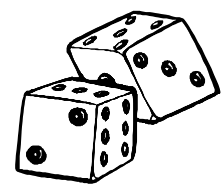
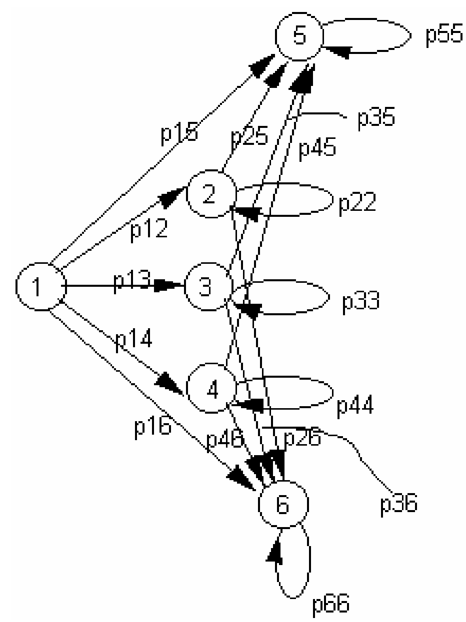
56:171 Operations Research
Homework #9 -- Due November 8, 2002

Computation may be done with the MARKOV workspace of APL functions, or any other software that you may wish to use.

1. ***Markov Chains***. In the game of "craps", one player rolls a pair of six-sided dice one or more times, until he or she either wins or loses. Suppose that we are the "roller":
   - On the first throw, if we roll a 7 or an 11, we *win* right away.
   - If we roll a 2, 3, or 12, we *lose* right away.
   - If we first roll a 4, 5, 6, 8, 9, or 10, we keep rolling the dice until we get either a 7 or the total rolled on the first throw:
     - If we get a 7, we *lose*.
     - If we roll the same total as the first throw, we *win*.





Define the states of a Markov chain model as :
state 1 : start the game,
state 2 : roll a 4 or 10,
state 3 : roll a 5 or 9,
state 4 : roll a 6 or 8,
state 5 : win,
state 6 : loss.
(We have simplified the model by combining states: for example, whether a 4 or 10 is rolled, the probability of "making one's point" is the same.)

There are 6×6=36 different outcomes (equally likely) of throwing the dice:

```
+|1 2 3  4  5  6
1|2 3 4  5  6  7
2|3 4 5  6  7  8
3|4 5 6  7  8  9
4|5 6 7  8  9 10
5|6 7 8  9 10 11
6|7 8 9 10 11 12
```

a. Complete the table with the probability distribution of the outcome of a toss:

Sum:	2	3	4	5	6	7	8	9	10	11	12
Probability											

The transition probability matrix is :

	1	2	3	4	5	6
1	0	6/36	8/36	10/36	8/36	4/36
2	0	27/36	0	0	3/36	6/36
3	0	0	26/36	0	4/36	6/36
4	0	0	0	25/36	5/36	6/36
5	0	0	0	0	1	0
6	0	0	0	0	0	1

b. Identify the following matrices:

Q =

From\to	1	2	3	4
1				
2				
3				
4				

R=

From\to	5	6
1		
2		
3		
4		

E = (I–Q)$^{-1}$ = *Expected number of visits*

```
    \  1     2          3           4
1   |  1   0.6666667  0.8    0.90909091
2   |  0   4          0      0
3   |  0   0          3.6    0
4   |  0   0          0      3.2727273
```

A = ER = *Absorption probabilities*

```
        \      5            6
  1     |  0.49292929  0.50707071
  2     |  0.33333333  0.66666667
  3     |  0.4         0.6
  4     |  0.45454545  0.54545455
```

c. What is the probability of winning for the roller (the person rolling the dice)?

d. What is the expected number of rolls of the dice in a crap game?

e. If the first roll is a "4", what is the roller's probability of winning?

f. If the first roll of the dice is a "4", how many additional rolls are expected before the game ends?

2. Light Bulb Replacement A bowling center has just purchased a new outdoor sign containing 1000 light bulbs. Based upon historical data, the manager has the following probability distribution of a bulb's failure. Failed bulbs are replaced monthly. The age of a bulb is its age at the beginning of a month, so when t=0, the bulb has just been placed in service and according to the table, has a 50% probability of failing during its first month of operation. (There is a high "infant mortality" rate.)

Age t (months)	0	1	2	3	4
Failure probability p(t)	0.5	0.1	0.1	0.1	0.2
Cumulative probability F(t)	0.5	0.6	0.7	0.8	1.0
Conditional probability of failure, $f(t\|n\geq t)$	0.5	0.2	0.25	0.333	1.0

For example, the probability that a bulb fails during the next month, if it has reached the age 2 months, is $f(2|n\geq 2) = 0.1/(0.1+0.1+0.2) = 0.25$.

We assume that a 4-month-old bulb is certain to fail during its next month of operation.

Define a discrete-time Markov chain model for one of positions in the sign,where the state t=0, 1, …4 is the age of the bulb in that position at the end of the month *after* all failed bulbs have been replaced.

a. Draw a transition diagram and write the transition probability matrix.

From\to	0	1	2	3	4
0					
1					
2					
3					
4					

b. What is the steadystate probability distribution?

t	0	1	2	3	4
π_t					

c. According to this model, what is the expected number of replacements when the sign is examined at the end of each month?

d. If each bulb costs $2 to replace, what is the monthly replacement cost?

Because of the high "infant mortality" rate, the manager is considering "burn-in"—for $2.40 the manufacturer will burn them for one month at the plant. The more expensive bulb will have one less month of life, but the first failure-prone month is eliminated. She wants to know if the extra $0.40 is worth it.

e. Define a new Markov chain model with states t=1, 2, 3, 4, and write the transition probability matrix:

From\to	1	2	3	4
1				
2				
3				
4				

f. What is the steadystate distribution of the new Markov chain?

t	1	2	3	4
π_t				

g. What is the expected number of failed bulbs each month?

h. What is the expected monthly replacement cost for the sign?

3. Consider a **two-stage production flow line**. At each stage, an operation is performed on the part being processed. Parts are introduced into the system at stage 1. Processing is serial, and each stage can hold only one part at a time.

For purposes of analysis, we discretize time into 1-minute intervals. At the beginning of an interval, stage 1 is empty, working, or blocked whereas stage 2 is either working or empty. The system operates under the following rules:

- If stage 1 is empty at the beginning of an intgerval, a new part is introduced into stage 1 with probability 0.9. Work begins on the part; however, it cannot be completed during the minute it is introduced. If stage 1 is working or blocked, no part enters.

- If stage 2 is working at the beginning of an interval, the part will be completed and will leave the system with probability 0.8. Alternatively, it will remain in stage 2 with probability 0.2.
- If stage 1 is working at the beginning of an interval, the part will be completed with probability 0.6. A completed part will move to stage 2 if stage 2 is empty. Otherwise, it is blocked and remains in stage 1 until stage 2 becomes empty.

Develop a discrete-time Markov chain model of this situation, defining the states

1. (empty, idle)
2. (working, empty)
3. (working, working)
4. (empty, working)
5. (blocked, working)

a. Draw the transition diagram, and write the transition probability matrix.

From\to	1	2	3	4	5
1					
2					
3					
4					
5					

b. Compute the steadystate probability distribution of this process.

i	1	2	3	4	5
π_i					

c. What is the expected idle time (empty or blocked) for each stage?

d. What is the expected throughput, i.e., # parts completed per hour?

Problems 2 & 3 are based upon exercises in the textbook *Operations Research Models & Methods*, by Paul Jensen & Jonathan Bard.