56:171 Operations Research
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• ***Write your name on the first page, and initial the other pages.***
• ***Answer both questions of Part One, and 2 problems from Part Two.***
• ***Any questions remaining may be considered a "take-home" exam, for ½ credit, making maximum 90.***

		Possible	Score
Part One:	1. True/False	15	_____
	2. Sensitivity analysis (LINDO)	25	_____
Part Two:	3. Geometry & Duality of LP	20	_____
	4. Decision Analysis	20	_____
	5. Transportation & Assignment problems	20	_____
	total:	80	_____

PART ONE

(1.) ***True/False:*** Indicate by "+" or "o" whether each statement is "true" or "false", respectively:

_+__ a. If the optimal value of a slack variable of a primal LP constraint is positive, then the optimal value of the dual variable for that same constraint must equal zero.

_+___ b. In reference to LP, the terms "dual variable", "shadow price", and "simplex multiplier" are synonymous.

_+___ c. If you make a mistake in choosing the pivot row in the simplex method, the next basic solution will have one or more negative basic variables.

_+___ d. If the primal LP feasible region is nonempty and bounded, then the dual LP cannot be unbounded nor infeasible.

_o ___ e. The number of basic variables of a transportation problem with m sources and n destinations is m+n+1.

_o ___ f. If the current basis is not degenerate, the dual variables at any iteration of the simplex method for solving a transportation problem are uniquely determined.

_o ___ g. If a basic feasible solution of a transportation problem is degenerate, the next iteration cannot result in an improvement of the objective.

_o ___ h. The two-phase simplex method solves for the dual variables in phase one, and then solves for the primal variables in phase two.

_o ___ i. If you make a mistake in choosing the pivot column in the simplex method, the next basic solution will be infeasible.

_+___ j. If there is a tie in the minimum ratio test of the simplex method, the tableau that follows will be degenerate.

_+___ k. During a change of basis in the simplex method for the transportation problem, the "substitution rates" are all +1, 0, or -1.

_o ___ l. If a slack variable of a primal LP constraint is zero in the optimal solution, then there is a corresponding dual variable whose optimal value is also zero.

_+___ m. In a transportation problem with 4 sources and 6 destinations, with total supply exceeding total demand, the number of basic variables will be 10.

_o__ n. If the right-hand-side of a "greater-than-or-equal" constraint is increased, the objective function will either remain the same or improve.

_+___ o. The "complementary slackness condition" of LP implies that in the output of the optimal solution, either the slack (or surplus) in a constraint or its dual variable (or both) must be zero.

2. *Sensitivity Analysis in LP.* Consult the LINDO output to answer the questions below:

- During the next two months, General Cars must meet (on time) the following demands for trucks and cars: Month 1: 400 trucks, 800 cars; Month 2: 300 trucks, 300 cars.
- During each month, at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel.
- During month 1, steel costs $400 per ton; during month 2, steel costs $600 per ton.

- At most 1500 tons of steel may be purchased each month (steel may only be used during the month in which it is purchased).
- At the beginning of month 1, 100 trucks and 200 cars are in inventory.
- At the end of each month, a holding cost of $150 per vehicle is assessed.
- Each car gets 20 mpg (miles per gallon), and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg.

The company wishes to meet the demand and mileage requirements at minimum cost (including steel costs and holding costs).

Define variables:

C1 = number of cars to be produced in month 1
C2 = number of cars to be produced in month 2
T1 = number of trucks to be produced in month 1
T2 = number of trucks to be produced in month 2
S1 = tons of steel used in month 1
S2 = tons of steel used in month 2
IC1 = number of cars in inventory at end of month 1
IT1 = number of trucks in inventory at end of month 1
IC2 = number of cars in inventory at end of month 2
IT2 = number of trucks in inventory at end of month 2

LINDO output:

```
  MIN     400 S1 + 600 S2 + 150 IC1 + 150 IT1 + 150 IC2 + 150 IT2
  SUBJECT TO
         2)   C1 + T1 <=   1000
         3)   C2 + T2 <=   1000
         4) - S1 + C1 + 2 T1 =    0
         5) - S2 + C2 + 2 T2 =    0
         6) - IC1 + C1 >=   600
         7) - IT1 + T1 >=   300
         8)   IC1 - IC2 + C2 >=   300
         9)   IT1 - IT2 + T2 >=   300
        10)   4 C1 - 6 T1 >=   0
        11)   4 C2 - 6 T2 >=   0
  END
  SUB       S1     1500.00000     ! Note simple upper bounds on S1 & S2
  SUB       S2     1500.00000

 LP OPTIMUM FOUND AT STEP      8

        OBJECTIVE FUNCTION VALUE
        1)      995000.0

  VARIABLE        VALUE          REDUCED COST
        S1      1400.000000          0.000000
        S2       700.000000          0.000000
       IC1         0.000000          0.000000
       IT1       100.000000          0.000000
       IC2         0.000000        750.000000
       IT2         0.000000       1350.000000
        C1       600.000000          0.000000
        T1       400.000000          0.000000
        C2       300.000000          0.000000
        T2       200.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)         0.000000        130.000000
        3)       500.000000          0.000000
        4)         0.000000        400.000000
        5)         0.000000        600.000000
        6)         0.000000       -450.000000
        7)         0.000000      -1050.000000
        8)         0.000000       -600.000000
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9)	0.000000	-1200.000000
10)	0.000000	-20.000000
11)	0.000000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
S1	400.000000	92.857147	INFINITY
S2	600.000000	INFINITY	92.857147
IC1	150.000000	216.666656	200.000000
IT1	150.000000	200.000000	INFINITY
IC2	150.000000	INFINITY	750.000000
IT2	150.000000	INFINITY	1350.000000
C1	0.000000	216.666656	200.000000
T1	0.000000	200.000000	INFINITY
C2	0.000000	200.000000	216.666656
T2	0.000000	INFINITY	200.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	1000.000000	71.428574	0.000000
3	1000.000000	INFINITY	500.000000
4	0.000000	1400.000000	100.000000
5	0.000000	700.000000	800.000000
6	600.000000	0.000000	0.000000
7	300.000000	0.000000	200.000000
8	300.000000	500.000000	0.000000
9	300.000000	0.000000	200.000000
10	0.000000	0.000000	0.000000
11	0.000000	0.000000	INFINITY

THE TABLEAU

ROW	(BASIS)	S1	S2	IC1	IT1	IC2	IT2
1	ART	0.000	0.000	0.000	0.000	750.000	1350.000
2	IC1	0.000	0.000	1.000	0.000	0.000	0.000
3	SLK 3	0.000	0.000	0.000	0.000	1.000	1.000
4	S1	1.000	0.000	0.000	0.000	0.000	0.000
5	S2	0.000	1.000	0.000	0.000	-1.000	-2.000
6	C1	0.000	0.000	0.000	0.000	0.000	0.000
7	T1	0.000	0.000	0.000	0.000	0.000	0.000
8	C2	0.000	0.000	0.000	0.000	-1.000	0.000
9	SLK 11	0.000	0.000	0.000	0.000	-4.000	6.000
10	IT1	0.000	0.000	0.000	1.000	0.000	0.000
11	T2	0.000	0.000	0.000	0.000	0.000	-1.000

ROW	C1	T1	C2	T2	SLK 2	SLK 3	SLK 6
1	0.000	0.000	0.000	0.000	130.000	0.000	450.000
2	0.000	0.000	0.000	0.000	0.600	0.000	1.000
3	0.000	0.000	0.000	0.000	1.000	1.000	1.000
4	0.000	0.000	0.000	0.000	1.400	0.000	0.000
5	0.000	0.000	0.000	0.000	-1.400	0.000	-1.000
6	1.000	0.000	0.000	0.000	0.600	0.000	0.000
7	0.000	1.000	0.000	0.000	0.400	0.000	0.000
8	0.000	0.000	1.000	0.000	-0.600	0.000	-1.000
9	0.000	0.000	0.000	0.000	0.000	0.000	-4.000
10	0.000	0.000	0.000	0.000	0.400	0.000	0.000
11	0.000	0.000	0.000	1.000	-0.400	0.000	0.000

ROW	SLK 7	SLK 8	SLK 9	SLK 10	SLK 11	RHS
1	0.10E+04	0.60E+03	0.12E+04	20.	0.00E+00	-0.10E+07
2	0.000	0.000	0.000	-0.100	0.000	0.000
3	1.000	1.000	1.000	0.000	0.000	500.000
4	0.000	0.000	0.000	0.100	0.000	1400.000

5	-2.000	-1.000	-2.000	-0.100	0.000	700.000
6	0.000	0.000	0.000	-0.100	0.000	600.000
7	0.000	0.000	0.000	0.100	0.000	400.000
8	0.000	-1.000	0.000	0.100	0.000	300.000
9	6.000	-4.000	6.000	1.000	1.000	0.000
10	1.000	0.000	0.000	0.100	0.000	100.000
11	-1.000	0.000	-1.000	-0.100	0.000	200.000

a. Suppose that the cost of steel in month 1 were to increase by $50/ton. Would the production plan need to be revised? _NO_

b. Would the production plan need to be revised if the cost of steel in month 1 were to increase by $100/ton? _YES

c. Suppose that the holding cost of vehicles is increased to $160/month. Should the production plan be revised? _____ *Note: cannot be determined without using "100%-rule" (see Winston's text)*

d. If the demand for trucks in month 1 were to increase by 10, what would be the effect on the total cost? $10,500 (increase or decrease?)

d. By using the *substitution rates* in the tableau, determine what would be the effect on the production plan if the demand for trucks in month 1 were to increase by 10.
Hint: What nonbasic variable would be changed by 10, and in which direction?
Answer: SLK_7 will increase!

	Var.	Change	direction
tons of steel used in month 1	S1	_____	increase? Decrease?
# of cars to be produced in month 1	C1	_____	increase? Decrease?
# of trucks to be produced in month 1	T1	_____	increase? Decrease?
# of cars in inventory, end of month 1	IC1	_____	increase? Decrease?
# of trucks in inventory, end of month 1	IT1	_10__	increase? **Decrease**?
tons of steel used in month 2	S2	_20__	**increase**? Decrease?
# of cars to be produced in month 2	C2	_____	increase? Decrease?
# of trucks to be produced in month 2	T2	_10__	**increase**? Decrease?

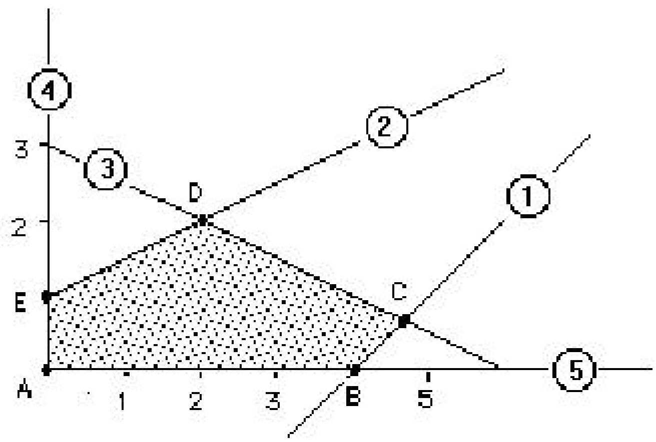
PART TWO

3. Geometry & Duality of the Linear Programming. Consider the following LP problem:

Maximize $3X_1 + 2X_2$

subject to
$X_1 - X_2 \leq 4$ (1)
$-X_1 + 2X_2 \leq 2$ (2)
$X_1 + 2X_2 \leq 6$ (3)
$X_1 \geq 0$ (4)
$X_2 \geq 0$ (5)

Let x_3, x_4, & x_5 be the slack variables for constraints (1)-(3). Below is a graph of the feasible region:



(a.) The feasible region is a polyhedron with 5 edges. Indicate which constraint defines each edge by labeling the edges (in the circles) on the graph, using the numbers (1) through (5) to the right of the constraints above. Note: X_1=0 corresponds to the vertical axis, X_2=0 to the horizontal axis!

(b.) How many basic variables must this LP problem have? __3 (plus the objective, -z)__

(c.) Which variables are basic at the extreme point labeled (D)? _X_1, X_2, X_3 (& objective –z)_

(d.) Suppose that during the simplex method, a move is made from the extreme point labeled (D), i.e., X=(2,2), to the extreme point labeled (C), i.e., X = ($^{14}/_3$, $^2/_3$). Which variable entered the basis? __X_4_ Which left the basis? __X_3___

(e.) What is the total number of basic solutions of the system? _10_ *(binomial coefficient, # of combinations of 5 objects, 3 at a time)*
How many of these are feasible? _5_ How many are infeasible? __5__ (Do NOT compute them!)

(f.) Write the dual of the LP above, using variables Y_1, Y_2, etc.

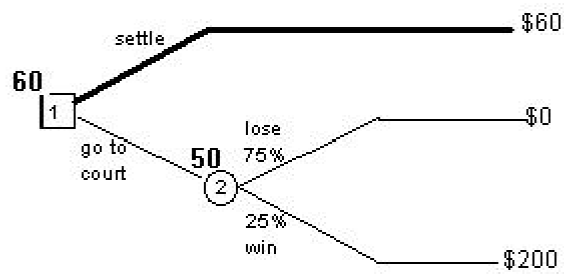
Given: Point C is optimal, with objective value 15 $^1/_3$.

(g.) What can be said about the optimal values of the dual variables?

Y_1 __ must be zero __ must be nonzero _X undetermined
Y_2 _X must be zero __ must be nonzero __ undetermined
Y_3 __ must be zero __ must be nonzero _X undetermined
Y_4 _X must be zero __ must be nonzero __ undetermined
Y_5 _X must be zero __ must be nonzero __ undetermined
Y_6 __ must be zero __ must be nonzero __ undetermined

Note: Use complementary slackness conditions:
X_1>0 ➔ zero slack or surplus in 1st dual constraint,i.e., Y_4=0. Likewise, X_2>0 ➔ Y_5=0
Positive slack in 2nd primal constraint ➔ 2nd dual variable (Y_2) = 0.
If one variable of the pair is zero, the corresponding variable may be either zero or positive. For example, the slack is zero in the 1st primal constraint, but this does not imply that Y_1 must be positive!

4. Decision Analysis. General Custard Corporation is being sued by Sue Smith. Sue can settle out of court and win $60,000, or she can go to court. If she goes to court, there is a 25% chance that she will win the case *(event W)* and a 75% chance she will lose *(event L).* If she wins, she will receive $200,000, and if she loses, she will net $0. A decision tree representing her situation appears below, where payoffs are in thousands of dollars:



__a__1. What is the decision which maximizes the expected value?

a. settle b. go to court

For $20,000, Sue can hire a consultant who will predict the outcome of the trial, i.e., either he predicts a loss of the suit *(event PL)*, or he predicts a win *(event PW)*. The consultant is correct 80% of the time.

__c__ 2. The probability that the consultant will predict a win, i.e. P{PW} is *(choose nearest value)*

a. ≤25% b. 30% c. 35%
d. 40% e. 45% f. ≥ 50%

$$P\{PW\} = P\{PW \; W\}P\{W\} + P\{PW \; L\}P\{L\}$$
$$= (0.8)(0.25) + (0.2)(0.75) = 0.35$$

Bayes' Rule states that if S_i is one of the *n* states of nature and O_j is the outcome of an experiment,

$$P\left\{S_i \middle| O_j\right\} = \frac{P\left\{O_j \middle| S_i\right\} P\left\{S_i\right\}}{P\left\{O_j\right\}}, \text{ where } P\left\{O_j\right\} = \sum_{k=1}^{n} P\left\{O_j \middle| S_k\right\} P\left\{S_k\right\}$$

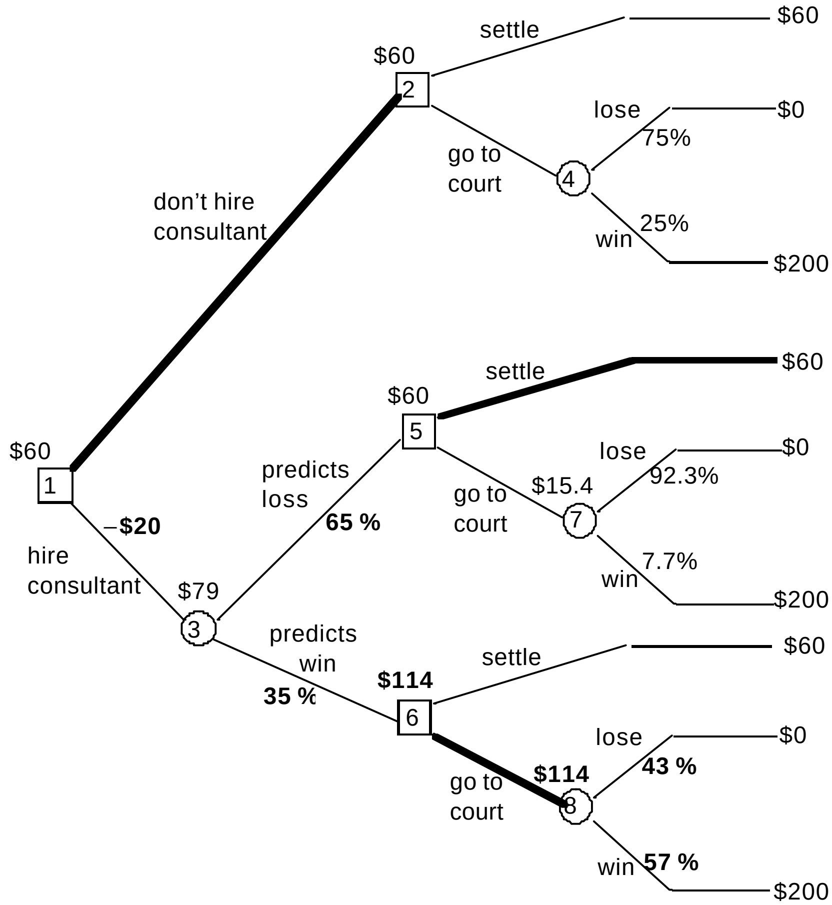
__d__ 3. According to Bayes' theorem, the conditional probability that, *if* the consultant predicts a win, then in fact Sue *will* win, i.e. P{W | PW}, is *(choose nearest value)*

a. ≤30% b. 40% c. 50 d. 60%
e. 70% f. 80% g. ≥ 90%

$$P\{W \mid PW\} = \frac{P\{PW \mid W\}P\{W\}}{P\{PW\}} = \frac{(0.8)(0.25)}{0.35} = 0.5714$$

The decision tree below includes Sue's decision as to whether or not to hire the consultant. *Note that the consultant's fee have not yet been deducted from the "payoffs" on the far right.*

4. Write the probabilities on the branches emanating from nodes 3 and 8.



Note that some of the nodes have been "folded back".

5. Should Sue hire the consultant? *Circle:* Yes [No]

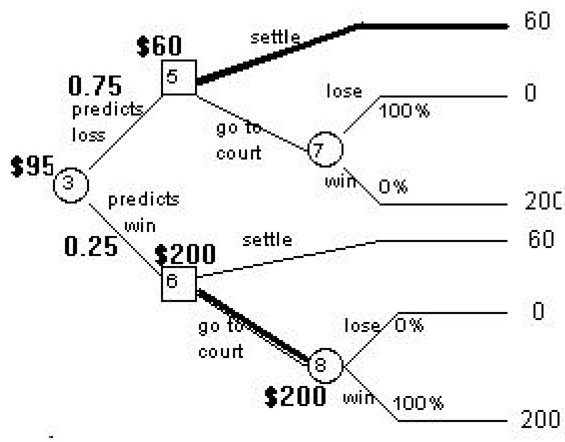
_d__ 6. The expected value of the consultant's opinion is (in <u>thousands</u> of $) *(Choose nearest value)*:

a. ≤16 b. 17 c. 18 d. 19

e. 20 f. 21 g. 22 h. ≥23

Note: the value is the difference between the expected payoff <u>with</u> the information ($79) minus the expected payoff <u>without</u> the information ($60)

Suppose that "perfect information" were given to Sue at no cost, i.e., a prediction which is *100% accurate*, so that the portion of the tree containing nodes 3, 5, 6, 7, & 8 would appear as below:



Note: if the prediction is <u>*perfect*</u>*, the probability that a win is predicted is 25% (the prior probability), etc.*

__h__ 7. What would be the expected value of node 3? *(Choose nearest value, in thousands of \$)*

a. ≤10	b. 15	c. 20	d. 25
e. 30	f. 35	g. 40	h. ≥45

__f__ 8. What would be the expected value of perfect information (EVPI)? *(Choose nearest value, in thousands of \$)*

a. ≤10	b. 15	c. 20	d. 25
e. 30	f. 35	g. 40	h. ≥45

Note: The expected value of perfect information is the difference between the values with and without that information, i.e., \$95 - \$60 = \$35.

5. ***(a) Transportation Problem.*** The following is a transportation tableau, with an initial set of shipments indicated:

destinations

[redacted]

				supply
4 [9]	14 [7]	[12]	[6]	18
[15]	[14]	4 [12]	[15]	4
2 [8]	[9]	4 [6]	[12]	6
[14]	[12]	7 [11]	5 [12]	12
demand: 6	14	15	5	

a. Is the solution above basic? _YES *The number of basic variables should be m+n-1=4+4-1=7, and there are exactly seven positive shipments indicated.*

c. Complete the computation of a set of dual variables for the above transportation tableau:

Dual variables for supply constraints: $U_1 =$ 0, $U_2 =$ __**5**_, $U_3 =$ __-1__, $U_4 =$ _**4**_

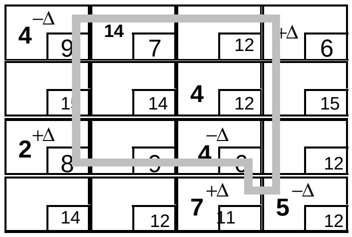
Dual variables for demand constraints: $V_1 =$ _9_ , $V_2 =$ _7_, $V_3 =$ _7_, $V_4 =$ _**8**_

c. Compute the reduced costs for X_{14} _–2_ & X_{32} _+3_ *Note: reduced cost of X_{ij} is $C_{ij} - (U_i + V_j)$*

d. Is the above solution optimal? *No, because there is (at least) one variable (X_{14}) with a negative reduced cost.*

e. If <u>not</u> optimal, perform one iteration to improve the solution, and write the result below:

Note: the nonbasic shipment in row 1, column 4, is increased by Δ, which requires the adjustments in the other cells as shown ([1,1], [3,1], [3,3], [4,3], and [4,4]). The increase is "blocked" when Δ reaches the value 4, and two of the basic shipments [1,1] and [3,3] simultaneously drop to zero. (Only one of these two shipments leave the basis, however, since the number of basic variables must always remain seven in this problem.)



9	**14** 7	12	**4** 6
15	14	**4** 12	15
6 8	9	**0** 6	12
14	12	**11** 11	**1** 12

(b) ***Assignment Problem.*** **Three machines are to be assigned to three jobs (one machine per job),** so that total machine hours used is minimized. The hours required by each machine for the jobs is given in the table below.

JOB

[redacted]		1	2	3
	A	4	2	9
	B	2	1	5
	C	5	2	10

a. Perform the row reduction step of the Hungarian method. (Write the updated matrix below.)

JOB

[redacted]		1	2	3
	A	2	0	7
	B	1	0	4
	C	3	0	8

b. Perform the column reduction step, and write the updated matrix below:

JOB

[redacted]

	1	2	3
A	1	0	3
B	0	0	0
C	2	0	4

c. Are any further steps required? If so, perform them, and write the resulting matrices below:

JOB

[redacted]

	1	2	3
A	1	0	3
B	0	0	0
C	2	0	4

→

JOB

[redacted]

	1	2	3
A	0	0	2
B	0	1	0
C	1	0	3

Note: Only two lines are sufficient to cover all the zeros, and therefore we subtract 1 from each uncovered cell and add 1 to the cell in which the lines cross, as shown above. After this step, three lines are required to cover the zeros, and the optimal assignment is as shown by the circled zeros.

f. Find the optimal assignment:

Machine A performs job _1_.
Machine B performs job _3_.
Machine C performs job _2_.

g. Total machine hours required is _11_ (= 4+5+2)

e. This assignment problem can be modeled as an LP with _6_ constraints (plus nonnegativity) and _9_ variables. The number of basic variables will be _5_. The number of variables which are positive will be __3__. The optimal solution would therefore be classified as a degenerate solution.