56:171 Operations Research
Homework #3 Solutions – Fall 2002

1. **Revised Simplex Method** Consider the LP problem

Maximize $z = 3x_1 - x_2 + 2x_3$
subject to

$$x_1 + x_2 + x_3 \le 15$$
$$2x_1 - x_2 + x_3 \le 2$$
$$-x_1 + x_2 + x_3 \le 4$$
$$x_j \ge 0,\ j = 1,2,3$$

a. Let $x_4, x_5, \&, x_6$ denote the slack variables for the three constraints, and write the LP with equality constraints.
***Answer*:**
Maximize $z = 3x_1 - x_2 + 2x_3$
subject to

$$x_1 + x_2 + x_3 + x_4 = 15$$
$$2x_1 - x_2 + x_3 + x_5 = 2$$
$$-x_1 + x_2 + x_3 + x_6 = 4$$
$$x_j \ge 0,\ j = 1,2,3,4,5,6$$

After several iterations of the revised simplex method,

the basis B={4,3,2} and the basis inverse matrix is $(A^B)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

b. Proceed with one iteration of the revised simplex method, by
i. Computing the simplex multiplier vector π
***Answer*:**

$$\pi = C_B (A^B)^{-1} = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0, & \frac{3}{2}, & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0, & 1.5, & 0.5 \end{bmatrix}$$

ii. "pricing", i.e., computing the "relative profits", of the non-basic columns.
***Answer*:**

$$C^N = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}, A^N = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\overline{C^N} = C^N - \pi A^N = \begin{bmatrix} 1/2 & -3/2 & -1/2 \end{bmatrix}$$

The relative profits for non-basic variables are $\overline{C_1} = 0.5$, $\overline{C_5} = -1.5$, $\overline{C_6} = -0.5$.

iii. Selecting the column to enter the basis.
Answer: Only the relative profit of X_1 is positive and the problem is Max problem, and so X_1 should enter the basic.

iv. Computing the substitution rates of the entering column.
Answer: The substitution rates of the entering variable X_1 is

$$\alpha = (A^B)^{-1} A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \\ -3/2 \end{bmatrix}$$

v. Select the variable to leave the basis.
Answer:

The current right-hand-side is $\beta = X_B = (A^B)^{-1} b = \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$ and the ratios (right-hand-side over *positive* substitution rates) are $\begin{bmatrix} 5.5 \\ 6 \\ \cdots \end{bmatrix}$. *(Note that the ratio is not computed for the last row.)*

So by the minimum ratio test, X_1 enters the basis, replacing the basic variable in the first row (the row in which the minimum ratio is found), namely X_4.

vi. Update the basis inverse matrix.
Answer: The new basis is B= {1, 3, 2}. The basis inverse can be updated by writing α, the column of substitution rates, alongside inverse matrix and pivoting in the first row, as shown:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \boxed{2} \\ 1/2 \\ -3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 & 0 & -1/2 \\ -1/4 & 1/2 & 3/4 \\ 3/4 & -1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

c. Write the dual of the above LP (i.e. with equality constraints & slack variables) in (a).
Answer: Minimize $z = 15y_1 + 2y_2 + 4y_3$
subject to
$y_1 + 2y_2 - y_3 \geq 3$

$$y_1 - y_2 + y_3 \geq -1$$
$$y_1 + y_2 + y_3 \geq 2$$
$$y_j \geq 0, j = 1,2,3$$

d. Substitute the vector π which you computed above in step (i) above to test whether it is feasible in the dual LP. Which constraint(s) if any are violated? How does this relate to the results in step (ii) above?
Answer: If we substitute $\pi = [0, \quad 1.5, \quad 0.5]$ for the dual variables y, the first constraint $y_1 + 2y_2 - y_3 \geq 3$ is violated.
Note: The simplex multipler vector π satisfies all the constraints in the dual problem *if & only if* the relative profits in (ii) are all non-positive (which implies that the solution is optimal).

2. LP formulation: *Staffing a Call Center* (Case 3.3, pages 106-108, *Intro. to O.R.* by Hillier & Lieberman) Answer parts (a), (b), & (c) on page 108, using LINGO with sets to enter the model.

For the following analysis, consider the labor cost for the time employees spent answering phones. The cost for paperwork time is charged to other cost centers.

a. How many Spanish-speaking operators and how many English-speaking operators does the hospital need to staff the call center during each 21-hour shift of the day in order to answer all calls? Please provide an integer number since half a human operator makes no sense.
***Answer*:**

Work shift	Spanish Calls/hr	English Calls/hr	Spanish Speaking Operators Required	English Speaking Operators Required
7 a.m. – 9 a.m.	8	32	2	6
9 a.m. – 11 a.m.	17	68	3	12
11 a.m. – 1 p.m.	14	56	3	10
1 p.m. – 3 p.m.	19	76	4	13
3 p.m. – 5 p.m.	16	64	3	11
5 p.m. – 7 p.m.	7	28	2	5
7 p.m. – 9 p.m.	2	8	1	2

b. Formulate a linear programming model of this problem.
***Answer*:** Define *decision variables*
E_i, S_i **:** the number of full-time English and Spanish speaking operators, respectively, whose starting shift is $i = 1,\ldots,5$, where starting shift means that the operator starts to answer calls at shift *i.*
P_1, P_2 **:** the number of part time employers beginning shift 3 p.m.- 5 p.m. and 5 p.m.-7 p.m, respectively.

There is a constraint for each language requirement and each 2-hour period:

Minimize $40E_1 + 40S_1 + 40E_2 + 40S_2 + 40E_3 + 40S_3 + 44E_4 + 44S_4 + 44E_5 + 44S_5 + 44P_1 + 48P_2$

subject to

$E_1 \geq 6$ *(English-speaking operator rqmts)*

$E_2 \geq 12$

$E_1 + E_3 \geq 10$

$E_2 + E_4 \geq 13$

$E_3 + E_5 + P_1 \geq 11$

$E_4 + P_1 + P_2 \geq 5$

$E_5 + P_2 \geq 2$

$S_1 \geq 2$ *(Spanish-speaking operator rqmts)*

$S_2 \geq 3$

$S_1 + S_3 \geq 3$

$S_2 + S_4 \geq 4$

$S_3 + S_5 \geq 3$

$S_4 \geq 2$

$S_5 \geq 1$

$E_i, S_i, P_j \geq 0$ for all i=1,…,7 and j=1,2.

c. Obtain an optimal solution for the LP model formulated in part (b)

***Answer*:**

```
OBJECTIVE FUNCTION VALUE

1)      1640.000

VARIABLE        VALUE          REDUCED COST
      E1          6.000000          0.000000
      S1          2.000000          0.000000
      E2         12.000000          0.000000
      S2          3.000000          0.000000
      E3          5.000000          0.000000
      S3          2.000000          0.000000
      E4          1.000000          0.000000
      S4          2.000000          0.000000
      E5          2.000000          0.000000
      S5          1.000000          0.000000
      P1          4.000000          0.000000
      P2          0.000000         40.000000

     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)          0.000000        -40.000000
      3)          0.000000          0.000000
      4)          1.000000          0.000000
      5)          0.000000        -40.000000
      6)          0.000000        -40.000000
      7)          0.000000         -4.000000
      8)          0.000000         -4.000000
```

9)	0.000000	-40.000000
10)	0.000000	-40.000000
11)	1.000000	0.000000
12)	1.000000	0.000000
13)	0.000000	-40.000000
14)	0.000000	-44.000000
15)	0.000000	-4.000000

LINGO model: This is a bigger challenge than most other models we've looked at. We define the sets LANGUAGE, PERIOD, & SHIFT, and then the derived sets which I've arbitrarily names A, B, & D. The attribute W (of set B) specifies which 2-hour period each shift is answering phones: W(i,j) = 1 if shift i is working in period j, and 0 otherwise. These binary values are then used to compute the pay for each shift and to impose the requirements for operators during each period. I have here defined the decision variables X(k,i) = # of operators speaking language k working shift i. Thus X(1,1) & X(2,1) are identical to the variables E_1 & S_1, respectively, in the model shown above. Because none of the part-time operators speak Spanish, the variables X(2,6) = X(2,7) = 0 (where Spanish is the 2nd language and the part-time shifts are #6&7).

```
MODEL:

SETS:
      LANGUAGE/E S/;
      PERIOD/1..7/: RATE  ;
      !  Shifts 1-5 are full-time, and 6&7 are part-time;
      SHIFT/1..7/: PAY ;
      A(LANGUAGE,PERIOD): REQMT;
      B(SHIFT,PERIOD): W;
      D(LANGUAGE,SHIFT): X;
ENDSETS

DATA:
      ! RATE is rate of pay for each 2-hour work period;
      RATE=20 20 20 20 20 24 24;
      ! REQMT(i,j) is requirement for operators speaking language i
            during 2-hour work period j;
      REQMT= 6 12 10 13 11  5  2
             2  3  3  4  3  2  1;
      ! W(j,k) indicates whether operator working shift j
            is answering phones during 2-hr work period k;
      W= 1 0 1 0 0 0 0
         0 1 0 1 0 0 0
         0 0 1 0 1 0 0
         0 0 0 1 0 1 0
         0 0 0 0 1 0 1
         0 0 0 0 1 1 0
         0 0 0 0 0 1 1;
ENDDATA
```

LINGO model

The LP model using these sets & data is

```
MIN = @SUM(LANGUAGE(I):
            @SUM(SHIFT(J): PAY(J) * X(I,J) ) );

!  Compute pay for each shift (both full-time & part-time);
@FOR(SHIFT(J):
      PAY(J) = @SUM(PERIOD(K): W(J,K)*RATE(K) ); );

!  No Spanish-speaking part-time operators (shifts 6&7) ;
      X(2,6)=0;
      X(2,7)=0;

!  For each language & work period, require needed # operators;
@FOR(LANGUAGE(I):
      @FOR(PERIOD(K):
            @SUM(SHIFT(J):
                   W(J,K)*X(I,J) )  >= REQMT(I,K); ); );

END
```

The solution found by LINGO is the same as that shown above.

```
 Objective value:                              1640.000

                      Variable           Value        Reduced Cost
                      X( E, 1)        6.000000           0.0000000
                      X( E, 2)        13.00000           0.0000000
                      X( E, 3)        4.000000           0.0000000
                      X( E, 5)        2.000000           0.0000000
                      X( E, 6)        5.000000           0.0000000
                      X( S, 1)        2.000000           0.0000000
                      X( S, 2)        3.000000           0.0000000
                      X( S, 3)        2.000000           0.0000000
                      X( S, 4)        2.000000           0.0000000
                      X( S, 5)        1.000000           0.0000000
```

3. Sensitivity Analysis

Ken and Larry, Inc., supplies its ice cream parlors with three flavors of ice cream: chocolate, vanilla, and banana. Because of extremely hot weather and a high demand for its products, the company has run short of its supply of ingredients: milk, sugar, & cream. Hence, they will not be able to fill all the orders received from their retail outlets, the ice cream parlors. Owing to these circumstances, the company has decided to choose the amount of each product to produce that will maximize total profit, given the constraints on supply of the basic ingredients. The chocolate, vanilla, and banana flavors generate, respectively, \$1.00, \$0.90, and \$0.95 per profit per gallon sold. The company has only 200 gallons of milk, 150 pounds of sugar, and 60 gallons of cream left in its inventory. The LP formulation for this problem has variables C, V, and B representing gallons of chocolate, vanilla, and banana ice cream produced, respectively.

```
!    Ken & Larry Ice Cream – from Intro to O.R. by
!    Hillier & Lieberman (7th ed) p. 296

MAXIMIZE  C+0.9V+0.95B
ST
      0.45C + 0.50V + 0.40B <= 200  ! milk resource
      0.50C + 0.40V + 0.40B <= 150  ! sugar resource
      0.10C + 0.15V + 0.20B <= 60   ! cream resource
END
```

LINDO model

```
        OBJECTIVE FUNCTION VALUE
        1)      341.2500

  VARIABLE        VALUE          REDUCED COST
         C         0.000000          0.037500
         V       300.000000          0.000000
         B        75.000000          0.000000

       ROW   SLACK OR SURPLUS     DUAL PRICES
        2)        20.000000          0.000000
        3)         0.000000          1.875000
        4)         0.000000          1.000000
```

```
RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
        C        1.000000         0.037500         INFINITY
        V        0.900000         0.050000         0.012500
        B        0.950000         0.021429         0.050000

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2      200.000000         INFINITY        20.000000
        3      150.000000        10.000000        30.000000
        4       60.000000        15.000000         3.750000
```

The LP formulation for this problem has variables C, V, and B representing gallons of chocolate, vanilla, and banana ice cream produced, respectively.

a. What is the optimal profit and the optimal solution?

***Answer*:** The optimal profit is \$*341.25*.

The optimal quantities of the products are 0 gallons of chocolate ice cream, 300 gallons of vanilla ice cream and 75 gallons of banana ice cream.

b. Suppose the profit per gallon of banana changes to \$1.00. Will the optimal solution change, and what can be said about the effect on total profit?

***Answer*:** An increase of profit of the banana ice cream to \$1.00 is an increase of \$0.05. This exceeds the "Allowable Increase" (**0.021429**) in which the basis is unchanged. So the basis changes, changing the optimal solution and the total profit (which would of course increase.)

c. Suppose the profit per gallon of banana changes to 92 cents. Will the optimal solution changes, and what can be said about the effect on total profit?

***Answer*:** Because the decrease (\$0.03) is less than the allowable decrease (\$**0.05**) for which the basis is unchanged, the basic variables (& their values) are unchanged, but the total profit decreases by \$0.03/gal. × 75 gal. = \$2.25.

d. Suppose the company discovers that 3 gallons of cream have gone sour and so must be thrown out. Will the optimal solution change, and what can be said about the effect on the total profit?
***Answer*:** The optimal solution would be changed because the quantity of cream whose slack is 0 is changed. Because the decrease (3 gal.) is less than the allowable decrease (which is **3.75**), the total profit would decrease by \$**3** (dual price of cream resource is \$1.0/gal. so 3 gal.× 1.0 \$/gal. = \$3).

e. Suppose that the company has the opportunity to buy an additional 15 pounds of sugar at a total cost of \$15. Should they buy it? Explain!

***Answer*:** Inside the allowable range, the dual price is **\$1.875** so if 10 pounds of sugar is bought and used the profit increase by 10 × \$1.875 = \$18.75 which is more than the price of 15 pounds of sugar and brings more profit (if 15 pounds of sugar is available, there would not be *less* profit than when 10 pounds is used, and there possibly will be an additional increase in profit.) So the company *should* buy the 15 pounds of sugar at the stated price, since they would obtain *at least* \$18.75−\$15.00 = \$3.75 in additional net profits.