<><><><><><> 56:171 Operations Research <><><><><><>
Final Exam - December 13, 1989
<><><><><><> Instructor: D.L. Bricker <><><><><><>

Do <u>all</u> of Part One (1 pt. each) , <u>one</u> from Part Two (15 pts.) , and <u>four</u> from Part Three (15 pts. each)

Score:	Part One:	0. _______	Multiple choice
	Part Two:	1. _______	Lindo Analysis
		2. _______	LP Formulation
	Part Three:	3. _______	Project scheduling
		4. _______	Integer Programming Formulation
		5. _______	Markov chain model of inventory system
		6. _______	Birth-death model of queue
		7. _______	Dynamic programming
	TOTAL:	_______	

<><><><><><> PART ONE <><><><><><>

Multiple Choice: Circle the letter for the best answer to each question. If you feel the statement is vague, you may explain what assumptions you are making or the reason for your answer, etc., for possible partial credit.

(1). When using the Hungarian method to solve assignment problems, if the number of lines drawn to cover the zeroes in the reduced matrix is larger than the number of rows,

a. a mistake has been made, and one should review previous steps.
b. this indicates no solution exists.
c. this means an optimal solution has been reached.
d. a dummy row or column must be introduced.

(2.) The probabilities in a Markov chain transition matrix are actually

a. simple probabilities.
b. joint probabilities.
c. conditional probabilities.
d. more than one of the above are correct.
e. none of the above.

(3.) Consider a discrete-time Markov chain with transition probability matrix :

$$\begin{bmatrix} .6 & .4 \\ .3 & .7 \end{bmatrix}$$

If the system is initially in state #1, the probability that the system will be in state 2 after exactly one step is:

a. 0.6 c. 0.7 e. none of the above
b. 0.4 d. 0.52

(4.) If the Markov chain in the previous problem was initially in state #1, the probability that the system will still be in state 1 after 2 transitions is

a. 0.36 c. 0 e. 0.52
b. 0.60 d. 0.48 f. none of the above

(5.) An absorbing state of a Markov chain is one in which the probability of

a. moving into that state is zero. c. moving out of that state is zero.
b. moving out of that state is one. d. none of the above.

(6.)The steady-state probability vector p of a discrete Markov chain with transition probability matrix P satisfies the matrix equation

a. $P\pi = 0$
b. $P\pi = \pi$
c. $\pi(I - P) = 0$
d. $P^t \pi = 0$
e. none of the above

(7.) The Poisson process is a special case of the birth-death process with

a. no births
b. no deaths
c. birth rate = death rate
d. death is by Poissoning
e. time between births &/or deaths has Poisson distribution
f. none of the above

(8.) For a continuous-time Markov chain, let L be the matrix of transition rates. The sum of each ...

a. row is 0
b. column is 0
c. row is 1
d. column is 1
e. none of the above

(9.) To compute the steady state distribution p of a continuous-time Markov chain, one must solve (in addition to sum of p components equal to 1) the matrix equation (where Λ^t is the transpose of Λ):

a. $\pi\Lambda = 1$
b. $\Lambda^t \pi = 1$
c. $\Lambda^t \pi = \pi$
d. $\pi\Lambda = \pi$
e. $\pi\Lambda = 0$
f. none of the above

(10.) In an M/M/1 queue, if $\lambda > \mu$,

a. $\pi_o = 1$ in steady state
b. no steady state exists
c. $\pi_i > 0$ for all i
d. $\pi_o = 0$ in steady state

(11.) In order for a function of two variables x_1 and x_2 subject to one constraint to have a local minimum at a point, all the derivatives of the Lagrangian must equal zero at that point. This is a:

a. Necessary condition for optimality
b. Sufficient condition for optimality
c. Both (a) & (b)
d. Neither (a) nor (b)

(12.) Given the following problem:

$$\text{Max } x_1^2+2x_2^2-x_1x_2 \text{ s.t. } x_1+x_2=4$$

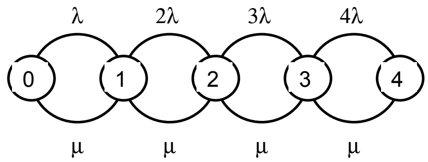
the Lagrangian function is:

a. $2x_1 + 4x_2 - x_1x_2$
b. $x_1^2+2x_2^2 - x_1x_2$
c. $2x_1 + 4x_2 - x_1x_2 + lx_1 + lx_2 - 4l$
d. $x_1^2+2x_2^2 - x_1x_2 + lx_1 + lx_2 - 4l$
e. none of the above

(13.) If at a particular point the first derivative of a function (of a single variable) equals zero, and the second derivative is greater than zero, then that point is a

a. local maximum
b. local minimum
c. global maximum
d. global minimum
e. saddle point
f. not necessarily any of the above

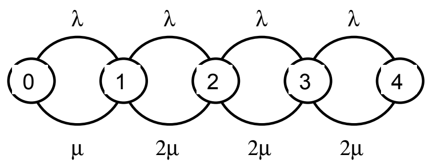
(14.) Consider the following queueing model:



The notation for this type of queue is:

a. M/M/1 c. M/M/2/4 e. M/M/1/4
b. M/M/2 d. M/M/4 f. none of the above

(15.) Consider the following queueing model:



The notation for this type of queue is:

a. M/M/1 c. M/M/2/4 e. M/M/1/4
b. M/M/2 d. M/M/4 f. none of the above

<><><><><> PART TWO <><><><><>

1. LINDO analysis

Problem Statement: McNaughton Inc. produces two steak sauces, spicy Diablo and mild Red Baron. These sauces are both made by blending two ingredients A and B. A certain level of flexibility is permitted in the formulas for these products. Indeed, the restrictions are that:

i) Red Baron must contain no more than 75% of A.
ii) Diablo must contain no less than 25% of A and no less than 50% of B

Up to 40 quarts of A and 30 quarts of B could be purchased. McNaughton can sell as much of these sauces as it produces at a price per quart of $3.35 for Diablo and $2.85 for Red Baron. A and B cost $1.60 and $2.05 per quart, respectively. McNaughton wishes to maximize its net revenue from the sale of these sauces.

Define *D = quarts of Diablo to be produced*
R = quarts of Red Baron to be produced
A_1 = quarts of A used to make Diablo
A_2 = quarts of A used to make Red Baron
B_1 = quarts of B used to make Diablo
B_2 = quarts of B used to make Red Baron

The LINDO output for solving this problem follows:

```
MAX     3.35 D + 2.85 R - 1.6 A1 - 1.6 A2 - 2.05 B1 - 2.05 B2
SUBJECT TO
       2) - D + A1 + B1 =    0
       3) - R + A2 + B2 =    0
       4)   A1 + A2 <=   40
       5)   B1 + B2 <=   30
       6) - 0.25 D + A1 >=   0
       7) - 0.5 D + B1 >=   0
```

```
       8) - 0.75 R + A2 <=   0
 END

          OBJECTIVE FUNCTION VALUE
  1)         99.0000000

VARIABLE            VALUE                      REDUCED COST
     D            50.000000                     0.000000
     R            20.000000                     0.000000
     A1           25.000000                     0.000000
     A2           15.000000                     0.000000
     B1           25.000000                     0.000000
     B2            5.000000                     0.000000
```

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	-2.350000
3)	0.000000	-4.350000
4)	0.000000	0.750000
5)	0.000000	2.300001
6)	12.500000	0.000000
7)	0.000000	-1.999999
8)	0.000000	2.000000

RANGES IN WHICH THE BASIS IS UNCHANGED

		OBJ COEFFICIENT RANGES	
VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
D	3.350000	0.750000	0.500000
R	2.850000	0.500000	0.375000
A1	-1.600000	1.500001	0.666666
A2	-1.600000	0.666666	0.500000
B1	-2.050000	1.500001	1.000000
B2	-2.050000	1.000000	1.500001

		RIGHTHAND SIDE RANGES	
ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	0.000000	10.000000	10.000000
3	0.000000	16.666668	3.333333
4	40.000000	50.000000	10.000000
5	30.000000	10.000000	16.666664
6	0.000000	12.500000	INFINITY
7	0.000000	6.250000	5.000000
8	0.000000	2.500000	12.500000

THE TABLEAU:

ROW	(BASIS)	D	R	A1	A2
1	ART	0.000	0.000	0.000	0.000
2	A1	0.000	0.000	1.000	0.000
3	R	0.000	1.000	0.000	0.000
4	A2	0.000	0.000	0.000	1.000
5	B2	0.000	0.000	0.000	0.000
6	SLK 6	0.000	0.000	0.000	0.000
7	D	1.000	0.000	0.000	0.000
8	B1	0.000	0.000	0.000	0.000

(tableau, continued)

ROW	B1	B2	SLK 4	SLK 5	SLK 6
1	0.000	0.000	0.750	2.300	0.000
2	0.000	0.000	-0.500	1.500	0.000
3	0.000	0.000	2.000	-2.000	0.000
4	0.000	0.000	1.500	-1.500	0.000
5	0.000	1.000	0.500	-0.500	0.000
6	0.000	0.000	-0.250	0.750	1.000
7	0.000	0.000	-1.000	3.000	0.000
8	1.000	0.000	-0.500	1.500	0.000

ROW	SLK 7	SLK 8	RHS
1	2.000	2.000	99.000
2	3.000	2.000	25.000
3	-4.000	-4.000	20.000
4	-3.000	-2.000	15.000
5	-1.000	-2.000	5.000
6	2.000	1.000	12.500
7	4.000	4.000	50.000
8	1.000	2.000	25.000

```
: PARARHS
ROW:5
NEW RHS VAL=150

     VAR     VAR         PIVOT        RHS        DUAL PRICE         OBJ
     OUT      IN          ROW         VAL        BEFORE PIVOT       VAL
                                     30.0000        2.30000         99.0000
     R        SLK 7         3        40.0000        2.30000        122.000
     SLK 6    SLK 8         6       120.000         1.30000        226.000
     A2       R             4       120.000         1.30000        226.000
                                    150.000         0.800000       250.000
```

a. How many quarts of Diablo are produced? ______________

b. How much profit does the firm make on these two products? ____________

c. What *additional* amount should the firm be willing to pay to have another quart of ingredient B available? _________ What is the *total* amount the firm should be willing to pay for another quart of ingredient B? ___________How many quarts should they be willing to buy at this cost? ______________

d. How much can the price of Diablo increase before the composition of the current optimal product mix changes? ________________

e. If one less quart of B were to be used, i.e., if we were to force an increase of one unit of slack in the availability constraint for row 5, what would be the changes in the quantity of Diablo produced? __________ of Red Baron? _________

f. Based on the LINDO output, draw a rough sketch below of as much as you can of the optimal profit vs. available supply of ingredient B (assuming that it is still available at the current cost of $2.05/qt.)

2. LP formulation I now have $100. The following investments are each available only once during the next three years:

Investment A: Every dollar invested now yields $0.10 a year from now and $1.30 three years from now.
Investment B: Every dollar invested now yields $0.20 a year from now and $1.10 two years from now.
Investment C: Every dollar invested a year from now yields $1.50 three years from now.

At most $50 may be placed in each of investments A, B, and C. During each year uninvested cash can be placed in money market funds, which yield 6% interest per year. Any return from previous investments may be reinvested immediately.

Define variables A, B, and C to be the dollars placed in these investments, and M_t the dollars invested in money market funds at the beginning of year t.

a. Write a constraint stating that the amount invested at the beginning of year #1 (now) is $100.

b. Write a constraint limiting the amount invested at the beginning of year #2 (i.e., a year from now).

c. Write an expression for the total cash on hand three years from now (beginning of year #4).

d. Write an LP model to maximize my cash on hand three years from now.

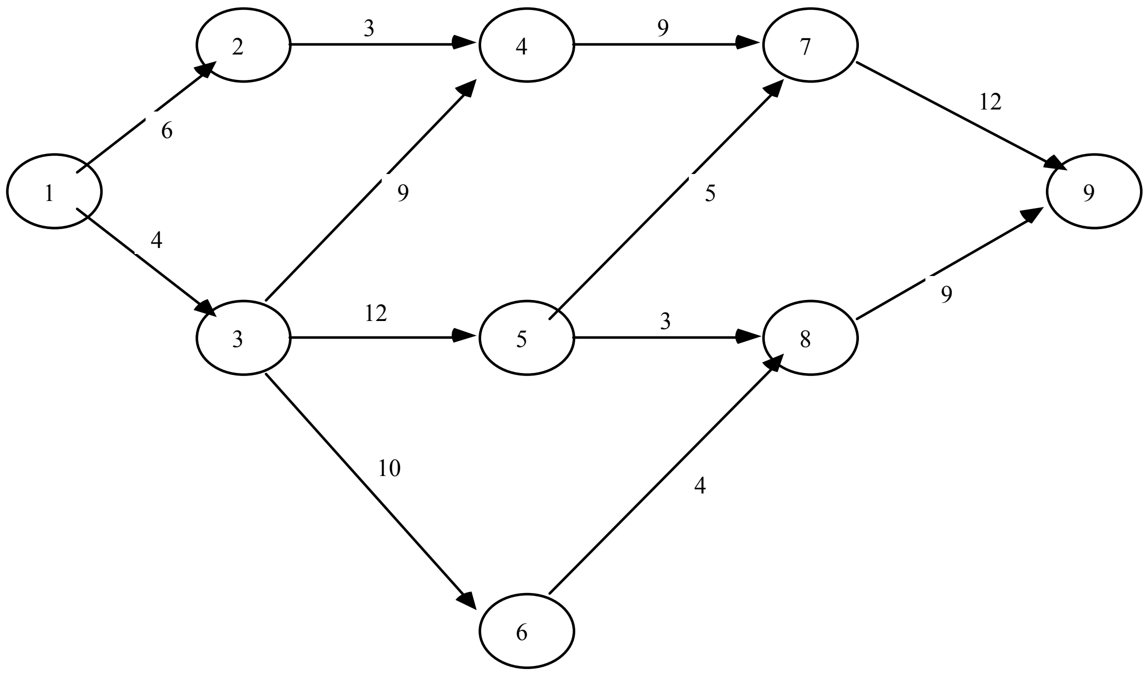
e. How many variables does the **Dual** of this LP have? ________________

f. How many constraints (other than non-negativity constraints) does the dual LP have? ____________

g. Write one of the dual constraints (other than the non-negativity constraints).

<><><><><><> PART THREE <><><><><><>

3. Project Scheduling: Consider the project consisting of eleven activities, represented by the AOA (activity-on-arrow) diagram below:



a. For each event node, compute Earliest Event Time.

b. For each event node, compute Latest Event Time

Write the values for (a) and (b) directly on the diagram.

c. Complete the table below of Early Start, Early Finish, Late Start, & Late Finish times for each activity:

Activity	duration	ES	EF	LS	LF
(1,2)	6	___	___	___	___
(1,3)	4	___	___	___	___
(2,4)	3	___	___	___	___
(3,4)	9	___	___	___	___
(3,5)	12	___	___	___	___
(3,6)	10	___	___	___	___
(4,7)	9	___	___	___	___
(5,7)	5	___	___	___	___
(5,8)	3	___	___	___	___
(6,8)	4	___	___	___	___
(7,9)	12	___	___	___	___
(8,9)	9	___	___	___	___

d. What activities are on the critical path? *(Circle activities in table above.)*

e. What is the earliest that the project can be completed, if it is begun at time zero? _________

f. What is the Total Float (or slack) of activity (3,4)? ________ of activity (3,5)? _________

g. What is the Free Float of activy (3,4)? ________ of activity (3,5)?_________

4. Integer Programming Model Formulation: The Tower Engineering Corporation is considering undertaking several proposed projects for the next fiscal year. The projects, together with the number of engineers, number of support personnel required for each project, and the expected project profit, are:

Project #:	1	2	3	4	5	6
Engineers req'd:	20	55	47	38	90	63
Support req'd:	15	45	50	40	70	70
Profit (x$\$10^6$)	1.0	1.8	2.0	1.5	3.6	2.2

The corporation faces the following restrictions:

- Only 175 engineers are available
- 150 support personnel are available
- Project #2 can be selected <u>only if</u> Project #1 is selected
- If project #5 is selected, project #3 cannot be selected, and vice versa
- No more than three projects may be selected in all
- If <u>both</u> projects 1 & 2 are selected, then project 3 cannot be selected
- If <u>both</u> projects 4&5 are selected, there is an extra bonus profit of 0.5 (x$\$10^6$)

Formulate an integer linear programming model to select the set of projects which will maximize the profits. Be sure to define your decision variables.

5. Markov Chain Model of Inventory System: Consider the following inventory system for a certain spare part for a company's 2 production lines. A maximum of four parts may be kept on the shelf. At the end of each day, the parts in use are inspected and, if worn, replaced with one off the shelf. The probability distribution of the number replaced each day is:

n=	0	1	2
P{n}=	0.3	0.5	0.2

To avoid shortages, the current policy is to restock the shelf at the end of each day (after spare parts have been removed) so that the shelf is again filled to its limit (i.e., 4) **if** there are fewer than 2 parts on the shelf.

The inventory system has been modeled as a Markov chain, with the state of the system defined as the end-of-day inventory level (before restocking). Refer to the computer output which follows to answer the following questions: **Note that in the computer output, state #1 is inventory level 0, state #2 is inventory level 1, etc.**

a. Explain the meaning of the value $P_{5,4}$ (where the subscripts refer to inventory levels of 4 & 3, respectively!) and explain why it is 0.5

b Explain the meaning of the value $P_{2,4}$ and explain why it is 0.5

c. Explain the meaning of the value $P_{3,1}$ and explain why it is 0.2

d. If the shelf is full Monday morning, what is:

the expected number of days until a stockout occurs? ________

the probability that the shelf is full Thursday night? ________

the probability that the next restocking of the shelf occurs Tuesday night? ________

the expected number of times that the shelf is restocked during the next four days? _______

The transition probability matrix, P:

from	1	2	3	4	5
1	0	0	0.2	0.5	0.3
2	0	0	0.2	0.5	0.3
3	0.2	0.5	0.3	0	0
4	0	0.2	0.5	0.3	0
5	0	0	0.2	0.5	0.3

The second power of P:

from	1	2	3	4	5
1	0.04	0.2	0.37	0.3	0.09
2	0.04	0.2	0.37	0.3	0.09
3	0.06	0.15	0.23	0.35	0.21
4	0.1	0.31	0.34	0.19	0.06
5	0.04	0.2	0.37	0.3	0.09

The third power of P:

from	1	2	3	4	5
1	0.074	0.245	0.327	0.255	0.099
2	0.074	0.245	0.327	0.255	0.099
3	0.046	0.185	0.328	0.315	0.126
4	0.068	0.208	0.291	0.292	0.141
5	0.074	0.245	0.327	0.255	0.099

The fourth power of P:

from	1	2	3	4	5
1	0.0654	0.2145	0.3092	0.2855	0.1254
2	0.0654	0.2145	0.3092	0.2855	0.1254
3	0.0656	0.227	0.3273	0.273	0.1071
4	0.0582	0.2039	0.3167	0.2961	0.1251
5	0.0654	0.2145	0.3092	0.2855	0.1254

The sum of the first four powers of P:

from	1	2	3	4	5
1	0.1794	0.6595	1.2062	1.3405	0.6144
2	0.1794	0.6595	1.2062	1.3405	0.6144
3	0.3716	1.062	1.1853	0.938	0.4431
4	0.2262	0.9219	1.4477	1.0781	0.3261
5	0.1794	0.6595	1.2062	1.3405	0.6144

The mean first passage time matrix:

from	1	2	3	4	5
1	15.76923077	4.641509434	3.076923077	2.571428571	8.367346939
2	15.76923077	4.641509434	3.076923077	2.571428571	8.367346939
3	12.69230769	2.754716981	3.153846154	4	9.795918367
4	15	3.396226415	2.307692308	3.514285714	10.81632653
5	15.76923077	4.641509434	3.076923077	2.571428571	8.367346939

The steady-state distribution:

i	Pi
1	0.063415
2	0.21545
3	0.31707
4	0.28455
5	0.11951

6. BIRTH/DEATH MODEL OF QUEUE: Customers arrive at a grocery checkout lane in a Poisson process at an average rate of one every two minutes if there are 2 or fewer customers already in the checkout lane, and one every four minutes if there are already 3 in the lane. If there are already 4 in the lane, no additional customers will join the queue. The service times are exponentially distributed, with an average of 1.5 minutes if there are fewer than 3 customers in the checkout lane. If three or four customers are in the lane, another clerk assists in bagging the groceries, so that the service time average is reduced to 1 minute. Consult the computer output which follows to answer the following questions:

(a.) Draw the diagram for this birth/death process, indicating the birth & death rates.

(b.) Explain the computation of π_0. (Write the numerical expression.)

(c.) What fraction of the time will the cashier be busy? _________

(d.) What fraction of the time will the second clerk be busy at this checkout lane?_________

(e.) Explain the computation of the average number of customers in this checkout lane. (Write the numerical expression.)

(f.) What is the average time that a customer spends in this checkout lane? _________

Steady-State Distribution

i	Lambda	Mu	ρ	Pi	CDF
0	0.5000	0.0000	1.000000	0.375367	0.375367
1	0.5000	0.6667	0.750000	0.281525	0.656891
2	0.5000	0.6667	0.750000	0.211144	0.868035
3	0.2500	1.0000	0.500000	0.105572	0.973607
4	0.0000	1.0000	0.250000	0.026393	1.000000

(ρ[0]≡1, ρ[1]=LAMBDA[0]÷MU[1], etc.)

7. Dynamic Programming: We wish to plan production of an expensive, low-demand item for the next three months (January, February, & March).

- the cost of production is $15 for setup, plus $5 per unit produced, up to a maximum of 4 units.
- the storage cost for inventory is $2 per unit, based upon the level at the <u>beginning</u> of the month.
- a maximum of 3 units may be kept in inventory at the end of each month; any excess inventory is simply discarded.
- the demand each month is random, with the same probability distribution:

d	0	1	2
P{D=d}	0.3	0.4	0.3

- there is a penalty of $25 per unit for any demand which cannot be satisfied. Backorders are <u>not</u> allowed.
- the inventory at the end of December is 1.
- a salvage value of $4 per unit is received for any inventory remaining at the end of the last month (March)

Consult the computer output which follows to answer the following questions: **Note that in the computer output, stage 3 = January, stage 2 = February, etc.** (i.e., n = # months remaining in planning period.)

a. What is the optimal production quantity for January? _______

b. What is the total expected cost for the three months? __________

c. If, during January, the demand is 1 unit, what should be produced in February? ______

d. Three values have been blanked out in the computer output, What are they?

i. the optimal value $f_2(1)$ ___________

ii. the optimal decision $x_2^*(1)$ ___________

iii. the cost associated with the decision to produce 1 unit in February when the inventory is 0 at the end of January. ____________

The table of costs for each combination of state & decision at stage 2 is:

---STAGE 2---

s \ x:	0	1	2	3	4
0	46.00	[ ]	34.62	31.89	33.60
1	26.69	31.62	28.89	30.60	35.00
2	13.62	25.89	27.60	32.00	37.00
3	7.89	24.60	29.00	34.00	39.00

The tables of the optimal value function $f_n(S_n)$ at each stage are:

Stage 3:

State	Optimal Values	Optimal Decisions
0	44.61	4
1	39.83	0
2	28.33	0
3	21.82	0

Stage 2:

State	Optimal Values	Optimal Decisions
0	31.89	3
1	[ ]	[ ]
2	13.62	0
3	7.89	0

Stage 1:

State	Optimal Values	Optimal Decisions
0	21.00	2
1	8.30	0
2	0.00	0
3	¯2.00	0