56:171 Operations Research

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• ***Write your name on the first page, and initial the other pages.***

• ***Answer both questions of Part One, and 5 problems from Part Two.***

		Possible	Score
Part One:	1. True/False	30	_____
	2. Sensitivity analysis (LINDO)	20	_____
Part Two:	3. Geometry of simplex method	10	_____
	4. LP duality	10	_____
	5. Revised simplex method	10	_____
	6. LP model formulation	10	_____
	7. Assignment problem	10	_____
	8. Project scheduling	10	_____
	total:	100	_____

PART ONE

(1.) ***True/False:*** Indicate by "+" or "o" whether each statement is "true" or "false", respectively:

____ a. If the optimal value of a slack variable of a primal LP constraint is positive, then the optimal value of the dual variable for that same constraint must equal zero.

____ b. In reference to LP, the terms "dual variable", "shadow price", and "simplex multiplier" are identical.

____ c. If you make a mistake in choosing the pivot row in the simplex method, the next basic solution will have one or more negative variables.

____ d. If the primal LP feasible region is nonempty and bounded, then the dual LP cannot be unbounded nor infeasible.

____ e. In PERT, the total completion time of the project is assumed to have a BETA distribution.

____ f. If the primal LP has an optimal solution, then its dual LP has also.

____ g. All tasks on the critical path have their latest finish time equal to their earliest start time.

____ h. If the current basis is not degenerate, the dual variables at any iteration of the simplex method for solving a transportation problem are uniquely determined.

____ i. The two-phase simplex method solves for the dual variables in phase one, and then solves for the primal variables in phase two.

____ j. In a minimization problem, the "Big-M" method assigns high costs to artificial variables to force them from the basis.

____ k. If you make a mistake in choosing the pivot column in the simplex method, the next basic solution will be infeasible.

____ l. During a change of basis in the simplex method for the transportation problem, the "substitution rates" are all +1, 0, or -1.

____ m. In the critical path method for project scheduling, the latest finish time for a task depends upon the earliest finish time for the project.

____ n. If a slack variable of a primal LP constraint is zero in the optimal solution, then there is a corresponding dual variable whose optimal value is also zero.

____ o. In a transportation problem with 4 sources and 6 destinations, with total supply exceeding total demand, the number of basic variables will be 10.

(2.) ***Sensitivity Analysis in LP.*** Recall the Gasoline Blending Problem discussed in class: A refinery takes four raw gasolines, blends them, and produces three types of fuel.

Raw Gas Type	Octane Rating	Available (Barrels/day)	Price ($/barrel)
1	68	4000	31.02
2	86	5050	33.15
3	91	7100	36.35
4	99	4300	38.75

Fuel blend Type	Minimum Octane rating	Selling price Price ($/barrel)	Demand Pattern (barrels/day)
1	95	45.15	≤ 10,000
2	90	42.95	any amt. can be sold
3	85	40.99	≥ 15,000

Raw gasolines not used in blending can be sold at
$38.95/barrel if octane rating ≥ 90, and $36.85/barrel if octane rating < 90

The LINDO output for this problem is as follows:

```
MAX     14.13 X11 + 12 X21 + 8.8 X31 + 6.4 X41 + 11.93 X12 + 9.8 X22
        + 6.6 X32 + 4.2 X42 + 9.97 X13 + 7.84 X23 + 4.64 X33 + 2.24 X43
        + 5.83 Y1 + 3.7 Y2 + 2.6 Y3 + 0.2 Y4
SUBJECT TO
            2)     - 27 X11 - 9 X21 - 4 X31 + 4 X41        >=   0
            3)     - 22 X12 - 4 X22 + X32 + 9 X42          >=   0
            4)     - 17 X13 + X23 + 6 X33 + 14 X43         >=   0
            5)             X11 + X12 + X13 + Y1            =    4000
            6)             X21 + X22 + X23 + Y2            =    5050
            7)             X31 + X32 + X33 + Y3            =    7100
            8)             X41 + X42 + X43 + Y4            =    4300
            9)             X11 + X21 + X31 + X41           <=   10000
            10)            X13 + X23 + X33 + X43           >=   15000

       OBJECTIVE FUNCTION VALUE
            1)    140216.500

   VARIABLE            VALUE              REDUCED COST
     X11            633.213900              .000000
     X21               .000000              .000000
     X31               .000000             0.000000
     X41           4274.193300              .000000
     X12               .000000              .000000
     X22               .000000              .542424
     X32               .000000              .693098
     X42               .000000              .934175
     X13           2824.194000              .000000
     X23           5050.000000              .000000
     X33           7100.000000              .000000
     X43             25.806451              .000000
     Y1             542.592600              .000000
```

Y2	.000000	5.533334
Y3	.000000	4.970370
Y4	.000000	7.429629

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	-.307407
3)	.000000	-.277273
4)	.000000	-.307407
5)	.000000	5.830000
6)	.000000	9.233334
7)	.000000	7.570370
8)	.000000	7.629630
9)	5092.593000	.000000
10)	.000000	-1.085926

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X11	14.130000	INFINITY	-.000002
X21	12.000000	.000000	INFINITY
X31	8.800000	0.000000	INFINITY
X41	6.400000	INFINITY	-.000001
X12	11.930000	2.283539	2.983333
X22	9.800000	.542424	INFINITY
X32	6.600000	.693098	INFINITY
X42	4.200000	.934175	INFINITY
X13	9.970000	-.000002	9.529629
X23	7.840000	INFINITY	.000000
X33	4.640000	INFINITY	0.000000
X43	2.240000	-.000001	INFINITY
Y1	5.830000	6.100000	2.931999
Y2	3.700000	5.533334	INFINITY
Y3	2.600000	4.970370	INFINITY
Y4	.200000	7.429629	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	.000000	17096.773000	14650.000000
3	.000000	.000000	11937.040000
4	.000000	87550.000000	800.000000
5	4000.000000	INFINITY	542.592600
6	5050.000000	44.444450	1627.778000
7	7100.000000	34.782610	3662.500000
8	4300.000000	3662.500000	4274.193300
9	10000.000000	INFINITY	5092.593000
10	15000.000000	1465.000000	47.058822

THE TABLEAU

ROW	(BASIS)	X11	X21	X31	X41	X12	X22
1	ART	0.000	0.000	.000	0.000	0.000	.542
2	X31	3.875	1.625	1.000	.000	.000	.333
3	X12	.000	.000	.000	.000	1.000	.182
4	X13	1.000	0.000	.000	.000	.000	-.333

5	Y1	.000	0.000	.000	.000	0.000	.152
6	X23	.000	1.000	.000	.000	.000	1.000
7	X41	-2.875	-.625	.000	1.000	.000	.333
8	X33	-3.875	-1.625	.000	.000	.000	-.333
9	SLK 9	0.000	.000	.000	.000	.000	-.667
10	X43	2.875	.625	.000	.000	.000	-.333

ROW	X32	X42	X13	X23	X33	X43	Y1
1	.693	.934	0.000	0.000	0.000	0.000	.000
2	.426	.574	0.000	.000	.000	0.000	.000
3	-.045	-.409	.000	.000	.000	.000	.000
4	-.148	.148	1.000	0.000	.000	0.000	.000
5	.194	.261	0.000	0.000	.000	0.000	1.000
6	.000	.000	.000	1.000	.000	.000	.000
7	.426	.574	0.000	.000	.000	0.000	.000
8	.574	-.574	0.000	.000	1.000	0.000	.000
9	-.852	-1.148	0.000	.000	.000	0.000	.000
10	-.426	.426	0.000	.000	.000	1.000	.000

ROW	Y2	Y3	Y4	SLK 2	SLK 3	SLK 4	SLK 9
1	5.533	4.970	7.430	.307	.277	.307	.000
2	.333	.426	.574	.144	.000	.019	.000
3	.000	.000	.000	.000	.045	.000	.000
4	-.333	-.148	.148	.037	.000	.037	.000
5	.333	.148	-.148	-.037	-.045	-.037	.000
6	1.000	.000	.000	.000	.000	.000	.000
7	.333	.426	.574	-.106	.000	.019	.000
8	-.333	.574	-.574	-.144	.000	-.019	.000
9	-.667	-.852	-1.148	-.037	.000	-.037	1.000
10	-.333	-.426	.426	.106	.000	-.019	.000

ROW	SLK 10	RHS
1	1.1	0.14E+06
2	.315	2453.704
3	.000	.000
4	-.370	3457.407
5	.370	542.593
6	.000	5050.000
7	.315	2453.704
8	-.315	4646.296
9	-.630	5092.593
10	-.315	1846.296

Consult the LINDO output to answer the following questions. *(If not enough information is available in the output, answer "no info".)*

a. Suppose that the market price of "raw gasoline #1" were to drop by $1.50 per barrel. Would the solution change?_____ If so, how? ____________________________

b. If the supplier for "raw gasoline #4" were to increase its availability by 1000 barrels per day (to 5300 barrels/day), would this increase the company's profits?_______ If so, by how much?________

c. If the demand for fuel blend #3 (which *must* be satisfied) increases by 100 barrels per day, what will be the change in: *(Be sure to specify whether **in**crease or **de**crease!)*

- the optimal profit? _________
- the quantity of "raw gasoline #4" used in making fuel blend #3? _________
- the quantity of "raw gasoline #1" sold on the market? _________

(Hint: the variable SLK10 is actually what we have called a "surplus" variable: converted to an equation, row #10 is: $X13+X23+X33+X43 - SLK10 = 15000$
If the sum ($X13+X23+X33+X43$*) is to be increased by 100, while the RHS remains 15000, what becomes of SLK10? According to the "substitution rates", how are the basic variables changed?)*

d. Type 2 "raw gasoline" is not sold on the market. If a previous commitment required the company to sell 100 barrels, at the given price, how much loss in profit would result?

____

e. Suppose that 100 barrels of type 2 "raw gasoline" is sold on the market. What are the resulting changes in the optimal values of the following variables? *(Be sure to specify whether **in**crease or **de**crease!)*

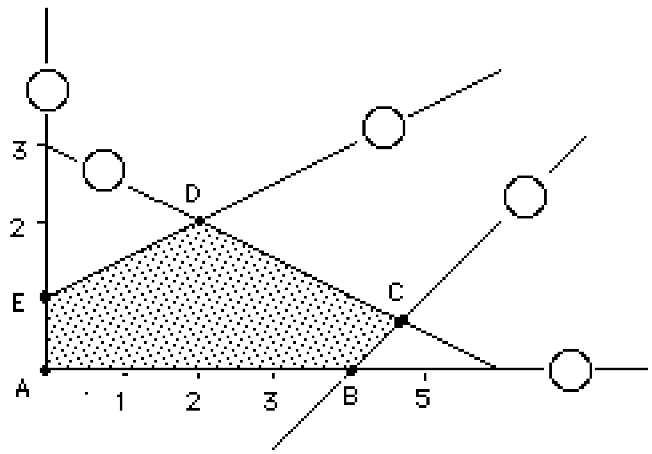
- the number of gallons of raw gas #1 sold on the market?_________
- the number of gallons of raw gas #2 used in blend #3?___________
- the number of gallons of raw gas #1 used in blend #3?___________
- the number of gallons of raw gas #2 used in blend #1?___________

❚❑❚❑❚❑❚❑❚ PART TWO ❚❑❚❑❚❑❚❑❚

(3.) Consider the following LP problem:

$$\begin{array}{llcll} \text{Maximize} & 3X_1 + 2X_2 & & & \\ \text{subject to} & X_1 + 2X_2 & \le & 6 & (1) \\ & X_1 - X_2 & \le & 4 & (2) \\ & -X_1 + 2X_2 & \le & 1 & (3) \\ & X_1 \ge 0 & & & (4) \\ & X_2 \ge 0 & & & (5) \end{array}$$

Below is a graph of the feasible region:



(a.) The feasible region is a polyhedron with 5 edges. Indicate which constraint defines each edge by labeling the edges (in the circles) on the graph, using the numbers (1) through (5) to the right of the constraints above.

(b.) How many basic variables must this LP problem have? ______________

(c.) Which variables are basic at the extreme point labeled (B)? _____________

(d.) Suppose that during the simplex method, a move is made from the extreme point labeled (B), i.e., X=(4,0), to the extreme point labeled (C), i.e., X = ($^{14}/_{3}$, $^{2}/_{3}$). Which variable entered the basis?________ Which left the basis? _________

(e.) Which extreme point is optimal for this problem? ____________

(f.) What is the total number of basic solutions of the system? How many of these are feasible? _______ How many are infeasible? _________ (Do NOT compute them!)

(4.) ***Revised Simplex Method.*** We wish to solve the LP problem

Max $z = cx$

subject to $Ax=b, x\geq 0$

where A is a 2x5 matrix. After several iterations, the current basic variables are (-z), X_1, and X_5. A portion of the current tableau is shown below:

-z	X_1	X_2	X_3	X_4	X_5	RHS
		0.875		-2.125		
		-0.125		0.875	0	9.75
		0.375	0.875	0.375		4.75

a. What is the "substitution rate" of X_2 for X_1? _______

b. If X_2 increases by 1 unit, X_1 (**in**creases/**de**creases) *(circle one)* by ________ units.

c. If the objective coefficient vector c is (3, 2, 6, 2, 4), and the current basis inverse matrix is

$$\begin{bmatrix} 3/8 & 1/4 \\ -1/8 & 1/4 \end{bmatrix}$$

compute the values of the simplex multipliers:____________

d. Using the results of (c), what is the relative profit of X_3, given that column 3 of the A matrix is the transpose of [3, 5]? __________

e. Complete the missing portions of the tableau above.

f. The current tableau is not optimal. Circle a pivot entry which will increase the profit.

g. Which variables will be basic at the next iteration? _______________

(5.) **LINEAR PROGRAMMING DUALITY**: Consider the following LP:

$$\begin{array}{lllllll}
\text{Minimize} & 2X_1 & + 5X_2 & + 3X_3 & & +X_5 & \\
\text{subject to} & X_1 & & + 2X_3 & -X_4 & & = 12 \\
 & -X_1 & + 2X_2 & & + X_4 & + X_5 & \leq 15 \\
 & & 6X_2 & - X_3 & & +2X_5 & \geq 8
\end{array}$$

$X_1 \geq 0$, $X_2 \geq 0$, $0 \leq X_3 \leq 4$, $X_4 \leq 0$ (X_5 is **unrestricted** in sign)

a. Write a dual of this LP problem. (Note the upper bound on X_3.)

b. The point (4,2,4,0,0) is feasible. What (if anything) does thisimply:
   -- about the feasibility of the dual problem?

   -- about the boundedness of the dual problem?

c. IF X=(4,2,4,0,0) is optimal in the primal problem, then what **dual** variables(including slack or surplus variables) must be zero in the dual optimal solution, according to the complementary slackness conditions for this primal-dual pair of problems?

(6.) Formulate the following problem as an LP: A manufacturer must plan production of a certain item over the next 4 quarters. Each unit of the item requires 1 man-hour of labor. Labor costs are $10 per hour regularly, or $15 per hour for overtime. Overtime is limited to 50% of regular time available. If a unit of the item is available for sale during a quarter but is not sold, an inventory carrying cost of $2 per unit is charged. Other data are:

Quarter	Regular Man-hrs Available	Demand	Selling Price($)
1	1000	1200	28
2	800	1100	26
3	900	1400	27
4	1000	1200	27

Demand limits the most which can be sold, but **need not** be satisfied.
Formulate a linear programming model which can determine how much should be produced each quarter, and how much should be sold each quarter.

Use the following decision variables in your model:

R_t = number of units of the item to be produced in quarter t, using regular time production

O_t = number of units of the item to be produced in quarter t, using overtime

S_t = number of units of the item sold during quarter t

I_t = number of units of the item in inventory at the end of quarter t

- *What is the total number of constraints (not including nonnegativity restrictions) in your model?*

- *What is the total number of variables (not including slack & surplus variables) in your model?*

(7.) ***Assignment Problem.*** Three machines are to be assigned to three jobs (one machine per job), so that total machine hours used is minimized. The hours required by each machine for the jobs is given in the table below.

MACHINE	JOB 1	JOB 2	JOB 3
A	4	2	9
B	2	1	5
C	5	2	10

a. Perform the row reduction step of the Hungarian method. (Write the updated matrix below.)

MACHINE	JOB 1	JOB 2	JOB 3
A			
B			
C			

b. Perform the column reduction step, and write the updated matrix below:

MACHINE	JOB 1	JOB 2	JOB 3
A			
B			
C			

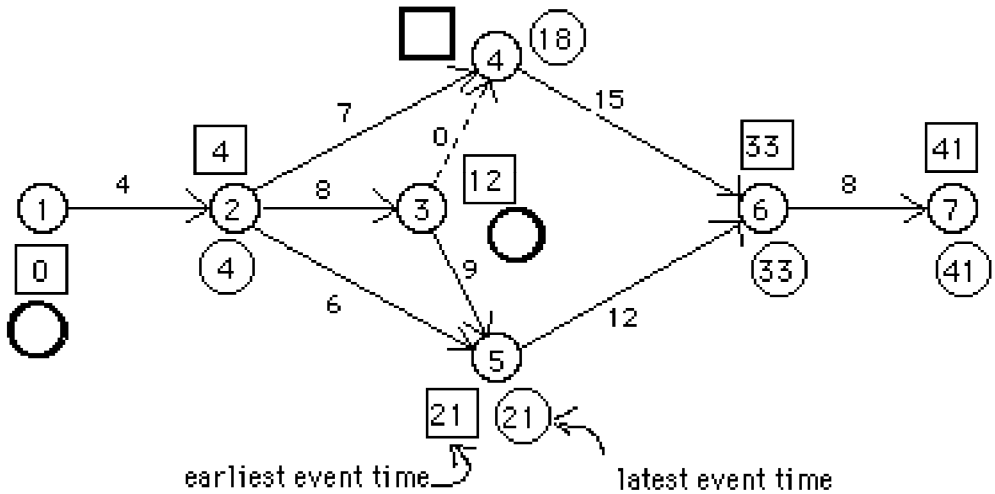
c. Are any further steps required? If so, perform them, and write the resulting matrices below:

MACHINE	JOB 1	JOB 2	JOB 3
A			
B			
C			

d. Find the optimal assignment: Machine A performs job ___. Machine B performs job ___. Machine C performs job ___. Total machine hours required is ____.

e. This assignment problem can be modeled as an LP with ___ constraints (plus nonnegativity) and ___ variables. The number of basic variables will be ___. The optimal solution is referred to as a(n) _____________ solution.

(8.) Consider the project with the network given below. Times required for the activities appear on the arrows.



a. How many activities (i.e., tasks), not including "dummies", are required to complete this project? ________

b. Complete the computation of the earliest & latest times for the events (indicated in the boxes & circles, respectively), and write the values in the circles & boxes above. *There are three values to be computed!*

c. Find the slack ("total float") for activity represented by the arrow (4,6). _________

d. How many activities are critical? _________

e. What is the earliest completion time for the project? ___________

f. Suppose that the times of activities (2,3) and (5,6) are not certain, but are random variables with expected values 8 and 12, respectively, and with standard deviations 4 and 5, respectively. Then, if the assumptions of PERT are satisfied, what is the probability distribution of the project completion time? *(Specify the type of distribution and its parameters.)* _________________