56:171 Operations Research
Homework #5 Solution– Fall 2002

1. Consider the transportation tableau:

dstn→ ↓source	1	2	3	4	5	Supply
A	12	8	9	15	11	9
B	10	11	12	11	14	7
C	9	7	11	14	8	4
D	13	12	13	12	12	7
E	8	9	10	9	10	3
Demand=	4	7	5	5	9	

a. Use the initial basic solution: X_{A3}=5, X_{A5}=4, X_{B1}=4, X_{B4}=3, X_{C4}=X_{C5}=2, X_{D2}=7, X_{E5}=3 & _____ = 0. *(Choose one more variable to complete the basis. Any choice is valid except one that would create a "cycle" of basic cells in the tableau!)*

Answer: Any cell *except* $X_{A4}, X_{B5}, X_{C1}, X_{C3}, X_{E3}$ or X_{E4}

dstn→ ↓source	1	2	3	4	5	Supply
A	12	8	9	15	11	9
B	10	11	12	11	14	7
C	9	7	11	14	8	4
D	13	12	13	12	12	7
E	8	9	10	9	10	3
Demand=	4	7	5	5	9	

Note that the diagonally shaded cells would create a cycle of basic cells if chosen to be basic.

b. Compute two different sets of values for the dual variables U & V (*simplex multipliers*) for this basis.

Answer: Let's choose X_{E2}=0 to be basic in (a) above.

If we arbitrarily choose $U_E = 0$ then $U_A = 1, U_B = -5, U_C = -2, U_D = 3$ and

$V_1 = 15, V_2 = 9, V_3 = 8, V_4 = 16, V_5 = 10$

		V_1 = 15	V_2 = 9	V_3 = 8	V_4 = 16	V_5 = 10	
		1	2	3	4	5	**Supply**
U_A = 1	**A**	12	8	5 / 9	15	4 / 11	9
U_B = -5	**B**	4 / 10	11	12	3 / 11	14	7
U_C = -2	**C**	9	7	11	2 / 14	2 / 8	4
U_D = 3	**D**	13	7 / 12	13	12	12	7
U_E = 0	**E**	8	0 / 9	10	9	3 / 10	3
	Demand	4	7	5	5	9	

If we instead arbitrarily choose $U_A = 0$ then we will obtain different values:

$U_B = -6, U_C = -3, U_D = 2, U_E = -1$ and $V_1 = 16, V_2 = 10, V_3 = 9, V_4 = 17, V_5 = 11$

		V_1 = 16	V_2 = 10	V_3 = 9	V_4 = 17	V_5 = 11	
		1	2	3	4	5	**Supply**
U_A = 0	**A**	12	8	5 / 9	15	4 / 11	9
U_B = -6	**B**	4 / 10	11	12	3 / 11	14	7
U_C = -3	**C**	9	7	11	2 / 14	2 / 8	4
U_D = 2	**D**	13	7 / 12	13	12	12	7
U_E = -1	**E**	8	0 / 9	10	9	3 / 10	3
	Demand	4	7	5	5	9	

c. Using each set of simplex multipliers, price all of the nonbasic cells. How do the reduced costs depend upon the choice of dual variables? Select the variable having the "most negative" reduced cost to enter the basis.

Answer: By calculating $\overline{C}_{ij} = C_{ij} - (U_i + V_j)$ for *i=A,B,C,D,E* and *j=1,2,3,4,5*

We can get the following reduced costs, when $U_E = 0$.

$\overline{C}_{A1} = -4, \overline{C}_{A2} = -2, \overline{C}_{A4} = -2,$

$\overline{C}_{B2} = 7, \overline{C}_{B3} = 9, \overline{C}_{B5} = 9,$

$$\overline{C}_{C1} = -4, \overline{C}_{C2} = 0, \overline{C}_{C3} = 5,$$
$$\overline{C}_{D1} = -5, \overline{C}_{D3} = 2, \overline{C}_{D4} = -7, \overline{C}_{D5} = -1,$$
$$\overline{C}_{E1} = -7, \overline{C}_{E3} == 2, \overline{C}_{E4} = -7$$

When $U_A = 0$, the results are *exactly* the same—the reduced costs depend on the sums ($U_i + V_j$), not on the values U_i & V_j individually!

The "most negative" (i.e., smallest) reduced cost is −7, which is that of each of the nonbasic variables X_{D4}, X_{E1}, X_{E4}.

d. What variable will leave the basis as the new variable enters the basis?

Answer: If, for example, we chose X_{E4} as a new basic variable then X_{C4} must leave the basis.

e. Complete the computation of the optimal solution, using the transportation simplex method.

Answer: The optimal solution is the following.

$X_{A2} = 4, X_{A3} = 5,$

$X_{B1} = 4, X_{B4} = 3,$

$X_{C2} = 3, X_{C5} = 1,$

$X_{D5} = 7,$

$X_{E4} = 2, X_{E5} = 1$

and all others are 0.

Cost = 291 (Solution is optimal!)

Next table is the following

		$V_1 =$ 9	$V_2 =$ 10	$V_3 =$ 9	$V_4 =$ 10	$V_5 =$ 11	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	3 / **12**	-2 / **8**	**5** / **9**	5 / **15**	**4** / **11**	**9**
$U_B =$ 1	**B**	**4** / **10**	0 / **11**	1 / **12**	**3** / **11**	2 / **14**	**7**
$U_C =$ -3	**C**	3 / **9**	0 / **7**	5 / **11**	7 / **14**	**4** / **8**	**4**
$U_D =$ 2	**D**	2 / **13**	**7** / **12**	2 / **13**	0 / **12**	-1 / **12**	**7**
$U_E =$ -1	**E**	0 / **8**	**0** / **9**	2 / **10**	**2** / **9**	**1** / **10**	**3**
	Demand	**4**	**7**	**5**	**5**	**9**	

The X_{A2} has the most negative reduced cost -2 and entering X_{A2} into the basis makes X_{E2} leave the basis:

		$V_1 =$ 7	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 8	$V_5 =$ 11	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	5 / **12**	0 / **8**	5 / **9**	7 / **15**	4 / **11**	**9**
$U_B =$ 3	**B**	4 / **10**	0 / **11**	0 / **12**	3 / **11**	0 / **14**	**7**
$U_C =$ -3	**C**	5 / **9**	2 / **7**	5 / **11**	9 / **14**	4 / **8**	**4**
$U_D =$ 4	**D**	2 / **13**	7 / **12**	0 / **13**	0 / **12**	-3 / **12**	**7**
$U_E =$ 1	**E**	0 / **8**	0 / **9**	0 / **10**	2 / **9**	1 / **10**	**3**
	Demand	4	7	5	5	9	

The X_{D5} has most negative reduced cost -3 and entering X_{D5} into the basis makes X_{A5} leave the basis:

		$V_1 =$ 6	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 7	$V_5 =$ 8	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	6 / **12**	4 / **8**	5 / **9**	8 / **15**	3 / **11**	**9**
$U_B =$ 4	**B**	4 / **10**	-1 / **11**	-1 / **12**	3 / **11**	2 / **14**	**7**
$U_C =$ 0	**C**	3 / **9**	-1 / **7**	2 / **11**	7 / **14**	4 / **8**	**4**
$U_D =$ 4	**D**	3 / **13**	3 / **12**	0 / **13**	1 / **12**	4 / **12**	**7**
$U_E =$ 2	**E**	0 / **8**	-1 / **9**	-1 / **10**	2 / **9**	1 / **10**	**3**
	Demand	4	7	5	5	9	

Continuing in the same way, we get the following table for which there is no variable having negative reduced cost—therefore it is the optimal solution.

		$V_1 =$ 7	$V_2 =$ 8	$V_3 =$ 9	$V_4 =$ 8	$V_5 =$ 9	
		1	**2**	**3**	**4**	**5**	**Supply**
$U_A =$ 0	**A**	5 **12**	4 **8**	5 **9**	7 **15**	2 **11**	**9**
$U_B =$ 3	**B**	4 **10**	0 **11**	0 **12**	3 **11**	2 **14**	**7**
$U_C =$ -1	**C**	3 **9**	3 **7**	3 **11**	7 **14**	1 **8**	**4**
$U_D =$ 3	**D**	3 **13**	1 **12**	1 **13**	1 **12**	7 **12**	**7**
$U_E =$ 1	**E**	0 **8**	0 **9**	0 **10**	2 **9**	1 **10**	**3**
	Demand	**4**	**7**	**5**	**5**	**9**	

2. **Production scheduling** (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 394) The MLK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

   For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced in Regular Time (RT) and in Overtime (OT). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced in regular time, (3) the cost (in thousands of dollars) per unit produced in overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for product 1 on the left and the number for product 2 on the right.

	Max combined production			Unit cost of production ($K)		
Month	RT	OT	Sales	RT	OT	Storage cost ($K)
1	10	3	5/3	15/16	18/20	1/2
2	8	2	3/5	17/15	20/18	2/1
3	10	3	4/4	19/17	22/22	

   The production manager wants a schedule developed for the number of units of each of the two products to be produced in regular time and (if regular time production capacity is used up) in overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

a. Formulate this problem as a balanced transportation problem by constructing the appropriate transportation tableau.

b. Use the Northwest Corner Method to find an initial basic feasible solution. Is it degenerate?

Answer for a) and b): The solution is *not* degenerate.

	1A	1B	2A	2B	3A	3B	EXCESS SUPPLY	
R1	**5** 15	**3** 16	**2** 16	18	18	19	0	10
O1	18	20	**1** 19	**2** 22	21	23	0	3
R2	inf	inf	17	**3** 15	**4** 19	**1** 16	0	8
O2	inf	inf	20	18	22	**2** 19	0	2
R3	inf	inf	inf	inf	19	**1** 17	**9** 0	10
O3	inf	inf	inf	inf	22	22	**3** 0	3
	5	3	3	5	4	4	12	**SUM=410**

c. Use the transportation simplex algorithm to find the optimal solution. Is it degenerate? Are there multiple optima?

Answer: The optimal solution is the following.

	1A	1B	2A	2B	3A	3B	EXCESS SUPPLY	
R1	**5** 15	**3** 16	**2** 16	18	18	19	0	10
O1	18	20	19	22	21	23	**3** 0	3
R2	inf	inf	**1** 17	**5** 15	19	**2** 16	0	8
O2	inf	inf	20	18	22	19	**2** 0	2
R3	inf	inf	inf	inf	**4** 19	**2** 17	**4** 0	10
O3	inf	inf	inf	inf	22	22	**3** 0	3
Demand	5	3	3	5	4	4	12	**SUM=389**

3. **Assignment Problem.** (adapted from O.R. text by Hillier & Lieberman, 7th edition, page 399.) Four cargo ships will be used for shipping goods from one port to four other ports (labeled 1, 2, 3, 4). Any ship can be used for making any one of these four trips. However, because of differences in the ships and cargoes, the total cost of loading, transporting, and unloading the goods for the different ship-port combinations varies considerably, as shown in the following table:

PORT→ ↓SHIP	1	2	3	4
1	$500	$400	$600	$700
2	$600	$600	$700	$500
3	$700	$500	$700	$600
4	$500	$400	$600	$600

The objective is to assign the four ships to four different ports in such a way as to minimize the total cost for all four shipments.

a. Use the Hungarian method to find an optimal solution.

Answer:

There are several optimal solutions:

After row reduction

PORT→ ↓ship	1	2	3	4
1	$100	$0	$200	$300
2	$100	$100	$200	$0
3	$200	$0	$200	$100
4	$100	$0	$200	$200

After column reduction

PORT→ ↓ship	1	2	3	4
1	$0	$0	$0	$300
2	$0	$100	$0	$0
3	$100	$0	$0	$100
4	$0	$0	$0	$200

For example, $X_{41}=X_{12}=X_{33}=X_{24}=1$ is optimal, as is $X_{11}=X_{32}=X_{43}=X_{24}=1$. (All optimal solutions have the assignment $X_{24}=1$.)

b. Reformulate this as an equivalent transportation problem.

Answer: Supplies & Demands are all 1!

dstn→ ↓source	1	2	3	4	Supply=
1	500	400	600	700	1
2	600	600	700	500	1
3	700	500	700	600	1
4	500	400	600	600	1
Demand=	1	1	1	1	

c. Use the Northwest Corner Method to obtain an initial basic feasible solution. (This will be a *degenerate* solution. Be sure to specify which variables are basic!)

Answer: Let the shaded cells form the initial basis.

	1	2	3	4	SUPPLY
1	1 / 500	/ 400	/ 600	/ 700	1
2	/ 600	1 / 600	/ 700	/ 500	1
3	/ 700	/ 500	1 / 700	/ 600	1
4	/ 500	/ 400	/ 600	1 / 600	1
Demand	1	1	1	1	

d. Use the transportation simplex method to find the optimal solution.

Answer:

		500	400	500	400	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	100 / 600	300 / 700	1
200	2	-100 / 600	1 / 600	0 / 700	-100 / 500	1
200	3	0 / 700	-100 / 500	1 / 700	0 / 600	1
200	4	-200 / 500	**-200** / **400**	-100 / 600	1 / 600	1
	Demand	1	1	1	1	

$X_{4,2}$ enters into the basis with value change and $X_{4,4}$ leaves the basis.

(Assignments, & therefore cost as well, have changed.)

		500	400	500	400	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	100 / 600	300 / 700	1
200	2	-100 / 600	0 / 600	1 / 700	**-100** / **500**	1
200	3	0 / 700	-100 / 500	0 / 700	1 / 600	1
0	4	0 / 500	1 / 400	100 / 600	200 / 600	1
	Demand	1	1	1	1	

$X_{2,4}$ has entered the basis with a value change and $X_{2,3}$ leaves the basis.

		500	400	400	300	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	200 / 600	400 / 700	1
200	2	-100 / 600	0 / 600	100 / 700	1 / 500	1
300	3	-100 / 700	**-200** / **500**	1 / 700	0 / 600	1
0	4	0 / 500	1 / 400	200 / 600	300 / 600	1
	Demand	1	1	1	1	

$X_{3,2}$ enters into the basis *without* value change and $X_{2,2}$ leaves the basis.

		500	400	500	400	
		1	2	3	4	SUPPLY
0	1	1 / 500	0 / 400	100 / 600	300 / 700	1
100	2	0 / 600	100 / 600	100 / 700	1 / 500	1
100	3	100 / 700	0 / 500	1 / 700	0 / 600	1
0	4	0 / 500	1 / 400	100 / 600	200 / 600	1
	Demand	1	1	1	1	

There is no negative reduced cost, i.e., this is optimal.

e. In how many iterations was the solution degenerate?
Answer: All the solutions are degenerate.

f. How many iterations produce a change in the *values* of the variables?
Answer: 2 iterations produce a change in the value of the variables.

g. How many iterations leave the variables *unchanged in value* (although the basis changes)?
Answer: 1 iteration leaves all variables unchanged in value.

4. Return of Marky D. Sod Recall the LP model for this problem in HW#4:
Buster Sod's younger brother, Marky Dee, operates three ranches in Texas. the acreage and irrigation water available for the three farms are shown below:

FARM	ACREAGE	WATER AVAILABLE (ACRE-FT)
1	400	1500
2	600	2000
3	300	900

Three crops can be grown. However, the maximum acreage that can be grown of each crop is limited by the amount of appropriate harvesting equipment available. The three crops are described below. Any combination of crops may be grown on a farm.

CROP	TOTAL HARVESTING CAPACITY (IN ACRES)	WATER REQMTS (ACRE-FT PER ACRE)	EXPECTED PROFIT ($/ACRE)
Milo	700	6	400
Cotton	800	4	300
Wheat	300	2	100

Decision variables: X_{ij} = # acreas of crop j planted on farm i.
The LINDO model (generated by LINGO) is:

```
MAX     400 X1MILO + 300 X1COTTON + 100 X1WHEAT + 400 X2MILO
      + 300 X2COTTON + 100 X2WHEAT + 400 X3MILO + 300 X3COTTON + 100 X3WHEAT
SUBJECT TO
        2)   X1MILO + X1COTTON + X1WHEAT <=   400
        3)   6 X1MILO + 4 X1COTTON + 2 X1WHEAT <=   1500
        4)   X2MILO + X2COTTON + X2WHEAT <=   600
        5)   6 X2MILO + 4 X2COTTON + 2 X2WHEAT <=   2000
        6)   X3MILO + X3COTTON + X3WHEAT <=   300
        7)   6 X3MILO + 4 X3COTTON + 2 X3WHEAT <=   900
        8)   X1MILO + X2MILO + X3MILO <=   700
        9)   X1COTTON + X2COTTON + X3COTTON <=   800
       10)   X1WHEAT + X2WHEAT + X3WHEAT <=   300
  END

       1)      320000.0

  VARIABLE        VALUE          REDUCED COST
    X1MILO         0.000000         0.000000
  X1COTTON       375.000000         0.000000
   X1WHEAT         0.000000        33.333332
    X2MILO        50.000000         0.000000
  X2COTTON       425.000000         0.000000
   X2WHEAT         0.000000        33.333332
    X3MILO       150.000000         0.000000
  X3COTTON         0.000000         0.000000
   X3WHEAT         0.000000        33.333332

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        25.000000         0.000000
        3)         0.000000        66.666664
        4)       125.000000         0.000000
        5)         0.000000        66.666664
        6)       150.000000         0.000000
        7)         0.000000        66.666664
        8)       500.000000         0.000000
        9)         0.000000        33.333332
       10)       300.000000         0.000000
```

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1MILO	400.000000	0.000000	INFINITY
X1COTTON	300.000000	INFINITY	0.000000
X1WHEAT	100.000000	33.333328	INFINITY
X2MILO	400.000000	0.000000	0.000000
X2COTTON	300.000000	0.000000	0.000000
X2WHEAT	100.000000	33.333328	INFINITY
X3MILO	400.000000	INFINITY	0.000000
X3COTTON	300.000000	0.000000	INFINITY
X3WHEAT	100.000000	33.333328	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	400.000000	INFINITY	25.000000
3	1500.000000	100.000000	300.000000
4	600.000000	INFINITY	125.000000
5	2000.000000	750.000000	300.000000
6	300.000000	INFINITY	150.000000
7	900.000000	900.000000	900.000000
8	700.000000	INFINITY	500.000000
9	800.000000	75.000000	425.000000
10	300.000000	INFINITY	300.000000

THE TABLEAU:

ROW	(BASIS)	X1MILO	X1COTTON	X1WHEAT	X2MILO	X2COTTON	X2WHEAT
1	ART	0.000	0.000	33.333	0.000	0.000	33.333
2	SLK 2	-0.500	0.000	0.500	0.000	0.000	0.000
3	X1COTTON	1.500	1.000	0.500	0.000	0.000	0.000
4	SLK 4	0.500	0.000	0.167	0.000	0.000	0.667
5	X2MILO	1.000	0.000	0.333	1.000	0.000	0.333
6	SLK 6	0.000	0.000	0.000	0.000	0.000	0.000
7	X3MILO	0.000	0.000	0.000	0.000	0.000	0.000
8	SLK 8	0.000	0.000	-0.333	0.000	0.000	-0.333
9	X2COTTON	-1.500	0.000	-0.500	0.000	1.000	0.000
10	SLK 10	0.000	0.000	1.000	0.000	0.000	1.000

ROW	X3MILO	X3COTTON	X3WHEAT	SLK 2	SLK 3	SLK 4	SLK 5
1	0.000	0.000	33.333	0.000	66.667	0.000	66.667
2	0.000	0.000	0.000	1.000	-0.250	0.000	0.000
3	0.000	0.000	0.000	0.000	0.250	0.000	0.000
4	0.000	-0.333	0.000	0.000	0.083	1.000	-0.167
5	0.000	-0.667	0.000	0.000	0.167	0.000	0.167
6	0.000	0.333	0.667	0.000	0.000	0.000	0.000
7	1.000	0.667	0.333	0.000	0.000	0.000	0.000
8	0.000	0.000	-0.333	0.000	-0.167	0.000	-0.167
9	0.000	1.000	0.000	0.000	-0.250	0.000	0.000
10	0.000	0.000	1.000	0.000	0.000	0.000	0.000

ROW	SLK 6	SLK 7	SLK 8	SLK 9	SLK 10	
1	0.00E+00	67.	0.00E+00	33.	0.00E+00	0.32E+06
2	0.000	0.000	0.000	0.000	0.000	25.000
3	0.000	0.000	0.000	0.000	0.000	375.000
4	0.000	0.000	0.000	-0.333	0.000	125.000
5	0.000	0.000	0.000	-0.667	0.000	50.000
6	1.000	-0.167	0.000	0.000	0.000	150.000

7	0.000	0.167	0.000	0.000	0.000	150.000
8	0.000	-0.167	1.000	0.667	0.000	500.000
9	0.000	0.000	0.000	1.000	0.000	425.000
10	0.000	0.000	0.000	0.000	1.000	300.000

a. Another farmer whose farm adjoins Sod Farm #3 might be willing to sell Marky a portion of his water rights. How much should Marky offer, and for how many acre-feet?

Answer: If the price is strictly less than $66.67 per acre-feet, he can buy up to 900 acre-feet.

b. What increase in the profit per acre for wheat is required in order for it to be profitable for Marky to plant any?

Answer: The profit per acre for wheat must increase by more than $33.33 for it to be profitable for Marky to plant any wheat on *any* farm.

c. If Marky were to plant 100 acres of wheat on Farm #1, how should he best adjust the optimal plan above?

Answer:

$$\begin{bmatrix} ART \\ SLK2 \\ X1COTTON \\ SLK4 \\ X2MILO \\ SKL6 \\ X3MILLO \\ SKL8 \\ X2COTTON \\ SLK10 \end{bmatrix} = \begin{bmatrix} 0.32E+06 \\ 25 \\ 375 \\ 125 \\ 50 \\ 150 \\ 150 \\ 500 \\ 425 \\ 300 \end{bmatrix} - \begin{bmatrix} 33.333 \\ 0.5 \\ 0.5 \\ 0.167 \\ 0.333 \\ 0 \\ 0 \\ -0.333 \\ -0.5 \\ 1 \end{bmatrix} X1WHEAT = \begin{bmatrix} 316670 \\ -25 \\ 325 \\ 108.3 \\ 16.7 \\ 150 \\ 150 \\ 533.3 \\ 475 \\ 200 \end{bmatrix}$$

An increase in X1WHEAT of 100 is impossible because *SLK2* would become negative. By performing the "minimum ratio test", we discover that for up to an increase of 50 acres of wheat he could adjust the optimal plan with this equation, but after that he would need to solve the problem again (adding the constraint X1WHEAT=100).

d. Is there another optimal basic solution, besides the one given above? If so, how does it differ from that given above?

Answer: Because there are non-basic variables with reduced cost 0 (namely X3COTTON and X1MILO), increasing either of these variables up to its allowable limit does not change the objective value, and is therefore also an optimal solution.