SOLUTIONS

56:171 Operations Research
Homework #7 Solutions -- Fall 2002

1. **Decision Analysis** (adapted from Exercise 15.2-7, page 784, *Operations Research*, 7th edition, by Hillier & Lieberman.)
Dwight Moody is the manager of a large farm with 1,000 acres of arable land. For greater efficiency, Dwight always devotes the farm to growing one crop at a time. He now needs to make a decision on which one of four crops to grow during the upcoming growing season. For each of these crops, Dwight has obtained the following estimates of crop yields and net incomes per bushel under various weather conditions.

Weather	Crop 1	Crop 2	Crop 3	Crop 4
Dry	20	15	30	40
Moderate	35	20	25	40
Damp	40	30	25	40
Net income/bushel	\$1.00	\$1.50	\$1.00	\$0.50

After referring to historical meteorological records, Dwight also estimated the following probabilities for the weather during the growing season:

Dry	0.3
Moderate	0.5
Damp	0.2

Using the criterion of "Maximize expected payoff", determine which crop to grow.

Solution: Expected payoffs

- Crop 1: $(20\times 0.3+35\times 0.5+40\times 0.2)\times \$1.00=\$31.50$
- Crop 2: $(15\times 0.3+20\times 0.5+30\times 0.2)\times \$1.50=\$30.75$
- Crop 3: $(30\times 0.3+25\times 0.5+25\times 0.2)\times \$1.10=\$26.50$
- Crop 4: $(40\times 0.3+40\times 0.5+40\times 0.2)\times \$0.50=\$20.00$

Dwight Moody should choose crop 1 with \$31.50 payoff.

2. **Bayes' Rule** (Exercise 15.3-15, pp. 788-789, *Operations Research*, 7th edition, by Hillier & Lieberman)

There are two biased coins, coin A with probability of landing heads equal to 0.8 and the coin B with probability of heads equal to 0.4. One coin is chosen at random (each with probability 50%) to be tossed twice. You are to receive \$100 if you correctly predict how many heads will occur in two tosses of this coin.

a. Using the "Maximum Expected Payoff" criterion, what is the optimal prediction, and what is the corresponding expected payoff?

Solution: We are given $P(H|A) = 0.8$ and $P(H|B) = 0.4$

$P(2H \mid A) = (0.8)^2 = 0.64$ $\qquad$ $P(2H \mid B) = (0.4)^2 = 0.16$

$P(1H \mid A) = 1-0.64-0.04 = 0.32$ $\qquad$ $P(2H \mid B) = 1-0.16-0.36 = 0.48$

$P(0H \mid A) = (0.2)^2 = 0.04$ $\qquad$ $P(2H \mid B) = (0.6)^2 = 0.36$

$P(2H) = P(2H \mid A)\times P(A) + P(2H \mid B)\times P(B) = 0.5\times 0.64 + 0.5\times 0.16 =$ **0.4**

$P(1H) = P(1H \mid A)\times P(A) + P(1H \mid B)\times P(B) = 0.5\times 0.32 + 0.5\times 0.48 =$ **0.4**

$P(0H) = P(0H \mid A)\times P(A) + P(0H \mid B)\times P(B) = 0.5\times 0.04 + 0.5\times 0.36 = 0.2$

Should predict either 1 *or* 2 heads, each with expected payoff \$40.00

Suppose now that you may observe a preliminary toss of the chosen coin before predicting.

b. Determine your optimal prediction after observing a head in the preliminary toss.
Solution: Let H_0 denote the event that the outcome of the preliminary toss is heads, and T_0 if tails. By the "law of total probability",

$$P(H_0) = P(H_0 \mid A)P(A) + P(H_0 \mid B)P(B) = 0.8\times 0.5 + 0.4\times 0.5 = 0.6$$

and $P(T_0) = 1 - P(H_0) = 0.4$

According to Bayes' Rule,

$$P(A \mid H_0) = \frac{P(H_0 \mid A)\times P(A)}{P(H_0)} = \frac{0.8\times 0.5}{0.6} = \frac{2}{3} \Rightarrow P(B \mid H_0) = 1 - \frac{2}{3} = \frac{1}{3}$$

Then the probabilities of the outcomes of the following tosses (given H_0) are

$P(0H|H_0) = P(0H|\text{when coin is A}) \times P(A|H_0) + P(0H|\text{when coin is B}) \times P(B|H_0)$
$= 0.04\times 2/3 + 0.36\times 1/3 = 0.1467$
$P(1H|H_0) = P(1H|\text{when coin is A}) \times P(A|H_0) + P(1H|\text{when coin is B}) \times P(B|H_0)$
$= 0.32\times 2/3 + 0.48\times 1/3 = 0.3733$
$P(2H|H_0) = P(2H|\text{when coin is A}) \times P(A|H_0) + P(2H|\text{when coin is B}) \times P(B|H_0)$
$= 0.64\times 2/3 + 0.16\times 1/3 = 0.48$

Expected maximal payoff, given H_0, is \$48.00, obtained if one predicts two heads.

…after observing a tail in the preliminary toss.
Solution: According to Bayes' Rule,

$$P(A \mid T_0) = \frac{P(T_0 \mid A)\times P(A)}{P(T_0)} = \frac{0.2\times 0.5}{0.4} = \frac{1}{4} \Rightarrow P(B \mid T_0) = 1 - \frac{1}{4} = \frac{3}{4}$$

Then the probabilities of the outcomes of the following tosses (given T_0) are

$P(0H|T_0) = P(0H|\text{when coin is A}) \times P(A|T_0) + P(0H|\text{when coin is B}) \times P(B|T_0)$
$= 0.04\times 1/4 + 0.36\times 3/4 = 0.28$
$P(1H|T_0) = P(1H|\text{when coin is A}) \times P(A|T_0) + P(1H|\text{when coin is B})) \times P(B|T_0)$
$= 0.32\times 1/4 + 0.48\times 3/4 = 0.44$
$P(2H|T_0) = P(2H|\text{when coin is A})) \times P(A|T_0) + P(2H|\text{when coin is B})) \times P(B|T_0)$
$= 0.64\times 1/4 + 0.16\times 3/4 = 0.28$

Expected maximal payoff, given T_0, is now \$44.00, again by predicting two heads.

c. What is the expected value of the preliminary toss?
Solution: The expected payoff if there is a preliminary toss is
EVWSI =expected value with sample information
= $E(\text{payoff}|H_0)P(H_0) + E(\text{payoff}|T_0)P(T_0)$
= \$48 ×0.6 + \$44 ×0.4 = \$46.40
EVWOI = expected value without information
= \$40.00 (from part (a).)
EVSI = expected value of sample information
= **EVWSI – EVWOI**
(i.e., expected value with sample info minus expected value w/o info)
= \$46.40 – \$40 = \$6.40

3. Integer Programming Model (based upon Case 12.3, page 649-653 of Operations Research, 7th edition, by Hillier & Lieberman. See the text for the complete case description. What follows is a condensed version.)

Brenda Sims, the saleswoman on the floor at Furniture City, understood that Furniture City required a new inventory policy. Not only was the megastore losing money by making customers unhappy with delivery delays, but it was also losing money by wasting warehouse space. By changing the inventory policy to stock only popular items and replenish them immediately when they are sold, Furniture City would ensure that the majority of customers receive their furniture immediately and that the valuable warehouse space was utilized effectively.

She decided… to use her kitchen department as a model for the new inventory policy. She would identify all kitchen sets comprising 85% of customers orders. Given the fixed amount of warehouse space allocated to the kitchen department, she would identify the items Furniture City should stock in order to satisfy the greatest number of customer orders.

Brenda analyzed her records over the past three years and determined that 20 kitchen sets were responsible for 85% of customer orders. These 20 kitchen sets were composed of up to eight features, usually with four styles of each feature (except for the dishwashers, with two styles.)

- Floor tile: styles T1, T2, T3, T4
- Wallpaper: styles W1, W2, W3, W4
- Light fixtures: styles L1, L2, L3, L4
- Cabinets: styles C1, C2, C3, C4
- Countertops: styles O1, O2, O3, O4
- Dishwashers: styles D1, D2
- Sinks: styles S1, S2, S3, S4
- Ranges: styles R1, R2, R3, R4

(Sets, 14 through 20, however, do not include dishwashers.)

The warehouse could hold 50 ft^2 of tile and 12 rolls of wallpaper in the inventory bins. the inventory shelves could hold two light fixtures, two cabinets, three countertops, and two sinks. Dishwashers and ranges are similar in size, so Furniture City stored them in similar locations. The warehouse floor could hold a total of four dishwashers and ranges.

Every kitchen set includes exactly 20 ft^2 of tile and exactly 5 rolls of wallpaper. Therefore, 20 ft^2 of a particular style of tile and five rolls of a particular style of wallpaper are required for the styles to be in stock.

a. Formulate and use LINGO to solve a binary integer programming model which will maximize the total number of kitchen sets (and thus the number of customer orders) Furniture City stocks in the local warehouse. Assume that when a customer orders a kitchen set, all the particular items composing that kitchen set are replenished at the local warehouse immediately. (The sets and data section of a LINGO model may be downloaded with this homework assignment.)

Solution

```
MODEL: ! Case: Stocking Kitchen Sets ;
! Solution provided by Grant Mast, Bart Sorensen, Dan Mullen;
SETS:
 KITCHSET/1..20/: s;
FEATURE/1..30/: X;
 BELONG(KITCHSET,FEATURE): A;
 FGROUP/1..7/:CAPACITY;
 ENDSETS
 DATA: ! A(i,j) = 1 if kitchen set i includes feature j ;
 !  T T T T W W W W L L L L C C C C O O O O S S S S D D R R R R;
 A= 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0 1 0 0
 0 1 0 0 1 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 0
 1 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0
 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0
 0 0 0 1 0 0 0 1 1 0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 1 0 1 0 0 0
 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 1 0 0 0 1
 1 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0
 0 1 0 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0 0 1 0 0 0 1
 0 1 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0
 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 1 0 0 0 1 0
 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
 0 1 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 1 0 0
 0 0 0 1 0 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0 0 1 0 0 1 0 0 0 1 0
 0 0 0 1 0 0 0 1 0 0 0 1 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0
 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0
 0 0 1 0 0 0 1 0 0 0 0 1 1 0 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0
 1 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0
 0 1 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0
 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 0 0 1
 0 1 0 0 0 0 1 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 0 1;
CAPACITY = 2 2 2 2 3 2 4;
ENDDATA
 MAX = @SUM(KITCHSET(i):s(i));
@FOR(KITCHSET(i)| i #LE# 13:
 @SUM(FEATURE(j):X(j)*A(i,j))>=8*s(i));
 @FOR(KITCHSET(i) | I #GT# 13:
@SUM(FEATURE(j):X(j)*A(i,j))>=7*s(i));
@FOR(FGROUP(K)| K #LT# 7:
@SUM(FEATURE(J)| J #GE# 4*(K-1)+1 #AND# J #LE# 4*K : X(J) ) <= CAPACITY(K));
 @FOR(FGROUP(K)| K #GE# 7:
@SUM(FEATURE(J)| J #GE# 4*K-3 #AND# J #LE# 4*K+2 : X(J) ) <= CAPACITY(K));

@FOR(KITCHSET(I):  @BIN(s(I)););
@FOR(FEATURE(J):   @BIN(X(J)););

END
```

The number of binary integer variables(20+30=50) exceeds the maximum number which is allowed by the student version of LINGO. The solution shown below was found by using LINGO to create a file in "MPS" format which can be read by most solvers—in this case CPLEX.

b. How many of each feature and style should Furniture City stock in the local warehouse? How many different kitchen sets are in stock?

Solution: Four kitchen sets (#8, 15, 18, and 20) are kept in stock in the solution which was found:

```
CPLEX> display solution variables 1-50
Variable Name           Solution Value
X(2                    1.000000
X(3                    1.000000
X(5                    1.000000
X(7                    1.000000
X(9                    1.000000
X(11                   1.000000
X(13                   1.000000
X(14                   1.000000
X(17                   1.000000
X(18                   1.000000
X(20                   1.000000
X(21                   1.000000
X(23                   1.000000
X(23                   1.000000
X(26                   1.000000
X(28                   1.000000
X(29                   1.000000
X(30                   1.000000
S(8                    1.000000
S(15                   1.000000
S(18                   1.000000
S(20                   1.000000
All other variables in the range 1-50 are zero.
```

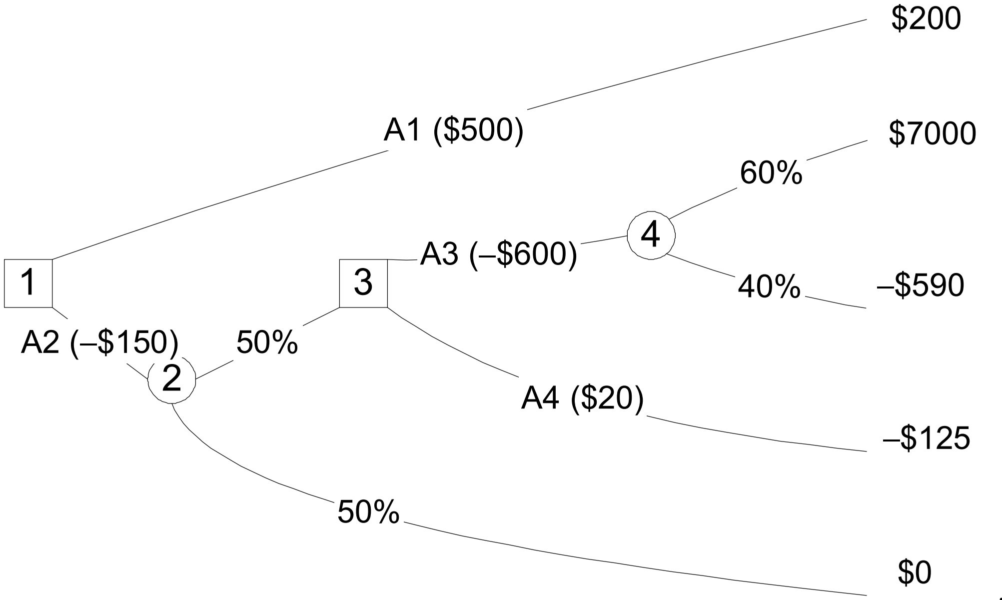
Furniture City decides to discontinue carrying nursery sets, and the warehouse space previously allocated to the nursery department is divided between the existing departments at Furniture City. The kitchen department receives enough additional space to allow it to stock both styles of dishwashers and three of the four styles of ranges.

c. How does the optimal inventory policy for the kitchen department change with this additional warehouse space?

	T1	T2	T3	T4	W1	W2	W3	W4	L1	L2	L3	L4	C1	C2	C3	C4	O1	O2	O3	O4	D1	D2	S1	S2	S3	S4	R1	R2	R3	R4
Set 1		X				X						X		X						X		X		X				X		
Set 2		X			X				X							X				X		X				X		X		
Set 3	X						X			X			X				X				X				X				X	
Set 4			X				X				X				X				X		X		X				X			
Set 5				X				X	X				X					X			X			X			X			
Set 6		X				X				X						X				X		X			X					X
Set 7	X						X					X			X			X			X		X				X			
Set 8		X			X						X		X				X					X			X					X
Set 9		X			X					X					X			X				X		X				X		
Set10	X				X				X				X						X		X					X			X	
Set11			X		X						X				X		X				X		X						X	
Set12		X				X			X					X				X				X				X		X		
Set13				X				X			X				X		X				X			X					X	
Set14				X				X				X	X						X				X				X			
Set15			X				X		X				X				X								X				X	
Set16			X				X					X	X						X					X			X			
Set17	X							X		X					X				X							X			X	
Set18		X					X				X			X						X			X					X		
Set19		X						X				X				X				X				X						X
Set20		X					X		X				X					X							X					X

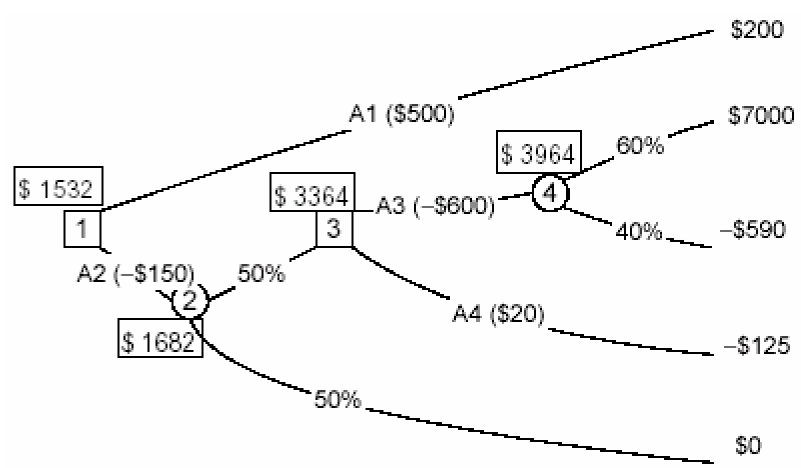
Table: Features composing each of twenty kitchen sets.

4. Decision Trees. Consider the decision tree below: On each decision branch, the immediate payoff (if +) or cost (if –) is shown. The probability is shown on each random branch. On the far right is the final payoff or cost



a. Fold back" the decision tree.

Solution:



b. What is the *expected payoff* at node 1?
Solution: $1532

c. What is the *optimal decision* at node 1?
Solution: Select alternative A2.