56:171 Operations Research
Homework #2 Solutions – Fall 2002

1. (Exercise 3.4-18, page 98, of Hillier&Lieberman text, 7th edition)
"Oxbridge University maintains a powerful mainframe computer for research use by its faculty, Ph.D. students, and research associates. During all working hours, an operator must be able to operate and maintain the computer, as well as to perform some programming services. Beryl Ingram, the director of the computer facility, oversees the operation.
It is now the beginning of the fall semester, and Beryl is confronted with the problem of assigning different working hours to her operators. Because all the operators are currently enrolled in the university, they are available to work only a limited number of hours each day, as shown in the following table.

		Maximum # hours available				
Name	Wage $/hr	Mon	Tues	Wed	Thur	Fri
K.C.	10.00	6	0	6	0	6
D.H.	10.10	0	6	0	6	0
H.B.	9.90	4	8	4	0	4
S.C.	9.80	5	5	5	0	5
K.S.	10.80	3	0	3	8	0
N.K.	11.30	0	0	0	6	2

There are six operators (four undergraduate students and two graduate students). They all have different wage rates because of differences in their experience with computers and in their programming ability. The above table shows their wage rates, along with the maximum number of hours that each can work each day.
Each operator is guaranteed a certain minimum number of hours per week that will maintain an adequate knowledge of the operation. This level is set arbitrarily at 8 hours per week for the undergraduate students (K.C,, D.H, H.B, and S.C.) and 7 hours per week for the graduate students (K.S. and N.K.).
The computer facility is to be open for operation from 8 a.m. to 10 p.m. Monday through Friday with exactly one operator on duty during these hours. On Saturdays and Sundays, the computer is to be operated by other staff.
Because of a tight budget, Beryl has to minimize cost. She wishes to determine the number of hours she should assign to each operator on each day."

a. Formulate a linear programming model for this problem. Be sure to define your variables!

Answer:
Define decision variables
X_{ij} = # hours operater *i* is assigned to work on day *j*
for all *i=1… 6 (where 1=KC, 2=DH,… 6= NK); j=1,… 5 (where 1=MON, 2=TUE, … 5=FRI)*

Minimize $z= 10(X_{11} + X_{13} + X_{15}) + 10.1(X_{22} + X_{24}) + 9.9(X_{31} + X_{32} + X_{33}+X_{35}) + 9.8(X_{41} + X_{42} + X_{43} + X_{45}) + 10.8(X_{51} + X_{53} + X_{54}) + 11.3(X_{64} + X_{65})$

subject to

maximum number hours available each day:

$X_{11}\le6$	$X_{22}\le6$	$X_{31}\le4$	$X_{41}\le5$	$X_{51}\le3$	$X_{64}\le6$
$X_{13}\le6$	$X_{24}\le6$	$X_{32}\le8$	$X_{42}\le5$	$X_{53}\le3$	$X_{65}\le2$
$X_{15}\le6$		$X_{33}\le4$	$X_{43}\le5$	$X_{54}\le8$	
		$X_{35}\le4$	$X_{45}\le5$		

number of hours guaranteed for each operator:

$X_{11} + X_{13} + X_{15} \ge 8$

$X_{22} + X_{24} \ge 8$

$X_{31} + X_{32} + X_{33} + X_{35} \ge 8$

$X_{41} + X_{42} + X_{43} + X_{45} \ge 8$

$X_{51} + X_{53} + X_{54} \ge 7$

$X_{64} + X_{65} \ge 7$

total number hours worked each day is 14:

$X_{11} + X_{31} + X41 + X_{51} = 14$

$X_{22} + X_{23} + X_{42} = 14$

$X_{13} + X_{33} + X_{43} + X_{53} = 14$

$X_{24} + X_{44} + X_{54} + X_{64} = 14$

$X_{15} + X_{35} + X_{45} + X_{65} = 14$

nonnegativity:

$Xij \ge 0$ for all i & j

b. Use an LP solver (e.g. LINDO or LINGO) to find the optimal solution.

The **LINGO** model is as follows:

```
MODEL:     ! Oxbridge University Computer Center;

SETS:
      OPERATOR / KC, DH, HB, SC, KS, NK/: MINIMUM, PAYRATE;
      DAY /MON, TUE, WED, THU, FRI/: REQUIRED;
      ASSIGN(OPERATOR,DAY): AVAILABLE, X;
ENDSETS

DATA:
      MINIMUM = 8 8 8 8 7 7;
      PAYRATE = 10.00  10.10  9.90  9.80  10.80  11.30;
      REQUIRED = 14 14 14 14 14;
      AVAILABLE=  6  0  6  0  6
                  0  6  0  6  0
                  4  8  4  0  4
                  5  5  5  0  5
                  3  0  3  8  0
                  0  0  0  6  2;

ENDDATA

MIN = TOTALPAY;

!     total weekly payroll cost;
      TOTALPAY = @SUM(ASSIGN(I,J)|AVAILABLE(I,J) #NE# 0:
PAYRATE(I)*X(I,J) );
```

```
!     must schedule required hours each day;
      @FOR(DAY(J):
            @SUM(OPERATOR(I)|AVAILABLE(I,J) #NE# 0:  X(I,J) ) =
REQUIRED(J) );

!     must schedule each operator at least minimum number of hours;
      @FOR(OPERATOR(I):
            @SUM(DAY(J)|AVAILABLE(I,J) #NE# 0:  X(I,J) ) >= MINIMUM(I)
);

!     upper (& lower) bounds on variables;
      @FOR(ASSIGN(I,J)| AVAILABLE(I,J) #NE# 0:
            @BND(0, X(I,J), AVAILABLE(I,J) ); );
END
```

Note the use of the logical expression "`AVAILABLE(I,J) #NE# 0`" in order to avoid defining and referencing assignments in which the operator is not available.
Note also the use of @BND to impose the upper bounds instead of

```
      @FOR(ASSIGN(I,J)| AVAILABLE(I,J) #NE# 0:
            X(I,J) <= AVAILABLE(I,J)   );
```

The syntax is @BND(*lower_bound*, *variable_name*, *upper_bound*);

```
Global optimal solution found at step:           10
 Objective value:                              709.6000

                       Variable           Value        Reduced Cost
                       TOTALPAY        709.6000           0.0000000
                    X( KC, MON)        3.000000           0.0000000
                    X( KC, WED)        2.000000           0.0000000
                    X( KC, FRI)        4.000000           0.0000000
                    X( DH, TUE)        2.000000           0.0000000
                    X( DH, THU)        6.000000          -0.1000000
                    X( HB, MON)        4.000000          -0.1000000
                    X( HB, TUE)        7.000000           0.0000000
                    X( HB, WED)        4.000000          -0.1000000
                    X( HB, FRI)        4.000000          -0.1000000
                    X( SC, MON)        5.000000          -0.2000000
                    X( SC, TUE)        5.000000          -0.1000000
                    X( SC, WED)        5.000000          -0.2000000
                    X( SC, FRI)        5.000000          -0.2000000
                    X( KS, MON)        2.000000           0.0000000
                    X( KS, WED)        3.000000           0.0000000
                    X( KS, THU)        2.000000           0.0000000
                    X( NK, THU)        6.000000           0.0000000
                    X( NK, FRI)        1.000000           0.0000000
```

2. (Exercise 4.4-9, page 176, of Hillier&Lieberman text, 7th edition)
Work through the simplex method step by step (in tabular form) to solve the following problem:

Maximize $Z=2X_1 - X_2 + X_3$
subject to

$$3X_1 + X_2 + X_3 \le 6$$
$$X_1 - X_2 + 2X_3 \le 1$$
$$X_1 + X_2 - X_3 \le 2$$

and

$$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0$$

Solution: Include a slack variable in each of the three inequality constraints, S_1, S_2, & S_3. Set up the initial tableau, and use (-Z) and the three slack variables for the initial basis.

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS
1	2	−1	1	0	0	0	0
0	3	1	1	1	0	0	6
0	1	−1	2	0	1	0	1
0	1	1	−1	0	0	1	2

We are maximizing, and so increasing either X_1 or X_3 (both of which have positive "relative profits") would improve, i.e., increase, the objective. We will arbitrarily choose X_1.

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS	$ratio$
1	2	−1	1	0	0	0	0	
0	3	1	1	1	0	0	6	6/3=2
0	**1**	−1	2	0	1	0	1	1/1=1
0	1	1	−1	0	0	1	2	2/1=2

As X_1 increases each of the basic variables S_1, S_2, & S_3 decrease (because of the positive *substitution rates*). The minimum ratio test indicates that the first to reach its lower bound of zero as X_1 increases is S_2, and hence X_1 replaces S_2 in the basis, and the pivot is to be done in the row in which the minimum ratio was computed.

The resulting tableau is:

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS	$ratio$
1	0	1	−3	0	−2	0	−2	
0	0	4	−5	1	−3	0	3	¾=0.75
0	1	−1	2	0	1	0	1	
0	0	**2**	−3	0	−1	1	1	½=0.5

The tableau is not optimal, since there is a positive relative profit in the X_2 column, which is therefore selected as the pivot column. As X_2 increases, S_1 and S_3 decrease (because of the positive substitution rates 4 & 2), and the minimum ratio test indicates that S3 is the first to reach zero. Hence the pivot is performed with the bottom row as the pivot row. The resulting tableau is:

$-Z$	X_1	X_2	X_3	S_1	S_2	S_3	RHS
1	0	0	−1.5	0	−1.5	−0.5	−2.5
0	0	0	1	1	−1	−2	1
0	1	0	0.5	0	0.5	0.5	1.5
0	0	1	−1.5	0	0	−0.5	0.5

There is now no positive relative profit in the objective row, and therefore the current basis is optimal and the optimal solution is $X_1 = 1.5$, $X_2 = 0.5$ and $X_3 = 0$ with slack 1, 0, & 0, respectively, in the three constraints. The optimal objective value $Z = 2.5$.