755 & 0.3654 & 0.2835 & 0.0756 \\ 0.3718 & 0.1872 & 0.2898 & 0.1512 \\ 0.433 & 0.252 & 0.063 & 0.252 \end{bmatrix} \quad P^5=\begin{bmatrix} 0.28882 & 0.36783 & 0.27279 & 0.07056 \\ 0.38287 & 0.24795 & 0.25578 & 0.1134 \\ 0.41842 & 0.33462 & 0.13104 & 0.11592 \\ 0.4087 & 0.3897 & 0.1764 & 0.0252 \end{bmatrix}$$

Steady State Distribution

i	P{i}
1	0.35945363
2	0.32350827
3	0.22645579
4	0.090582315

Mean First Passage Times

$$M=\begin{bmatrix} 2.782 & 1.1111111 & 3.015873 & 10.039683 \\ 1.98 & 3.0911111 & 1.9047619 & 8.9285714 \\ 1.4 & 2.5111111 & 4.415873 & 7.0238095 \\ 1 & 2.1111111 & 4.015873 & 11.039683 \end{bmatrix}$$

First Visit Probabilities

$$F^{(2)}=\begin{bmatrix} 0.27 & 0.09 & 0.63 & 0 \\ 0.42 & 0.27 & 0 & 0.28 \\ 0.4 & 0.54 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \end{bmatrix} \quad F^{(3)}=\begin{bmatrix} 0.378 & 0.009 & 0.063 & 0.252 \\ 0.28 & 0.405 & 0.189 & 0 \\ 0 & 0.414 & 0.378 & 0 \\ 0 & 0.09 & 0.63 & 0 \end{bmatrix}$$

$$F^{(4)}=\begin{bmatrix} 0.252 & 0.0009 & 0.1764 & 0.0252 \\ 0 & 0.2925 & 0.0189 & 0.0756 \\ 0 & 0.0414 & 0.2898 & 0.1512 \\ 0 & 0.009 & 0.063 & 0.252 \end{bmatrix} \quad F^{(5)}=\begin{bmatrix} 0 & 0.00009 & 0.03465 & 0.07056 \\ 0 & 0.02925 & 0.05292 & 0.1134 \\ 0 & 0.00414 & 0.13104 & 0.01512 \\ 0 & 0.0009 & 0.1764 & 0.0252 \end{bmatrix}$$

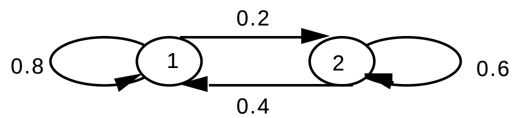
◊◊◊◊◊◊◊◊ Homework #9 Solutions ◊◊◊◊◊◊◊◊

1. A city's water supply comes from a reservoir. Careful study of this reservoir over the past 20 years has shown that, if the reservoir was full at the beginning of the summer, then the probability it would be full at the beginning of the following summer is 80%, independent of its status in previous years. On the other hand, if the reservoir was <u>not</u> full at the beginning of one summer, the probability it would be full at the beginning of the following summer is only 40%.

a. Defining the states to be "full" and "not full", draw a transition diagram for a Markov chain model of this reservoir.
b. Write the transition probability matrix.
c. If the reservoir was full at the beginning of summer 1993, what is the probability that it will be full at the beginning of summer 1995?
d. Write equations which determine the steadystate probability distribution for this Markov chain, and solve them.
e. In what percent of the years will the reservoir be full at the beginning of the summer, according to this model?

Solutions.

(a). Define state 1 be full and state 2 be not full.



(b). $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$.

(c). $P^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$. The probability $P_{11}^2 = 0.72$.

(d). $\pi = \pi P$, i.e.,

$$(\pi_1, \pi_2) = (\pi_1, \pi_2)\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \text{ or } \begin{cases} 0.8\pi_1 + 0.4\pi_2 = \pi_1 \\ 0.2\pi_1 + 0.6\pi_2 = \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

Thus, $(\pi_1, \pi_2) = (2/3, 1/3)$.

(e) 66.67% .

2. A distributing company owns a fleet of ten cars for use by its salesmen. A car's "birthday" is always considered to be July 1, when decisions are made concerning replacement of used cars in the fleet. Thus, a new car purchased in the spring of 1993 (for example), becomes one year old on its first birthday (July 1, 1993). On July 1 they replace any car which is four years old with a new car costing $10,000. Cars which break down and cannot be easily repaired are replaced immediately. (Assume that the $10,000 is in addition to the trade-in value of the car being replaced.) The probabilities of such a breakdown and the costs of operation & maintenance if <u>not</u> replaced are:

Age	P{breakdown}	Operating cost
Less than 1 year old	0%	$ 800
One year old	10%	$1500
Two years old	30%	$1800
Three years old	60%	$2200

The company has modeled each car as a Markov chain, with the state of the system being the "age" of the car on July 1. Consult the output of the APL workspace "Markov" below to answer the following questions:

a. Draw a diagram of the Markov chain model.
b. Explain the computation of the probabilities p_{11}, p_{31}, p_{34}, and p_{41}.
c. Find the average annual cost of the entire fleet of ten cars (purchasing, operating, and maintenance).
d. Find the average age of the fleet on July 1.
e. Find the average lifetime of a car in the fleet.
f. Find the probability that, if a car is 2 years old on July 1, 1993, its replacement will be a one-year-old car on July 1, 1996.
g. Find the probability that, if a car is 2 years old on July 1, 1993, its <u>next</u> replacement will be a one-year-old car on July 1, 1996

$$P=\begin{bmatrix} 0.1 & 0.9 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad P^2=\begin{bmatrix} 0.28 & 0.09 & 0.63 & 0 \\ 0.45 & 0.27 & 0 & 0.28 \\ 0.46 & 0.54 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \end{bmatrix} \quad P^3=\begin{bmatrix} 0.433 & 0.252 & 0.063 & 0.252 \\ 0.406 & 0.405 & 0.189 & 0 \\ 0.208 & 0.414 & 0.378 & 0 \\ 0.28 & 0.09 & 0.63 & 0 \end{bmatrix}$$

$$P^4=\begin{bmatrix} 0.4087 & 0.3897 & 0.1764 & 0.0252 \\ 0.2755 & 0.3654 & 0.2835 & 0.0756 \\ 0.3718 & 0.1872 & 0.2898 & 0.1512 \\ 0.433 & 0.252 & 0.063 & 0.252 \end{bmatrix} \quad P^5=\begin{bmatrix} 0.28882 & 0.36783 & 0.27279 & 0.07056 \\ 0.38287 & 0.24795 & 0.25578 & 0.1134 \\ 0.41842 & 0.33462 & 0.13104 & 0.11592 \\ 0.4087 & 0.3897 & 0.1764 & 0.0252 \end{bmatrix}$$

Steady State Distribution

i	P{i}
1	0.35945363
2	0.32350827
3	0.22645579
4	0.090582315

Mean First Passage Times

$$M=\begin{bmatrix} 2.782 & 1.1111111 & 3.015873 & 10.039683 \\ 1.98 & 3.0911111 & 1.9047619 & 8.9285714 \\ 1.4 & 2.5111111 & 4.415873 & 7.0238095 \\ 1 & 2.1111111 & 4.015873 & 11.039683 \end{bmatrix}$$

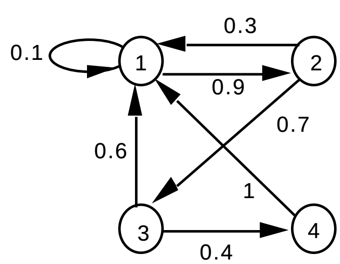
First Visit Probabilities

$$F^{(2)}=\begin{bmatrix} 0.27 & 0.09 & 0.63 & 0 \\ 0.42 & 0.27 & 0 & 0.28 \\ 0.4 & 0.54 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \end{bmatrix} \quad F^{(3)}=\begin{bmatrix} 0.378 & 0.009 & 0.063 & 0.252 \\ 0.28 & 0.405 & 0.189 & 0 \\ 0 & 0.414 & 0.378 & 0 \\ 0 & 0.09 & 0.63 & 0 \end{bmatrix}$$

$$F^{(4)}=\begin{bmatrix} 0.252 & 0.0009 & 0.1764 & 0.0252 \\ 0 & 0.2925 & 0.0189 & 0.0756 \\ 0 & 0.0414 & 0.2898 & 0.1512 \\ 0 & 0.009 & 0.063 & 0.252 \end{bmatrix} \quad F^{(5)}=\begin{bmatrix} 0 & 0.00009 & 0.03465 & 0.07056 \\ 0 & 0.02925 & 0.05292 & 0.1134 \\ 0 & 0.00414 & 0.13104 & 0.01512 \\ 0 & 0.0009 & 0.1764 & 0.0252 \end{bmatrix}$$

Solutions.

(a). State 1: age 1.
State 2: age 2.
State 3: age 3.
State 4: age 4.



(b). p_{11}=probability that a car that is one year old will breakdown before its next birthday.
p_{31}=probability that a car is three years old will brekdown before its next birthday.
p_{34}=probability that a car is three years old will **not** breakdown before its next birthday.
p_{41}=probability that a car is four years old will have to be replaced immediately after its fourth birthday.

(c). Assume that the cost for the year which is just ending is determined by the state of the car on July 1. The replacement cost, \$10,000, will be have been incurred by a car just reaching its first birthday, i.e., a car in state 1. The operating cost during the year just ending for a car reaching its n^{th} birthday is the annual operating cost for a car of age (n-1).

Therefore, the average annual cost of one car is :

$$\pi_1(800+10000)+\pi_2(1500)+\pi_3(1800)+\pi_4(2200) = 4974.26$$

Thus the average annual cost for ten cars is 10(4974.26)=49742.6.

(d). Average age: $\pi_1(1) + \pi_2(2) + \pi_3(3) + \pi_4(4) = 2.048$.

(e). Average lifetime=M_{11}=2.782.
(Note that the average age ≠ average lifetime!)

(f). $P^3_{21} = 0.406$.

(g) $F^{(3)}_{21} = 0.28$.

Homework # 10

*For this homework assignment, you may either do the computations manually, or use the APL workspace MARKOV which is on the ICAEN fileserver. (Requires that you have a copy of the APL*PLUS interpreter on a floppy disk.)*

1. (#6, page 943 of the text) At the beginning of a period, a company observes its inventory level. Then an order may be placed (and is instantaneously received). Finally, the period's demand is observed. We are given the following information:

- A $2 cost is assessed against each unit of inventory on hand at the end of a period.

- A $3 penalty is assessed against each unit of demand that is not met on time. Assume that all shortages result in lost sales.

- Placing an order costs 50¢ per unit plus a $5 ordering cost.

- During each period, demand is equally likely to equal 1, 2, or 3 units.

The company is considering the following ordering policy: At the end of any period, if the on-hand inventory is 1 unit or less, order sufficient units to bring the on-hand inventory level at the beginning of the next period up to 4 units.

(a.) What fraction of the time will the on-hand inventory level at the end of each period be 0 units? 1 unit? 2 units? 3 units? 4 units?

(b.) Determine the average cost per period incurred by this ordering policy.

(c.) Answer parts (a) and (b) if all shortages are backlogged. Assume that the cost for each unit backlogged is $3.

2. (#9, page 944 of the text) In the game of craps, we roll a pair of six-sided dice. On the first throw, if we roll a 7 or an 11, we win right away. If we roll a 2, 3, or 12, we lose right away. If we first roll a 4, 5, 6, 8, 9, or 10, we keep rolling the dice until we get either a 7 or the total rolled on the first throw. If we get a 7, we lose. If we roll the same total as the first throw, we win.

(a.) Model a crap game as a Markov chain with absorbing states. (Define the states, draw the transition diagram, and write the transition probability matrix.)

(b.) What is the probability of winning for the roller (the person rolling the dice)?

(c.) If the first roll is a "4", what is his/her probability of winning?

(d.) What is the expected number of rolls of the dice in a crap game?

(e.) If the first roll of the dice is a "4", how many additional rolls are expected before the game ends?

◊◊◊◊◊◊◊◊ Homework #10 Solutions ◊◊◊◊◊◊◊◊

1. (#6, page 943 of the text) At the beginning of a period, a company observes its inventory level. Then an order may be placed (and is instantaneously received). Finally, the period's demand is observed. We are given the following information:

- A \$2 cost is assessed against each unit of inventory on hand at the end of a period.

- A \$3 penalty is assessed against each unit of demand that is not met on time. Assume that all shortages result in lost sales.

- Placing an order costs 50¢ per unit plus a \$5 ordering cost.

- During each period, demand is equally likely to equal 1, 2, or 3 units.

The company is considering the following ordering policy: At the end of any period, if the on-hand inventory is 1 unit or less, order sufficient units to bring the on-hand inventory level at the beginning of the next period up to 4 units.

(a.) What fraction of the time will the on-hand inventory level at the end of each period be 0 units? 1 unit? 2 units? 3 units? 4 units?

(b.) Determine the average cost per period incurred by this ordering policy.

(c.) Answer parts (a) and (b) if all shortages are backlogged. Assume that the cost for each unit backlogged is \$3.

Solutions.

(a.) Define the states as :
state 1: the on-hand inventory is zero at the end of the period,
state 2: the on-hand inventory is one at the end of the period,
state 3: the on-hand inventory is two at the end of the period,
state 4: the on-hand inventory is three at the end of the period,
state 5: the on-hand inventory is four at the end of the period.

The transition matrix is :

$$P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

The steady-state probabilities are :

Steady State Distributions

i	P(i)
1	0.22916667
2	0.3333333
3	0.25
4	0.1875
5	0

Thus, P(0 unit on-hand)=P(1)=0.22916667,
P(1 unit on-hand)=P(2)=0.3333333,
P(2 units on-hand)=P(3)=0.25,
P(3 units on-hand)=P(4)=0.1875,
P(4 units on-hand)=P(5)=0.

(b.) Per period ordering cost=5[P(1)+P(2)]=2.81
Per period holding cost=2[P(2)+2p(3)+3P(4)+4P(5)]=2.79
Per period penalty cost=3(1/3)P(3)=0.25
Per period purchasing cost=0.5[4P(1)+3P(2)]=0.96
Thus, the total cost=6.81.

(c.) If all shortages are backlogged, we add one state (state 0: the on-hand inventory is -1) to the prevous state set. The transition becomes :

$$P = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

The steady state probabilities are :

Steady State Distributions

i	P(i)
0	0.083
1	0.146
2	0.333
3	0.25
4	0.188
5	0

P(0 unit on-hand)=P(1)=0.146,
P(1 unit on-hand)=P(2)=0.333,
P(2 units on-hand)=P(3)=0.25,
P(3 units on-hand)=P(4)=0.188,
P(4 units on-hand)=P(5)=0.

Per period ordering cost=5[P(0)+P(1)+P(2)]=2.81,
Per period holding cost=2[P(2)+2P(3)+3P(4)+4P(5)]=2.79,
Per period backlogging cost=0.5[5P(0)+4P(1)+3P(2)]=1,
Per period penalty cost=3(1/3)P(3)=0.25.
Thus, the total cost=7.01.

2. (#9, page 944 of the text) In the game of craps, we roll a pair of six-sided dice. On the first throw, if we roll a 7 or an 11, we win right away. If we roll a 2, 3, or 12, we lose right away. If we first roll a 4, 5, 6, 8, 9, or 10, we keep rolling the dice until we get either a 7 or the total rolled on the first throw. If we get a 7, we lose. If we roll the same total as the first throw, we win.

(a.) Model a crap game as a Markov chain with absorbing states. (Define the states, draw the transition diagram, and write the transition probability matrix.)

(b.) What is the probability of winning for the roller (the person rolling the dice)?

(c.) If the first roll is a "4", what is his/her probability of winning?

(d.) What is the expected number of rolls of the dice in a crap game?

(e.) If the first roll of the dice is a "4", how many additional rolls are expected before the game ends?

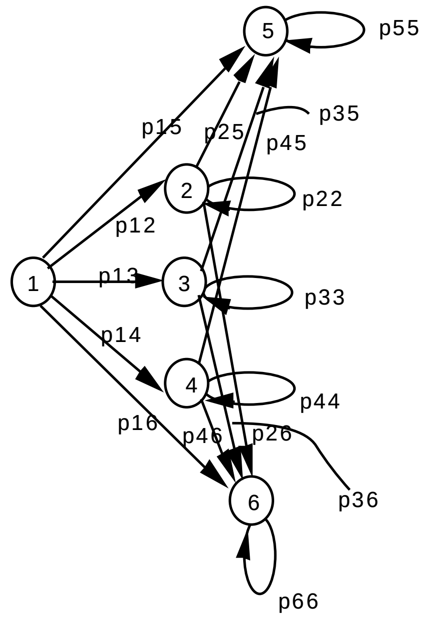
Solutions.

(a.) Define the states of this model as :
state 1 : start the game,
state 2 : roll a 4 or 10,
state 3 : roll a 5 or 9,
state 4 : roll a 6 or 8,
state 5 : win,
state 6 : loss.
Note: states 5 and 6 are absorbing states.

The transition probability matrix is :

$$P = \begin{bmatrix} 0 & 6/36 & 8/36 & 10/36 & 8/36 & 4/36 \\ 0 & 27/36 & 0 & 0 & 3/36 & 6/36 \\ 0 & 0 & 26/36 & 0 & 4/36 & 6/36 \\ 0 & 0 & 0 & 25/36 & 5/36 & 6/36 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The transition diagram is :



(b.) Using the MARKOV workspace (APL) in Macintosh SE, we obtain the following outputs :

A = Absorbing Probabilities

from	5	6
1	0.49292929	0.50707071
2	0.33333333	0.66666667
3	0.4	0.6
4	0.45454545	0.54545455

E = Expected NO. Visits to Transient States

from	1	2	3	4
1	1	0.6666667	0.8	0.90909091
2	0	4	0	0
3	0	0	3.6	0
4	0	0	0	3.2727273

Hence, the probability of winning is 0.49292929.

(c.) From above outputs, we have
P(win | first roll is 4)=P(absorbed by state 5 | state 2)=0.3333333.

(d.) Expected number of rolls=1+0.6666667+0.8+0.90909091=3.37576.

(e.) E(# of additional rolls | first roll is 4)=0+4+0+0+0=4.

Homework # 11

*For this homework assignment, you may either do the computations manually, or use the APL workspace BIRTH/DEATH which is on the ICAEN fileserver. (Requires that you have a copy of the APL*PLUS interpreter on a floppy disk.)*

1. Consider a manufacturing process in which raw parts (blanks) are machined on three machines, and inspected after each machining operation. (Cf. the class handout on a similar situation.) The relevant data is as follows:

Manufacturing System Parameters:

Station	Machine Operation			Inspection			
i	T	C	S	T	C	S	R
1	0.5	20	10	0.1	15	10	5
2	0.75	20	5	0.2	15	10	3
3	0.25	20	2	0.25	15	5	2

Pack & Ship: 0.1 hrs at 10 $/hr
Cost per blank: $50; Scrap Value: $10

T = time (hrs) per operation
C = cost ($/hr) of operation
S = scrap rate (%)
R = rework rate (%)

For example, machine #1 requires 0.5 hrs, at $20/hr., and has a 10% scrap rate. Those parts completing this operation are inspected, requiring 0.1 hr. at $15/hr. The inspector scraps 10%, and sends 5% back to machine #1 for rework (after which it is again inspected, etc.)

The Markov chain model of a part moving through this system has transition probability matrix:

Transition Probabilities

from \ to	1	2	3	4	5	6	7	8
1	0	0.9	0	0	0	0	0	0.1
2	0.05	0	0.85	0	0	0	0	0.1
3	0	0	0	0.95	0	0	0	0.05
4	0	0	0.03	0	0.87	0	0	0.1
5	0	0	0	0	0	0.98	0	0.02
6	0	0	0	0	0.02	0	0.93	0.05
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

a. Draw the diagram for this Markov chain and describe each state.
b. Which states are transient? which are absorbing?

The absorption probabilities are:

A = Absorption Probability Matrix

OK	Scrap
0.6335	0.3665
0.7039	0.2961
0.7909	0.2091
0.8325	0.1675
0.9296	0.07038
0.9486	0.05141

The matrix E is as follows:

E = Expected No. Visits

1	2	3	4	5	6
1.047	0.9424	0.8245	0.7833	0.6951	0.6812
0.05236	1.047	0.9162	0.8704	0.7723	0.7569
0	0	1.029	0.9779	0.8678	0.8504
0	0	0.03088	1.029	0.9134	0.8952
0	0	0	0	1.02	0.9996
0	0	0	0	0.0204	1.02

c. What percent of the parts which are started are successfully completed?
d. What is the expected number of blanks which should be required to fill an order for 100 completed parts?
e. What percent of the parts arriving at machine #2 will be successfully completed?
f. What is the expected total number of inspections which entering parts will undergo?
g. Explain the meaning of the number appearing in row 3, column 2 of the A matrix.
h. Explain the meaning of the number appearing in row 3, column 3 of the E matrix.
i. To fill the order for 100 completed parts, what is the expected man-hour requirement for each machine? for each inspection station?
j. What are the expected direct costs (row materials + operating costs - scrap value of rejected parts) per completed part?

2. My home uses only two light bulbs, which are turned on continuously. On average, a light bulb lasts for 20 days (exponentially distributed). When a light bulb fails, it takes me an average of 2 days (exponentially distributed) before I replace the bulb. (If both have failed at that time, I replace both of them.)

a. Formulate a three-state continuous-time Markov chain model of this situation.
b. Determine the fraction of the time that both light bulbs are working simultaneously.
c. Determine the fraction of the time that both light bulbs have failed simultaneously.

3. The college is trying to decide whether to rent a slow or fast copy machine. It is believed that an employee's time is worth \$15/hour. The slow copier rents for \$4 per hour, and it takes an employee an average of 10 minutes to complete a copy job (exponentially distributed). The fast copier rents for \$15 per hour, and it takes an employee an average of 6 minutes to complete a copy job (also exponentially distributed). An average of 4 employees per hour need to use the copying machines (interarrival times are exponentially distributed). Which machine should the college rent? (Justify your answer.)

◊◊◊◊◊◊◊◊ Homework #11 Solutions ◊◊◊◊◊◊◊◊

1. Consider a manufacturing process in which raw parts (blanks) are machined on three machines, and inspected after each machining operation. (Cf. the class handout on a similar situation.) The relevant data is as follows:

Manufacturing System Parameters:

Station	Machine Operation			Inspection			
i	T	C	S	T	C	S	R
1	0.5	20	10	0.1	15	10	5
2	0.75	20	5	0.2	15	10	3
3	0.25	20	2	0.25	15	5	2

Pack & Ship: 0.1 hrs at 10 $/hr
Cost per blank: $50; Scrap Value: $10

T = time (hrs) per operation
C = cost ($/hr) of operation
S = scrap rate (%)
R = rework rate (%)

For example, machine #1 requires 0.5 hrs, at $20/hr., and has a 10% scrap rate. Those parts completing this operation are inspected, requiring 0.1 hr. at $15/hr. The inspector scraps 10%, and sends 5% back to machine #1 for rework (after which it is again inspected, etc.)
The Markov chain model of a part moving through this system has transition probability matrix:

Transition Probabilities

from \ to	1	2	3	4	5	6	7	8
1	0	0.9	0	0	0	0	0	0.1
2	0.05	0	0.85	0	0	0	0	0.1
3	0	0	0	0.95	0	0	0	0.05
4	0	0	0.03	0	0.87	0	0	0.1
5	0	0	0	0	0	0.98	0	0.02
6	0	0	0	0	0.02	0	0.93	0.05
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

a. Draw the diagram for this Markov chain and describe each state.
b. Which states are transient? which are absorbing?

The absorption probabilities are:

A = Absorption Probability Matrix

OK	Scrap
0.6335	0.3665
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0.7909	0.2091
0.8325	0.1675
0.9296	0.07038
0.9486	0.05141

The matrix E is as follows:

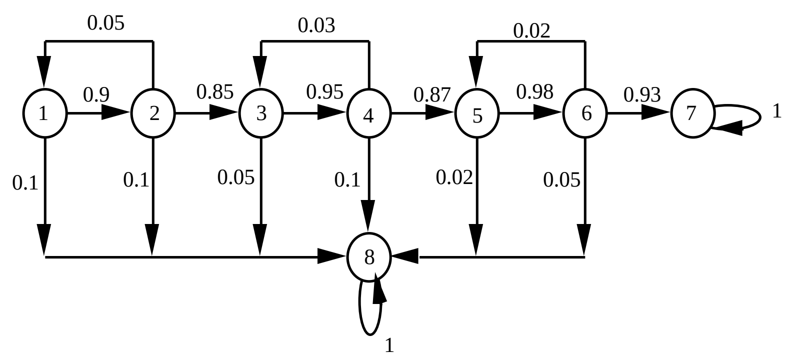
E = Expected No. Visits

1	2	3	4	5	6
1.047	0.9424	0.8245	0.7833	0.6951	0.6812
0.05236	1.047	0.9162	0.8704	0.7723	0.7569
0	0	1.029	0.9779	0.8678	0.8504
0	0	0.03088	1.029	0.9134	0.8952
0	0	0	0	1.02	0.9996
0	0	0	0	0.0204	1.02

c. What percent of the parts which are started are successfully completed?
d. What is the expected number of blanks which should be required to fill an order for 100 completed parts?
e. What percent of the parts arriving at machine #2 will be successfully completed?
f. What is the expected total number of inspections which entering parts will undergo?
g. Explain the meaning of the number appearing in row 3, column 2 of the A matrix.
h. Explain the meaning of the number appearing in row 3, column 3 of the E matrix.
i. To fill the order for 100 completed parts, what is the expected man-hour requirement for each machine? for each inspection station?
j. What are the expected direct costs (row materials + operating costs - scrap value of rejected parts) per completed part?

Solutions.

(a.)



State 1: machine operation, station 1. State 2: machine inspection, station 1.
State 3: machine operation, station 2. State 4: machine inspection, station 2.
State 5: machine operation, station 3 State 6: machine inspection, station 3.
State 7: pack and ship. State 8: scrap.

(b.) States 1,2,3,4,5, and 6 are transient, 7 and 8 are absorbing.

(c.) 63.35% (From absorbing probability matrix).

(d.) (1/0.6335)100=157.8 or 158.

(e.) 79.09% (From the row 3, column 1 of absorbing probability matrix).

(f.) 0.9424+0.7833+0.6812=2.4069 (From the first row of matrix E).

(g.) 0.2091 means that the probability of a part entering machine #2 going to be scraped.

(h.) 1.209 means that the expected number of visiting state 3 before being absorbed, given that the current state is 3.

(i.) Machine (1): 1.047(0.5/0.6335)(100)=82.64

Inspection (1): 1.047(0.1/0.7039)(100)=14.87
Machine (2): 1.029(0.75/0.7903)(100)=97.61
Inspection (2): 1.029(0.2/0.8325)(100)=24.73
Machine (3): 1.02(0.25/0.9296)(100)=27.43
Inspection (3): 1.02(0.25/0.9486)(100)=26.88
Pack & Ship: 1(0.1/1)(100)=10

Total=284.17

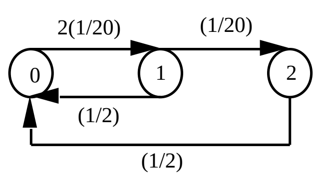
(j.) For a completed part :
Operating cost=20(0.8264+0.9761+0.2743)+15(0.1487+0.2473+0.2688)+10(0.1)=52.51
Material cost=50(1/0.6335)=78.93
Scrap recovered=10(1/0.6335)(0.3665)=5.785
Total direct cost=Operating cost + Material cost - Scrap recovered
=52.51+78.93-5.785=125.6

2. My home uses only two light bulbs, which are turned on continuously. On average, a light bulb lasts for 20 days (exponentially distributed). When a light bulb fails, it takes me an average of 2 days (exponentially distributed) before I replace the bulb. (If both have failed at that time, I replace both of them.)

a. Formulate a three-state continuous-time Markov chain model of this situation.
b. Determine the fraction of the time that both light bulbs are working simultaneously.
c. Determine the fraction of the time that both light bulbs have failed simultaneously.

Solutions.

(a.) Define state 1: two light bulbs are good,
state 2: one light bulb is good and another is broken,
state 3: two light bulbs are broken.



(b.& c.)
From the diagram in (a), we have

$(1/20)\pi 1=(1/2)\pi 2$, (balance equation for node 2)

$[(1/20)+(1/2)]\pi 1=2(1/20)\pi 0$, (balance equation for node 1)

$\pi 0+\pi 1+\pi 2=1$. (normalization equation)

Solving these three equations, we obtain the steady state probabilities as

$$(\pi 0,\pi 1,\pi 2)=(5/6,5/33,1/66)$$

Thus,

P(both light bulbs are working)=$\pi 0$=83.3333%

P(both light bulbs are broken)=$\pi 2$=1.5151%.

3. The college is trying to decide whether to rent a slow or fast copy machine. It is believed that an employee's time is worth $15/hour. The slow copier rents for $4 per hour, and it takes an employee an average of 10 minutes to complete a copy job (exponentially distributed). The fast copier rents for $15 per hour, and it takes an employee an average of 6 minutes to complete a copy job (also exponentially distributed). An average of 4 employees per hour need to use the copying machines (interarrival times are exponentially distributed). Which machine should the college rent? (Justify your answer.)

Solutions.

Slow copy machine : (M/M/1 model)

$\lambda=2,\ \mu=4$

$W=1/(\mu-\lambda)=0.5$ (hr/job)

Average cost per hour=(# jobs/hr)($15/hr)(hr/job)+($4/hr)=4(15)(0.5)+4=34.

Fast copy machine : (M/M/1 model)

$\lambda=4,\ \mu=10$

$W=1/(\mu-\lambda)=1/6$ (hr/job)

Average cost per hour=(# jobs/hr)($15/hr)(hr/job)+($15/hr)=4(15)(1/6)+15=25.

Thus, we choose the fast copy machine.

Homework # 12

1. *(Exercise 6, page 1083-1084 of text by Winston)* Bectol, Inc. is building a dam. A total of 10,000,000 cu ft of dirt is needed to construct the dam. A bulldozer is used to collect dirt for the dam. Then the dirt is moved via dumpers to the dam site. Only one bulldozer is available, and it rents for $100 per hour. Bectol can rent, at $40 per hour, as many dumpers as desired. Each dumper can hold 1000 cu ft of dirt. It takes an average of 12 minutes for the bulldozer to load a dumper with dirt, and it takes each dumper an average of five minutes to deliver the dirt to the dam and return to the bulldozer. Making appropriate assumptions about exponentiality so as to obtain a birth/death model, determine the optimal number of dumpers and the minimum total expected cost of moving the dirt needed to build the dam.

2. *(Exercise #2, page 975 of text by Winston, with numbers modified)* Suppose that a new car costs $12,000 and that the annual operating cost and resale value of the car are as shown in the table below:

Age of Car (years)	Resale Value	Operating Cost
1	$9000	$400 (year 1)
2	$8000	$600 (year 2)
3	$6000	$900 (year 3)
4	$4000	$1200 (year 4)
5	$3000	$1600 (year 5)
6	$2000	$2200 (year 6)

If I have a new car now, determine a replacement policy that minimizes the net cost of owning and operating a car for the next six years.

◊◊◊◊◊◊◊◊ Homework #12 Solutions ◊◊◊◊◊◊◊◊

1. (Exercise 6, page 1083-1084 of text by Winston) Bectol, Inc. is building a dam. A total of 10,000,000 cu ft of dirt is needed to construct the dam. A bulldozer is used to collect dirt for the dam. Then the dirt is moved via dumpers to the dam site. Only one bulldozer is available, and it rents for $100 per hour. Bectol can rent, at $40 per hour, as many dumpers as desired. Each dumper can hold 1000 cu ft of dirt. It takes an average of 12 minutes for the bulldozer to load a dumper with dirt, and it takes each dumper an average of five minutes to deliver the dirt to the dam and return to the bulldozer. Making appropriate assumptions about exponentiality so as to obtain a birth/death model, determine the optimal number of dumpers and the minimum total expected cost of moving the dirt needed to build the dam.

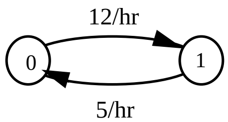
Solutions.

We have to use 10,000,000/1000=10,000 times of dumper to deliver all the dirt.

Case 1 : One dumper :

Define state 0 : no dumper in the system,

state 1 : one dumper in the system.



Steady-state Distribution

i	Pi	CDF
0	0.294118	0.294118
1	0.705882	1.000000

The average departure rate of dumper is $(1-\pi_0)5$=0.705882(5)=3.52941(times/hr)

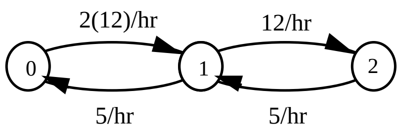
The total cost = (10,000/3.52941)($100+$40)=396667.

Case 2 : Two dumpers.

Define state 0 : no dumper in the system,

state 1 : one dumper in the system,

state 2 : two dumpers in the system, one is being served and another is waiting.



Steady-state Distribution

i	Pi	CDF
0	0.057737	0.057737
1	0.277136	0.334873
2	0.665127	1.000000

The total cost={10,000/[$(1-\pi_0)$(5)]}{$100+2($40)}=382059.

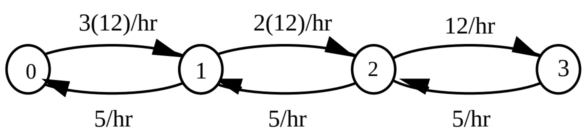
Case 3 : Three dumpers :

Define state 0 : no dumper in the system,

state 1 : one dumper in the system,

state 2 : two dumpers in the system, one is being served and another is waiting.

state 3 : three dumpers in the system, one is being served, and the other two are waiting.



Steady-state Distribution

i	Pi	CDF
0	0.007955	0.007955
1	0.057277	0.065233
2	0.659836	0.340164
3	0.659836	1.000000

The total cost = {10.000/[$(1-\pi_0)$(5)]}{$100+3($40)}=443528

Thus, the optimal number of dumpers is 2.

2. (Exercise #2, page 975 of text by Winston, with numbers modified) Suppose that a new car costs $12,000 and that the annual operating cost and resale value of the car are as shown in the table below:

Age of Car (years)	Resale Value	Operating Cost

1	$9000	$400 (year 1)
2	$8000	$600 (year 2)
3	$6000	$900 (year 3)
4	$4000	$1200 (year 4)
5	$3000	$1600 (year 5)
6	$2000	$2200 (year 6)

If I have a new car now, determine a replacement policy that minimizes the net cost of owning and operating a car for the next six years.

Solutions.

We use the dynamic programming to solve this problem.

Let G[x]=minimal total cost until end of time period 6, given a new machine at the end of time x.

Stage 6 : G[6]=0.

Stage 5 :

x	C	C+G
6	-8600	-8600

G[5]=-8600, and the optimal replace time for stage 5 is 6.

Stage 4 :

x	C	C+G
5	3400	-5200
6	-7000	-7000

G[4]=-7000, and the optimal replace time for stage 4 is 6.

Stage 3 :

x	C	C+G
4	3400	-3600
5	5000	-3600
6	-4100	-4100

G[3]=-4100, and the optimal replace time for stage 3 is 6.

Stage 2 :

x	C	C+G
3	3400	-700
4	5000	-2000
5	7900	-700
6	-900	-900

G[2]=-2000, and the optimal replace time for stage 2 is 4.

Stage 1 :

x	C	C+G
2	3400	1400
3	5000	900
4	7900	900
5	11100	2500
6	1700	1700

G[1]=900, and the optimal replace time for stage 1 is 3.

Stage 0 :

x	C	C+G
1	3400	4300
2	5000	3000

3	7900	3800
4	11100	4100
5	13700	5100
6	4900	4900

G[0]=3000, and the optimal replace time for stage 0 is 2.

Summary

t	x	G
0	2	3000
1	3	900
2	4	-2000
3	6	-4100
4	6	-7000
5	6	-8600
6	0	0

Thus the optimal policy is to replace the car when it is two years old, and the minimal total cost=3,000.

Homework # 13

1. A system consists of 4 devices, each subject to possible failure, all of which must function in order for the system to function. In order to increase the reliability of the system, redundant units may be included, so that the system continues to function if at least one of the redundant units remains functional. The data are:

Device	Reliability (%)	Weight (kg.)
1	80	1
2	90	3
3	75	1
4	85	2

If we include a single unit of each device, then the system reliability will be only 45.9%.

a. Explain how it is determined that the reliability is only 45.9%.

However, by including redunant units of one or more devices, we can substantially increase the reliability.

Suppose that the system may weigh no more than 12 kg. (Since at least one of each device *must* be included, a total of 7 kg, this leaves 5 kg available for redundant units.) Assume that no more than 3 units of any type need be considered. We wish to compute the number of units of each device type to be installed in order to maximize the system reliability, subject to the maximum weight restriction.

The dynamic programming model arbitrarily assumes that the devices are considered in the order: #4, #3, #2, and finally, #1. The optimal value function is defined to be:

$F_n(S)$ = maximum reliability which can be achieved for devices #n, n-1, ... 1, given that the weight used by these devices cannot exceed S *(the state variable)*

Note that the computation is done in the backward order, i.e., first the optimal value function $F_1(S)$ is computed for each value of the available weight S, then $F_2(S)$, etc., until finally $F_4(S)$ has been computed.

Data

Weight

i	1	2	3	4
Wt[i]	1	3	1	2

Reliability (%) vs # redundant units:

i	1	2	3
1	80	96	99.2
2	90	99	99.9
3	75	93.75	98.4375
4	85	97.75	99.6625

b. Explain the computation of the 97.75% reliability for 2 units of device #4 above.

Details of Computations at each Stage:

———Stage 1———

s \ x:	1	2	3
1	0.80	¯99999999.99	¯99999999.99
2	0.80	0.96	¯99999999.99
3	0.80	0.96	0.99
4	0.80	0.96	0.99
5	0.80	0.96	0.99
6	0.80	0.96	0.99
7	0.80	0.96	0.99
8	0.80	0.96	0.99
9	0.80	0.96	0.99
10	0.80	0.96	0.99
11	0.80	0.96	0.99
12	0.80	0.96	0.99

———Stage 3———

s \ x:	1	2	3
5	0.54	¯99999999.99	¯99999999.99
6	0.65	0.68	¯99999999.99
7	0.67	0.81	0.71
8	0.67	0.84	0.85
9	0.71	0.84	0.88
10	0.74	$\boxed{}$	0.88
11	0.74	0.92	0.94
12	0.74	0.92	0.97

———Stage 2———

s \ x:	1	2	3
4	0.72	¯99999999.99	¯99999999.99
5	0.86	¯99999999.99	¯99999999.99
6	0.89	¯99999999.99	¯99999999.99
7	0.89	0.79	¯99999999.99
8	0.89	0.95	¯99999999.99
9	0.89	0.98	¯99999999.99
10	0.89	0.98	0.80
11	0.89	0.98	0.96
12	0.89	0.98	0.99

———Stage 4———

s \ x:	1	2	3
7	0.46	¯99999999.99	¯99999999.99
8	0.57	¯99999999.99	¯99999999.99
9	0.69	0.53	¯99999999.99
10	0.72	0.66	¯99999999.99
11	0.75	0.79	0.54
12	0.76	0.83	0.67

Optimal values & decisions at each stage:

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
7	0.46	1	5
8	0.57	1	6
9	0.69	1	7
10	0.72	1	8
11	0.79	2	7
12	0.83	2	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
4	0.72	1	1
5	0.86	1	2
6	0.89	1	3
7	0.89	1	4
8	0.95	2	2
9			
10	0.98	2	4
11	0.98	2	5
12	0.99	3	3

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
5	0.54	1	4
6	0.68	2	4
7	0.81	2	5
8	0.85	3	5
9	0.88	3	6
10	0.89	2	8
11	0.94	3	8
12	0.97	3	9

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
1	0.80	1	0
2	0.96	2	0
3	0.99	3	0
4	0.99	3	1
5	0.99	3	2
6	0.99	3	3
7	0.99	3	4
8	0.99	3	5
9	0.99	3	6
10	0.99	3	7
11	0.99	3	8
12	0.99	3	9

c. What is the maximum reliability that can be achieved allowing 12 kg. total weight?

d. How many units of each device should be included in the system?

e. Four values have been blanked out in the output. What are they?

i. the optimal value $f_2(9)$ _____________

ii. the optimal decision $x_2^*(9)$ _____________

iii. the state which results from the optimal decision $x_2^*(9)$ _____________

iv. the value associated with the decision to include 2 units of device #3, given that 10 kg. is still available _____________

f. Suppose that only 11 kg. of capacity were available. What is the achievable system reliability? How many units of each device should be included?

2. We wish to plan production of an expensive, low-demand item for the next three months (January, February, & March).

- the cost of production is $15 for setup, plus $5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is $2 per unit, based upon the level at the beginning of the month.

- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand each month is random, with the same probability distribution:

d	0	1	2
P{D=d}	0.3	0.4	0.3

- there is a penalty of $25 per unit for any demand which cannot be satisfied. Backorders are not allowed.
- the inventory at the end of December is 1.
- a salvage value of $4 per unit is received for any inventory remaining at the end of the last month (March)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 3 = January, stage 2 = February, etc.** (i.e., n = # months remaining in planning period.)

a. What is the optimal production quantity for January? _______

b. What is the total expected cost for the three months? __________

c. If, during January, the demand is 1 unit, what should be produced in February? ______

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ ___________

ii. the optimal decision $x_2^*(1)$ ___________

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ____________

The table of costs for each combination of state & decision at stage 2 is:

---STAGE 2---

s \ x:	0	1	2	3	4
0	46.00		34.62	31.89	33.60
1	26.69	31.62	28.89	30.60	35.00
2	13.62	25.89	27.60	32.00	37.00
3	7.89	24.60	29.00	34.00	39.00

The tables of the optimal value function $f_n(S_n)$ at each stage are:

```
---------------------------------------
Stage 3:
              Optimal    Optimal
       State  Values    Decisions
         0     44.61        4
         1     39.83        0
         2     28.33        0
         3     21.82        0
---------------------------------------
Stage 2:
              Optimal    Optimal
       State  Values    Decisions
         0     31.89        3
         1     [     ]     [ ]
         2     13.62        0
         3      7.89        0
---------------------------------------
Stage 1:
              Optimal    Optimal
       State  Values    Decisions
         0     21.00        2
         1      8.30        0
         2      0.00        0
         3     ¯2.00        0
```

◊◊◊◊◊◊◊◊ Homework #13 Solutions ◊◊◊◊◊◊◊◊

1. A system consists of 4 devices, each subject to possible failure, all of which must function in order for the system to function. In order to increase the reliability of the system, redundant units may be included, so that the system continues to function if at least one of the redundant units remains functional. The data are:

Device	Reliability (%)	Weight (kg.)
1	80	1
2	90	3
3	75	1
4	85	2

If we include a single unit of each device, then the system reliability will be only 45.9%.

a. Explain how it is determined that the reliability is only 45.9%.

Solution.

(0.8)(0.9)(0.75)(0.85)=0.459.

However, by including redundant units of one or more devices, we can substantially increase the reliability.

Suppose that the system may weigh no more than 12 kg. (Since at least one of each device *must* be included, a total of 7 kg, this leaves 5 kg available for redundant units.) Assume that no more than 3 units of any type need be considered. We wish to compute the number of units of each device type to be installed in order to maximize the system reliability, subject to the maximum weight restriction.

The dynamic programming model arbitrarily assumes that the devices are considered in the order: #4, #3, #2, and finally, #1. The optimal value function is defined to be:

$F_n(S)$ = maximum reliability which can be achieved for devices #n, n-1, ... 1, given that the weight used by these devices cannot exceed S *(the state variable)*

Note that the computation is done in the backward order, i.e., first the optimal value function $F_1(S)$ is computed for each value of the available weight S, then $F_2(S)$, etc., until finally $F_4(S)$ has been computed.

Data

Weight

i	1	2	3	4
Wt[i]	1	3	1	2

Reliability (%) vs # redundant units:

i	1	2	3
1	80	96	99.2
2	90	99	99.9
3	75	93.75	98.4375
4	85	97.75	99.6625

b. Explain the computation of the 97.75% reliability for 2 units of device #4 above.

Solution. 1- (1- 0.85)(1- 0.85)=0.9775 or 97.75%.

Details of Computations at each Stage:

Stage 1

s \ x:	1	2	3
1	0.80	¯99999999.99	¯99999999.99
2	0.80	0.96	¯99999999.99
3	0.80	0.96	0.99
4	0.80	0.96	0.99
5	0.80	0.96	0.99
6	0.80	0.96	0.99
7	0.80	0.96	0.99
8	0.80	0.96	0.99
9	0.80	0.96	0.99
10	0.80	0.96	0.99
11	0.80	0.96	0.99
12	0.80	0.96	0.99

Stage 3

s \ x:	1	2	3
5	0.54	¯99999999.99	¯99999999.99
6	0.65	0.68	¯99999999.99
7	0.67	0.81	0.71
8	0.67	0.84	0.85
9	0.71	0.84	0.88
10	0.74	$\boxed{0.89}$	0.88
11	0.74	0.92	0.94
12	0.74	0.92	0.97

Stage 2

s \ x:	1	2	3
4	0.72	¯99999999.99	¯99999999.99
5	0.86	¯99999999.99	¯99999999.99
6	0.89	¯99999999.99	¯99999999.99
7	0.89	0.79	¯99999999.99
8	0.89	0.95	¯99999999.99
9	0.89	0.98	¯99999999.99
10	0.89	0.98	0.80
11	0.89	0.98	0.96
12	0.89	0.98	0.99

Stage 4

s \ x:	1	2	3
7	0.46	¯99999999.99	¯99999999.99
8	0.57	¯99999999.99	¯99999999.99
9	0.69	0.53	¯99999999.99
10	0.72	0.66	¯99999999.99
11	0.75	0.79	0.54
12	0.76	0.83	0.67

Optimal values & decisions at each stage:

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
7	0.46	1	5
8	0.57	1	6
9	0.69	1	7
10	0.72	1	8
11	0.79	2	7
12	0.83	2	8

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
4	0.72	1	1
5	0.86	1	2
6	0.89	1	3
7	0.89	1	4
8	0.95	2	2
9	0.98	2	3
10	0.98	2	4
11	0.98	2	5
12	0.99	3	3

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
5	0.54	1	4
6	0.68	2	4
7	0.81	2	5
8	0.85	3	5
9	0.88	3	6
10	0.89	2	8
11	0.94	3	8
12	0.97	3	9

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
1	0.80	1	0
2	0.96	2	0
3	0.99	3	0
4	0.99	3	1
5	0.99	3	2
6	0.99	3	3
7	0.99	3	4
8	0.99	3	5
9	0.99	3	6
10	0.99	3	7
11	0.99	3	8
12	0.99	3	9

c. What is the maximum reliability that can be achieved allowing 12 kg. total weight?

Solution. 0.83.

d. How many units of each device should be included in the system?

Solution.

Device #	# of units
1	1
2	1
3	3
4	2

e. Four values have been blanked out in the output. What are they?

i. the optimal value $f_2(9)$ ___0.98_____

ii. the optimal decision $x_2^*(9)$ ____2_____

iii. the state which results from the optimal decision $x_2^*(9)$ _____3_____

iv. the value associated with the decision to include 2 units of device #3, given that 10 kg. is still available ______0.98____

f. Suppose that only 11 kg. of capacity were available. What is the achievable system reliability? How many units of each device should be included?

Solution. Reliability=0.79.

Device #	# of units
1	1
2	1
3	2
4	2

2. We wish to plan production of an expensive, low-demand item for the next three months (January, February, & March).

- the cost of production is $15 for setup, plus $5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is $2 per unit, based upon the level at the beginning of the month.
- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand each month is random, with the same probability distribution:

d	0	1	2
P{D=d}	0.3	0.4	0.3

- there is a penalty of $25 per unit for any demand which cannot be satisfied. Backorders are not allowed.
- the inventory at the end of December is 1.
- a salvage value of $4 per unit is received for any inventory remaining at the end of the last month (March)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 3 = January, stage 2 = February, etc.** (i.e., n = # months remaining in planning period.)

a. What is the optimal production quantity for January? ___0____

b. What is the total expected cost for the three months? ___39.83____

c. If, during January, the demand is 1 unit, what should be produced in February? ___3__

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ ____26.69____

ii. the optimal decision $x_2^*(1)$ ___0_____

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ____44.69____

Solution.

P(d)	d	inventory cost	production cost	f_1^*+penalty
0.3	0	0	15+5	8.3
0.4	1	0	15+5	21
0.3	2	0	15+5	21+25

$(0.3)(15+5+8.3)+(0.4)(15+5+21)+(0.3)(15+5+25+21)=44.69.$

The table of costs for each combination of state & decision at stage 2 is:

---STAGE 2---

s \ x:	0	1	2	3	4
0	46.00	$\boxed{44.69}$	34.62	31.89	33.60
1	26.69	31.62	28.89	30.60	35.00
2	13.62	25.89	27.60	32.00	37.00
3	7.89	24.60	29.00	34.00	39.00

The tables of the optimal value function $f_n(S_n)$ at each stage are:

Stage 3:

State	Optimal Values	Optimal Decisions
0	44.61	4
1	39.83	0
2	28.33	0
3	21.82	0

Stage 2:

State	Optimal Values	Optimal Decisions
0	31.89	3
1	$\boxed{26.69}$	$\boxed{0}$
2	13.62	0
3	7.89	0

Stage 1:

State	Optimal Values	Optimal Decisions
0	21.00	2
1	8.30	0
2	0.00	0
3	¯2.00	0

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