56:171 Operations Research
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	Possible
1. True/False & multiple choice	20
2. Sensitivity analysis (LINDO)	20
3. Geometry & duality of LP	20
4. Revised Simplex Method	10
5. Transportation problem	10
Total:	80

There are two versions of the exam, which differ in sections 4 & 5!

(1.) ***True/False:*** Indicate by "+" or "o" whether each statement is "true" or "false", respectively:

_o_ a. If the optimal value of a slack variable of a primal LP constraint is zero, then the optimal value of the dual variable for that same constraint must be positive.

_+_ b. In reference to LP, the terms "dual variable", "shadow price", and "simplex multiplier" are all synonymous.

_o_ c. If you make a mistake in choosing the pivot column in the simplex method, the next basic solution will have one or more negative basic variables.

_+_ d. If the primal LP feasible region is nonempty and bounded, then the dual LP can be neither unbounded nor infeasible.

_+_ e. An assignment problem is a special case of a transportation problem

_o_ f. A degenerate solution of an LP has fewer basic than nonbasic variables.

_o_ g. If a basic feasible solution of a transportation problem is degenerate, the next iteration cannot result in an improvement of the objective.

_o_ h. The two-phase simplex method solves for the dual variables in phase one, and then solves for the primal variables in phase two.

_o_ i. In a "balanced" transportation problem, the number of sources equals the number of destinations

_+_ j. If there is a tie in the minimum ratio test of the simplex method, the tableau that follows will be degenerate.

_o_ k. A dual variable for an equality constraint is always zero.

_+_ l. The slack variable and the dual variable for a constraint cannot both be positive.

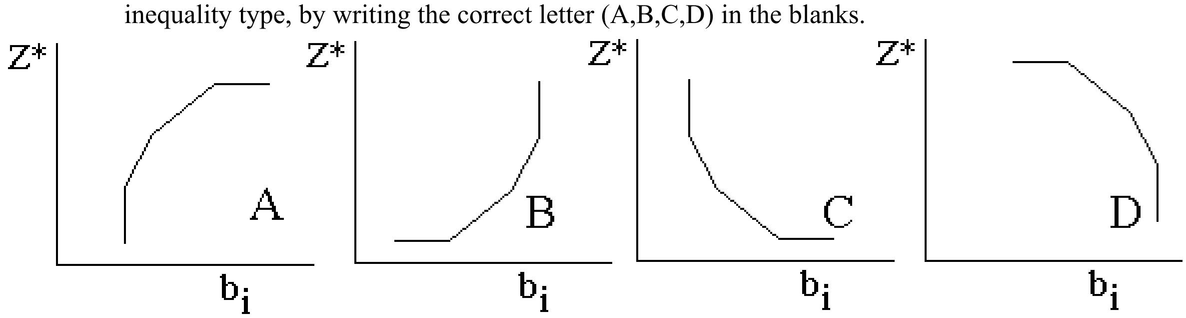
_+_ m. In a transportation problem with 4 sources and 6 destinations, with total supply exceeding total demand, the number of basic variables will be 10.

_+_ n. In a minimization LP problem, if the right-hand-side of a “greater-than-or-equal” constraint is increased, the objective function will either remain the same or increase.

_+_ o. The “complementary slackness condition” of LP implies that in the output of the optimal solution, either the slack (or surplus) in a constraint or its dual variable (or both) must be zero.

_+_ p. Every basic feasible solution of an assignment problem is degenerate.

Match the four hypothetical graphs of optimal value vs right-hand-side to the appropriate combination of min/max and inequality type, by writing the correct letter (A,B,C,D) in the blanks.



__C_ q. Min cx st $Ax \leq b$

__B__ r. Min cx st $Ax \geq b$

_A_ s. Max cx st $Ax \leq b$

_D_ t. Max cx st $Ax \geq b$

2. *Sensitivity Analysis in LP.* Consult the LINDO output to answer the questions below:
A refinery takes four raw gasolines, blends them, and produces three types of fuel.

Raw Gasoline Type	Octane Rating	Available (barrels/day)	Price ($/barrel)
1	68	4000	31.02
2	86	5050	33.15
3	91	7100	36.35
4	99	4300	38.75

Fuel Blend Type	Min Octane Rating	Selling price ($/barrel)	Demand (barrels/day)
1	95	45.15	≤ 10,000
2	90	42.95	any amt can be sold
3	85	40.99	≥ 15,000

Raw gasolines not used in blending can be sold at

- $38.95/barrel if octane rating ≥90
- $36.85/barrel if octane rating <90

Decision variables:

XIJ = barrels/day of raw gasoline of type I (1≤I≤4) used in making fuel type J (1≤J≤3)
YI = barrels/day of raw gasoline type I sold "as is" on the market (I=1,2,3,4)

a. In the optimal solution, neither raw gasolines #2 , #3, or #4 is sold on the market (without blending). If you were *required* to sell 10 barrels of one of these raw gasoline types, which one would you select? _#3_

b. If you sold 10 barrels of this raw gasoline on the market, what would be the effect on the quantity of raw gasoline #1 sold on the market? __1.48__ (check:__increase or Xdecrease?)

c. If 100 additional barrels/day of fuel blend #3 must be produced, what is
… the effect on the profit? $108.59 (check:__increase or Xdecrease?)
…the effect on the quantity of raw gasoline #1 sold on the market? _37 (check:__increase or Xdecrease?)
…the effect on the quantity of fuel blend #1 which is produced? _62.9_ (check:__increase or Xdecrease?)
(Note: X11 increases by 8.1 and X41 by 54.8, a total of 62.9)

d. If the minimum octane rating for fuel blend #3 were 87 rather than 85, what would be the coefficients in the constraint of row #4: –19 X13 + (__–1__)X23 + (__4__)X33 + 12 X43 ≥ 0

```
MAX      14.13 X11 + 12 X21 + 8.8 X31 + 6.4 X41 + 11.93 X12 + 9.8 X22
       + 6.6 X32 + 4.2 X42 + 9.97 X13 + 7.84 X23 + 4.64 X33 + 2.24 X43
       + 5.83 Y1 + 3.7 Y2 + 2.6 Y3 + 0.2 Y4
  SUBJECT TO
         2) - 27 X11 - 9 X21 - 4 X31 + 4 X41 >=   0
         3) - 22 X12 - 4 X22 + X32 + 9 X42   >=   0
         4) - 17 X13 + X23 + 6 X33 + 14 X43  >=   0
         5)   X11 + X12 + X13 + Y1           <=    4000
         6)   X21 + X22 + X23 + Y2           <=    5050
         7)   X31 + X32 + X33 + Y3           <=    7100
         8)   X41 + X42 + X43 + Y4           <=    4300
         9)   X11 + X21 + X31 + X41          <=   10000
        10)   X13 + X23 + X33 + X43          >=   15000
  END
        OBJECTIVE FUNCTION VALUE
        1)      140216.5
```

Gasoline Blending

VARIABLE	VALUE	REDUCED COST
X11	633.213867	0.000000
X21	0.000000	0.000000
X31	0.000000	0.000000
X41	4274.193359	0.000000
X12	0.000000	0.000000
X22	0.000000	0.542424
X32	0.000000	0.693098
X42	0.000000	0.934175
X13	2824.193604	0.000000
X23	5050.000000	0.000000
X33	7100.000000	0.000000
X43	25.806452	0.000000
Y1	542.592590	0.000000
Y2	0.000000	5.533333
Y3	0.000000	4.970370
Y4	0.000000	7.429630

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	-0.307407
3)	0.000000	-0.277273
4)	0.000000	-0.307407
5)	0.000000	5.830000
6)	0.000000	9.233334
7)	0.000000	7.570370
8)	0.000000	7.629630
9)	5092.592773	0.000000
10)	0.000000	-1.085926

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X11	14.130000	INFINITY	0.000000
X21	12.000000	0.000000	INFINITY
X31	8.800000	0.000000	INFINITY
X41	6.400000	INFINITY	0.000000
X12	11.930000	2.283539	2.983334
X22	9.800000	0.542424	INFINITY
X32	6.600000	0.693098	INFINITY
X42	4.200000	0.934175	INFINITY
X13	9.970000	0.000000	9.529630
X23	7.840000	INFINITY	0.000000
X33	4.640000	INFINITY	0.000000
X43	2.240000	0.000000	INFINITY
Y1	5.830000	6.100000	2.932000
Y2	3.700000	5.533334	INFINITY
Y3	2.600000	4.970370	INFINITY
Y4	0.200000	7.429630	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	0.000000	17096.773438	14650.000000
3	0.000000	0.000000	11937.037109
4	0.000000	87550.007812	800.000000
5	4000.000000	INFINITY	542.592590
6	5050.000000	44.444447	1627.777710
7	7100.000000	34.782608	3662.500000
8	4300.000000	3662.500000	4274.193359
9	10000.000000	INFINITY	5092.592773
10	15000.000000	1465.000000	47.058823

THE TABLEAU

ROW	(BASIS)	X11	X21	X31	X41	X12	X22
1	ART	0.000	0.000	0.000	0.000	0.000	0.542
2	X11	1.000	0.419	0.258	0.000	0.000	0.086
3	X12	0.000	0.000	0.000	0.000	1.000	0.182
4	X13	0.000	-0.419	-0.258	0.000	0.000	-0.419
5	X33	0.000	0.000	1.000	0.000	0.000	0.000
6	X23	0.000	1.000	0.000	0.000	0.000	1.000
7	X41	0.000	0.581	0.742	1.000	0.000	0.581
8	X43	0.000	-0.581	-0.742	0.000	0.000	-0.581
9	SLK 9	0.000	0.000	0.000	0.000	0.000	-0.667
10	Y1	0.000	0.000	0.000	0.000	0.000	0.152

ROW	X32	X42	X13	X23	X33	X43	Y1
1	0.693	0.934	0.000	0.000	0.000	0.000	0.000
2	0.110	0.148	0.000	0.000	0.000	0.000	0.000
3	-0.045	-0.409	0.000	0.000	0.000	0.000	0.000
4	-0.258	0.000	1.000	0.000	0.000	0.000	0.000
5	1.000	0.000	0.000	0.000	1.000	0.000	0.000
6	0.000	0.000	0.000	1.000	0.000	0.000	0.000
7	0.742	1.000	0.000	0.000	0.000	0.000	0.000
8	-0.742	0.000	0.000	0.000	0.000	1.000	0.000
9	-0.852	-1.148	0.000	0.000	0.000	0.000	0.000
10	0.194	0.261	0.000	0.000	0.000	0.000	1.000

ROW	Y2	Y3	Y4	SLK 2	SLK 3	SLK 4	SLK 5
1	5.533	4.970	7.430	0.307	0.277	0.307	5.830
2	0.086	0.110	0.148	0.037	0.000	0.005	0.000
3	0.000	0.000	0.000	0.000	0.045	0.000	0.000
4	-0.419	-0.258	0.000	0.000	0.000	0.032	0.000
5	0.000	1.000	0.000	0.000	0.000	0.000	0.000
6	1.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.581	0.742	1.000	0.000	0.000	0.032	0.000
8	-0.581	-0.742	0.000	0.000	0.000	-0.032	0.000
9	-0.667	-0.852	-1.148	-0.037	0.000	-0.037	0.000
10	0.333	0.148	-0.148	-0.037	-0.045	-0.037	1.000

ROW	SLK 6	SLK 7	SLK 8	SLK 9	SLK10	
1	9.2	7.6	7.6	0.000	1.1	0.14E+06
2	0.086	0.110	0.148	0.000	0.081	633.214
3	0.000	0.000	0.000	0.000	0.000	0.000
4	-0.419	-0.258	0.000	0.000	-0.452	2824.194
5	0.000	1.000	0.000	0.000	0.000	7100.000
6	1.000	0.000	0.000	0.000	0.000	5050.000
7	0.581	0.742	1.000	0.000	0.548	4274.193
8	-0.581	-0.742	0.000	0.000	-0.548	25.806
9	-0.667	-0.852	-1.148	1.000	-0.630	5092.593
10	0.333	0.148	-0.148	0.000	0.370	542.593

3. Geometry & Duality of the Linear Programming. Consider the following LP problem: Consider the ***primal*** LP problem:

$$\text{Max } z = 10X_1 + 8X_2$$
$$\text{s.t.} \quad X_1 + X_2 \geq 3$$
$$2X_1 - X_2 \geq 2$$
$$2X_1 + 4X_2 \leq 10$$
$$X_1 \geq 0,\ X_2 \geq 0$$

a. Write the dual of the above problem, filling the blanks with numbers and the boxes with ≤, =, ≥, or "U"
(unrestricted in sign):

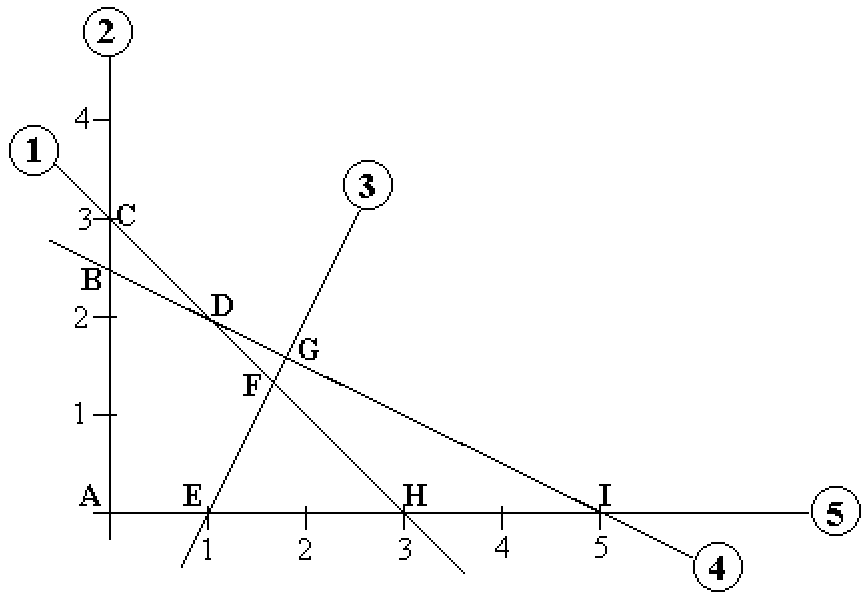
Min __3__ Y_1 + ___2__ Y_2 + ___10__ Y_3

s.t __1__ Y_1 + ___2__ Y_2 + __2__ Y_3 [≥] __10__

__1__ Y_1 + _(−1_)__ Y_2 + __4___ Y_3 [≥] _8___

sign restrictions: Y_1 [≤] 0, Y_2 [≤] 0 , Y_3 [≥] 0

Let x_3, x_4, & x_5 be the slack/surplus variables for constraints (1)-(3). Below is a graph of the feasible region:



(b.) The primal feasible region is a polyhedron. Which edges (#1 through #5) form its boundary? #1,3,4, &5

(c.) How many basic variables must this primal LP problem have? _3__

(d.) Of the nine points labeled **A** through **I,** what is the number of them which correspond to basic solutions? _9__

(e.) Which variables are basic at the point labeled **G**? X1, X2, & X3

(f.) Suppose that during the simplex method, a move is made from the extreme point labeled (H), i.e., X=(3,0), to the extreme point labeled (F), i.e., X = ($^5/_3$, $^4/_3$).
Which variable entered the basis? __X2__ Which left the basis? __X4___

(g.) What is the total number of basic solutions of the system? ___
How many of these are feasible? ______ How many are infeasible? ________ (Do NOT compute them!)

Given: Point **G** is optimal,
(h.) Based upon complementary slackness principles, what can be said about the optimal values of the dual variables (where Y_3 & Y_4 are slack/surplus in dual constraints 1 & 2, respectively)?

Y_1 _X_ must be zero __ must be nonzero __ undetermined
Y_2 __ must be zero __ must be nonzero _X_ undetermined
Y_3 __ must be zero __ must be nonzero _X_ undetermined
Y_4 _X_ must be zero __ must be nonzero __ undetermined
Y_5 _X_ must be zero __ must be nonzero __ undetermined

❚❏❚❏❚ VERSION A ❚❏❚❏❚

4. Revised Simplex Method. Consider the initial LP tableau for a MINIMIZATION problem:

-z	X1	X2	X3	X4	X5	X6	X7	b
1	1	15	8	0	0	0	0	*0*
0	–1	1	1	1	0	0	0	4
0	1	–1	0	0	1	0	0	1
0	2	10	1	0	0	1	0	5
0	0	–1	2	0	0	0	1	10

At a later iteration, besides -z, the basic variables are (in order) X_4, X_3, X_1, X_7 (i.e., the basis is B={4, 3, 1, 7} and the basis inverse matrix is

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix}$$

a. What are the values of the basic variables at this iteration?

$X_4 = 2$, $X_3 = \boxed{3}$, $X_1 = \boxed{1}$, $X_7 = 4$

a. What are the values of the simplex multipliers?

$\pi_1 = 0$, $\pi_2 = -15$, $\pi_3 = \boxed{8}$, $\pi_4 = \boxed{0}$

b. What is the *reduced cost* of the variable X_2? __–80___

c. Will entering X_2 into the basis result in an improvement? __YES___

d. What is the substitution rate of the nonbasic variable X_2 for the basic variable X_3? __12___

(This means that an increase of 1 unit of X_2 will result in a (*check*: __increase or _X_decrease?) of __12_ units in X_3).

❚❏❚❏❚VERSION B ❚❏❚❏❚

4. Revised Simplex Method. Consider the initial LP tableau for a MINIMIZATION problem:

-z	X1	X2	X3	X4	X5	X6	X7	b
1	15	1	8	0	0	0	0	*0*
0	1	–1	1	1	0	0	0	8
0	1	1	0	0	1	0	0	1
0	10	2	1	0	0	1	0	14
0	0	–1	2	0	0	0	1	10

At a later iteration, besides -z, the basic variables are (in order) X_4, X_3, X_1, X_7 (i.e., the basis is B={4, 3, 1, 7} and the basis inverse matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 9 & -1 & 0 \\ 0 & -10 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 20 & -2 & 1 \end{bmatrix}$$

a. What are the values of the basic variables at this iteration?

$X_4 = 3$, $X_3 = \boxed{4}$, $X_1 = \boxed{1}$, $X_7 = 2$

a. What are the values of the simplex multipliers?

$\pi_1 = 0$, $\pi_2 = -65$, $\pi_3 = \boxed{8}$, $\pi_4 = \boxed{0}$

b. What is the *reduced cost* of the variable X_2? __50__

c. Will entering X_2 into the basis result in an improvement? NO

d. What is the substitution rate of the nonbasic variable X_2 for the basic variable X_3? –8

(This means that an increase of 1 unit of X_2 will result in a (*check*: X increase or __decrease?) of __8_ units in X_3).

▮❑▮❑▮VERSION A ▮❑▮❑▮

5. ***Transportation Problem.*** The following is a transportation tableau, with an initial set of shipments indicated:

DESTINATIONS

SOURCES	1	2	3	4	supply
A	3 [6]	6 [4]	[8]	1 [9]	10
B	[3]	[5]	3 [2]	4 [7]	7
C	5 [5]	[4]	[6]	[5]	5
D	[4]	3 [7]	[9]	[6]	3
demand	8	9	3	5	

a. Is the solution above a basic feasible solution? YES *If not, explain why!*

b. Complete the computation of a set of dual variables for the above transportation tableau, starting by assigning the dual variable for source #1 equal to zero:

Dual variables for supply constraints: $U_1 = 0$, $U_2 = \boxed{-2}$, $U_3 = -1$, $U_4 = +3$

Dual variables for demand constraints: $V_1 = 6$, $V_2 = 4$, $V_3 = \boxed{4}$, $V_4 = 9$

c. Compute the reduced costs for X_{13} __4__ & X_{44} __–6__

Which of these two variables should enter the basis? X_{44}

Which basic variable should leave the basis? X_{14}

d. Suppose that the supply of source C were to increase from 5 to 10.

Why is the problem no longer "balanced"? sum of supplies not equal to sum of demands

What must be done to balance the problem? add a "dummy" destination with demand = 5

VERSION B

5. ***Transportation Problem.*** The following is a transportation tableau, with an initial set of shipments indicated:

DESTINATIONS

(Each cell: shipment, cost)

SOURCES	1	2	3	4	supply
A	3, 8	6, 5	6	1, 9	10
B	3	4	3, 2	4, 7	7
C	5, 5	4	6	5	5
D	4	3, 7	9	6	3
demand	8	9	3	5	

a. Is the solution above a basic feasible solution? YES *If not, explain why!*

b. Complete the computation of a set of dual variables for the above transportation tableau, starting by assigning the dual variable for source #1 equal to zero:

Dual variables for supply constraints: $U_1 = 0$, $U_2 =$ [2], $U_3 = +3$, $U_4 = +2$

Dual variables for demand constraints: $V_1 = 8$, $V_2 = 5$, $V_3 =$ [0], $V_4 = 9$

c. Compute the reduced costs for X_{13} _6__ & X_{44} _–5_

Which of these two variables should enter the basis? X_{44}

Which basic variable should leave the basis? X_{14}

d. Suppose that the supply of source C were to increase from 5 to 10.

Why is the problem no longer "balanced"? sum of supplies not equal to sum of demands

What must be done to balance the problem? add "dummy" destination with demand = 5