UPPER BOUNDING TECHNIQUE

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

Upper Bounding Technique

Consider the LP:

$$\text{Max } 2x_1 + 4x_2 + 5x_3 + 3x_4$$

subject to

$$x_1 + 3x_2 + 6x_3 + 2x_4 \le 24$$

$$5x_1 + 4x_2 + 4x_4 \le 20$$

Simple upper bounds

$$\begin{cases} 0 \le x_1 \le 3 \\ 0 \le x_2 \le 4 \\ 0 \le x_3 \le 3 \\ 0 \le x_4 \le 3 \end{cases}$$

If the upper bounding technique (UBT) is NOT used, the tableau is

2	4	5	3	0	0	0	0	0	0	0
1	3	6	2	1	0	0	0	0	0	24
5	4	0	4	0	1	0	0	0	0	20
1	0	0	0	0	0	1	0	0	0	3
0	1	0	0	0	0	0	1	0	0	4
0	0	1	0	0	0	0	0	1	0	3
0	0	0	1	0	0	0	0	0	1	3

and a 6x6 basis inverse matrix must be maintained.

When using UBT, only a 2x2 "working basis" is used.

When using the Upper Bounding Technique,

- **nonbasic variable** may be at *either*
  - lower bound

  or
  - upper bound

- a variable **enters the basis** by
  - increasing if it is at its lower bound

  or
  - decreasing if it is at its upper bound

When using the Upper Bounding Technique,

- choice of the **pivot column** :

reduced cost $\begin{cases} <0 \text{ if at lower bound} \\ >0 \text{ if at upper bound} \end{cases}$ *for minimization problem*

relative profit $\begin{cases} >0 \text{ if at lower bound} \\ <0 \text{ if at upper bound} \end{cases}$ *for maximization problem*

When using the Upper Bounding Technique,

- Choice of the **pivot row**:

The variable entering the basis from one bound (either lower or upper) is "blocked" whenever

it reaches its other bound

or a variable currently in the basis reaches its lower bound

or a variable currently in the basis reaches its upper bound

Upper Bounding Technique

Consider the LP

$$\begin{aligned} &\text{Maximize} \quad cx \\ &\text{subject to} \quad Ax = b \\ &\qquad\qquad \mathcal{L}_i \le x_i \le \mathcal{U}_i \end{aligned}$$

$\mathcal{L}_i$ might be, but need not be, zero !

Define a basis ("working basis") B such that

$\left(A^B\right)^{-1}$ exists, i.e., $\det\left(A^B\right) \neq 0$

and partition the non-basic variables into subsets

$L = \{i \mid x_i = \mathcal{L}_i\}$ and $U = \{i \mid x_i = \mathcal{U}_i\}$

$$A^B x_B + A^L x_L + A^U x_U = b$$

$$A^B x_B = b - A^L x_L - A^U x_U$$

$$x_B = \left(A^B\right)^{-1} b - \left(A^B\right)^{-1} A^L x_L - \left(A^B\right)^{-1} A^U x_U$$

The current basic solution is

non-basic variables

$$x_U = \mathcal{U}_U$$

$$x_L = \mathcal{L}_L$$

basic variables

$$x_B = \left(A^B\right)^{-1} b - \left(A^B\right)^{-1} A^L \mathcal{L}_L - \left(A^B\right)^{-1} A^U \mathcal{U}_U$$

Selection of Variable to Enter Basis

A nonbasic variable may be in either set L (at lower bound) or set U (at upper bound)

The "relative profit" ("reduced cost" if minimizing) specifies the change in the objective function per unit *increase* in the nonbasic variable.

Non basic set	Change in x_j if entering basis	sign of $\bar{c}_j = c_j - \pi A^j$	change in objective
U	decrease	positive	decrease
		negative	increase
L	increase	positive	increase
		negative	decrease

Selection of Variable to Enter Basis

Suppose that a nonbasic variable x_j were to be selected to enter the basis...

Nonbasic Variable	Substitution Rate in Row i α_{ij}	Effect on Basic Variable X_k in Row i	Blocking Value
in L INCREASING	positive	decrease	$(X_k - L_k)/\alpha_{ij}$
	negative	increase	$(U_k - X_k)/\lvert\alpha_{ij}\rvert$
in U DECREASING	positive	increase	$(U_k - X_k)/\lvert\alpha_{ij}\rvert$
	negative	decrease	$(X_k - L_k)/\alpha_{ij}$

Selection of Pivot Row

If "blocking value" is greater than $U_j - L_j$, then the nonbasic variable is moved from L to U (or vice-versa), but the basis B is unchanged!

When the nonbasic variable X_j is increasing from its LOWER BOUND:

The bound on the increase is θ, where:

$$\theta_0 = U_j - L_j$$

$$\theta_1 = \underset{\alpha_{ij} > 0}{\text{minimum}} \left\{ \frac{X_k - L_k}{\alpha_{ij}} \right\}$$

$$\theta_2 = \underset{\alpha_{ij} < 0}{\text{minimum}} \left\{ \frac{U_k - X_k}{|\alpha_{ij}|} \right\}$$

$$\theta = \text{minimum} \{\theta_0 , \theta_1 , \theta_2\}$$

Selection of Pivot Row

$\theta = \text{minimum}\ \{\theta_0, \theta_1, \theta_2\}$

θ	blocking value	change in partition:
θ_0	$U_j - L_j$	j transfers from L to U
θ_1	$\dfrac{X_k - L_k}{\alpha_{ij}}$ (>0)	j enters B k leaves B, enters L
θ_2	$\dfrac{U_k - X_k}{\lvert\alpha_{ij}\rvert}$ (<0)	j enters B k leaves B, enters U

When the nonbasic variable X_j is increasing from its LOWER BOUND

Selection of Pivot Row

When the nonbasic variable is decreasing from its UPPER BOUND:

The bound on the decrease is θ, where:

$$\begin{cases} \theta_0 = U_j - L_j \\ \theta_1 = \underset{\alpha_{ij} > 0}{\text{minimum}} \left\{ \dfrac{U_k - X_k}{\alpha_{ij}} \right\} \\ \theta_2 = \underset{\alpha_{ij} < 0}{\text{minimum}} \left\{ \dfrac{X_k - L_k}{|\alpha_{ij}|} \right\} \\ \theta = \text{minimum} \{\theta_0, \theta_1, \theta_2\} \end{cases}$$

Selection of Pivot Row

$$\theta = \text{minimum}\ \{\theta_0, \theta_1, \theta_2\}$$

θ	blocking value	change in partition:
θ_0	$U_j - L_j$	j transfers from L to U B unchanged
θ_1	$\dfrac{U_k - X_k}{\alpha_{ij}}$ (>0)	j enters B k leaves B, enters U
θ_2	$\dfrac{X_k - L_k}{\lvert\alpha_{ij}\rvert}$ (<0)	j enters B k leaves B, enters L

When the nonbasic variable is decreasing from its UPPER BOUND

Selection of Pivot Row

Examples, with output from APL workspace UBT

Minimize $18X_1 + 25X_2$

subject to

$$\begin{cases} 30 \le 5X_1 + 4X_2 \le 55 \\ -12 \le 4X_1 - 3X_2 \le 4 \end{cases}$$

$$2 \le X_1 \le 5 \,,\; 4 \le X_2 \le 8$$

Max $2x_1 + 4x_2 + 5x_3 + 3x_4$

subject to

$$x_1 + 3x_2 + 6x_3 + 2x_4 \le 24$$

$$5x_1 + 4x_2 + 4x_4 \le 20$$

$$x_1 \le 3 \,,\; x_2 \le 4 \,,\; x_3 \le 3 \,,\; x_4 \le 3$$

$$x_j \ge 0 \quad \forall\, j$$

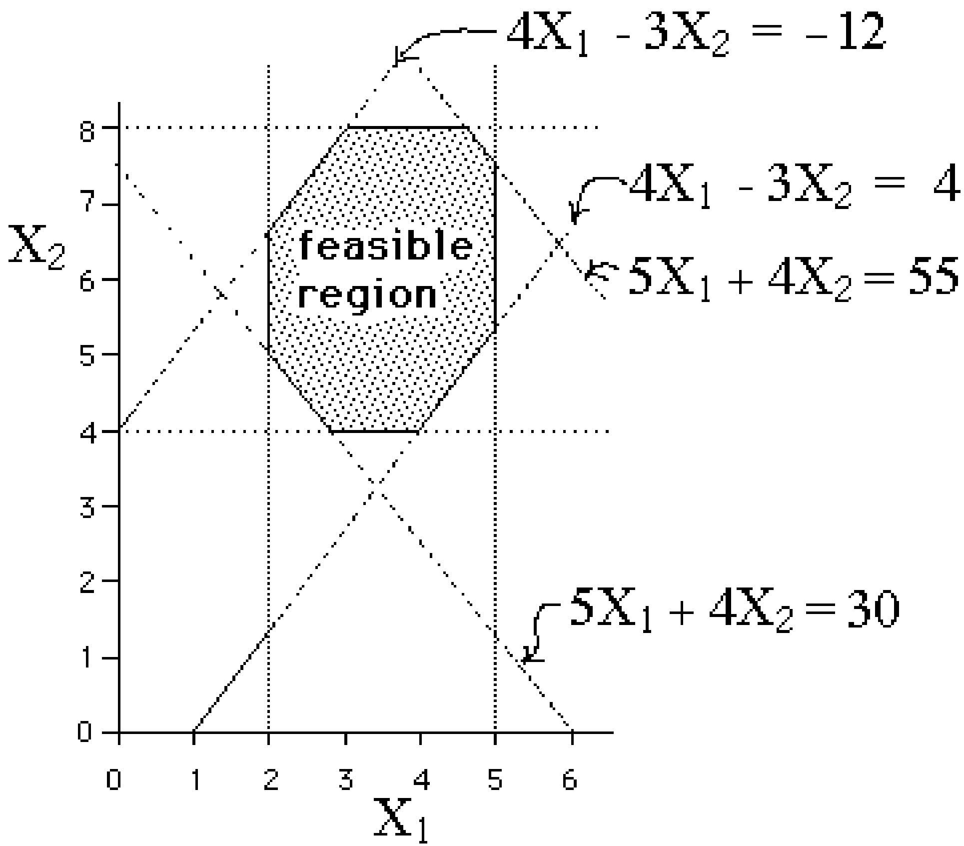
Example

Minimize $18X_1 + 25X_2$
subject to

$$\begin{cases} 30 \le 5X_1 + 4X_2 \le 55 \\ -12 \le 4X_1 - 3X_2 \le 4 \end{cases}$$

$$2 \le X_1 \le 5\ ,\ 4 \le X_2 \le 8$$

The ordinary simplex or revised simplex method would require a tableau with 8 constraints, and 8 slack &/or surplus variables (in addition to X_1 and X_2). That is, an 8x8 basis matrix is required.


4X1 - 3X2 = −12
4X1 - 3X2 = 4
5X1 + 4X2 = 55
5X1 + 4X2 = 30
feasible region
X2
X1
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6

Add slack variables to the ≤ constraints to create equalities. Then put upper bounds on these slack variables:

Maximize $18X_1 + 25X_2 + 0X_3 + 0X_4$

subject to

$$\begin{cases} 5X_1 + 4X_2 + X_3 = 55 \\ 4X_1 - 3X_2 + X_4 = 4 \end{cases}$$

upper & lower bounds

$$\begin{cases} 2 \le X_1 \le 5 \\ 4 \le X_2 \le 8 \\ 0 \le X_3 \le 25 \\ 0 \le X_4 \le 16 \end{cases}$$

Using UBT, only a 2x2 basis matrix is required, i.e. a reduction of nearly 94% in the number of elements in the inverse matrix!

1	2	3	4	b
5	4	1	0	55
4	¯3	0	1	4

Constraints

i	1	2	3	4
c[i]	18	25	0	0
L[i]	2	4	0	0
U[i]	5	8	25	16

Objective & Bounds

Iteration 1

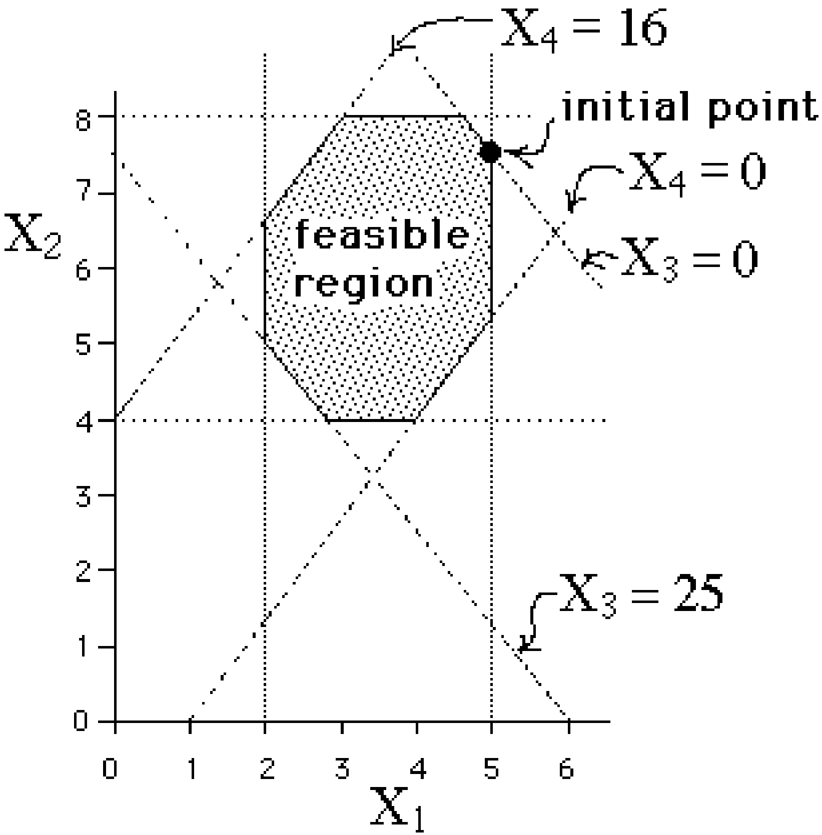
Current partition:

B= 2 4 / L= 3 / U= 1

Basis inverse matrix = $\begin{bmatrix} 0.25 & 0 \\ 0.75 & 1 \end{bmatrix}$

	X_1	X_2	X_3	X_4	
Basic solution=	5	7.5	0	6.5	with Z = 277.5
	‖	‖	‖	‖	
	upper bound	intermediate value	lower bound	intermediate value	

B = {2,4}
L = { 3 }
U = { 1 }



Initial partition

B = {2,4}
L = {3}
U = {1}

$$\pi = C_B\left(A^B\right)^{-1} = [25,\ 0]\begin{bmatrix} 1/4 & 0 \\ 3/4 & 1 \end{bmatrix}$$

```
Simplex multipliers= 6.25 0
```

$$C - \pi A = [18,25,0,0] - [25/4,0]\begin{bmatrix} 5 & 4 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$$

```
Reduced costs=    ¯13.25 0 ¯6.25 0
```

Since we wish to minimize, we would choose to increase either X_1 or X_3... however, X_1 is already at its upper bound (U={1}) and so we choose to enter X_3 into the basis.

```
Entering variable is X[3] from set L
Substitution Rates= 0.25 0.75
```

The substitution rates indicate that for each unit increase by X_3, the first basic variable (X_2) will be reduced by 0.25 and the second (X_4) will be reduced by 0.75.

X_2 is currently 7.5, and its lower bound is 4, so that it must leave the basis when it is decreased by 3.5, i.e., when X_3 is increased by $\frac{7.5 - 4}{0.25} = 14$

Likewise, X_4 can decrease by only 6.5 before it must leave the basis, i.e., X can increase by only $\frac{6.5 - 0}{0.75} = 26/3$

```
Entering variable is X[3] from set L

Substitution Rates= 0.25 0.75
```

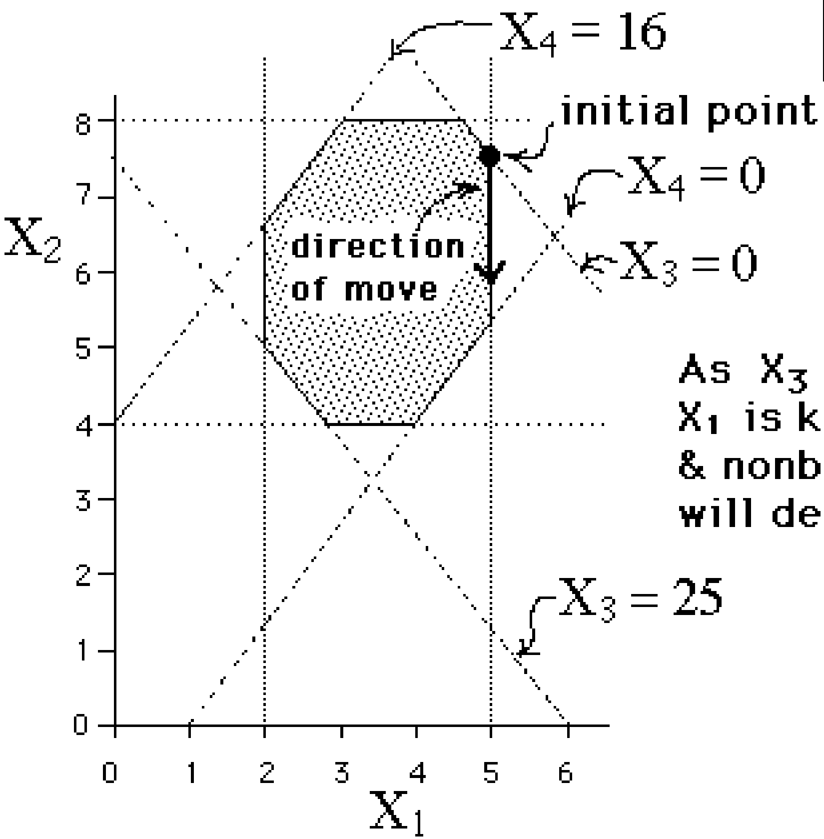
The substitution rates indicate that for each unit increase by X_3, the first basic variable (X_2) will be reduced by 0.25 and the second (X_4) will be reduced by 0.75.

```
Decreasing variables:        2        4
Block at value:         14.000    8.667
```

$$\frac{7.5 - 4}{0.25} \qquad \frac{6.5 - 0}{0.75}$$

```
Block at X[4] at value 8.66667
```

← **the minimum ratio!**



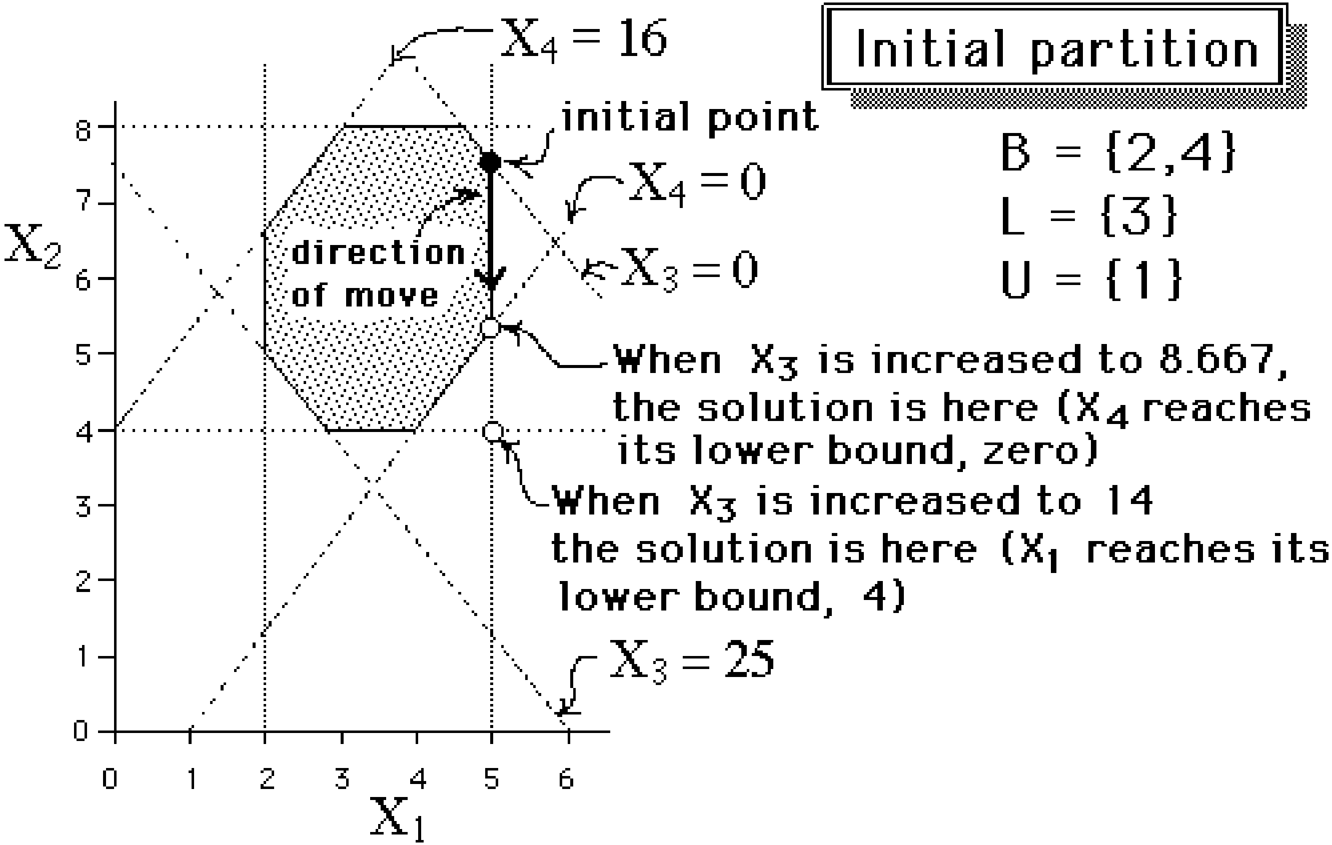
Initial partition

B = {2,4}
L = {3}
U = {1}

As X_3 is increased (while X_1 is kept at its upper bound & nonbasic), both X_2 and X_4 will decrease.

Initial partition

B = {2,4}
L = {3}
U = {1}



```
Current partition:
B= 2 3 /  L= 4 /  U= 1
```

Basis inverse matrix = $\begin{bmatrix} 0 & ^-0.33333 \\ 1 & 1.33333 \end{bmatrix}$

B={2,3}, L={4}, U={1}

```
Basic solution= 5 5.33333 8.66667 0 with Z = 223.333
Simplex multipliers= 0 ¯8.33333
Reduced costs=    51.3333 0 0 8.33333
```

Since we are minimizing, we would choose to decrease either X_1 or X_4.

However, X_4 is already at its lower bound, and so we choose to enter X_1 into the basis (from set U).

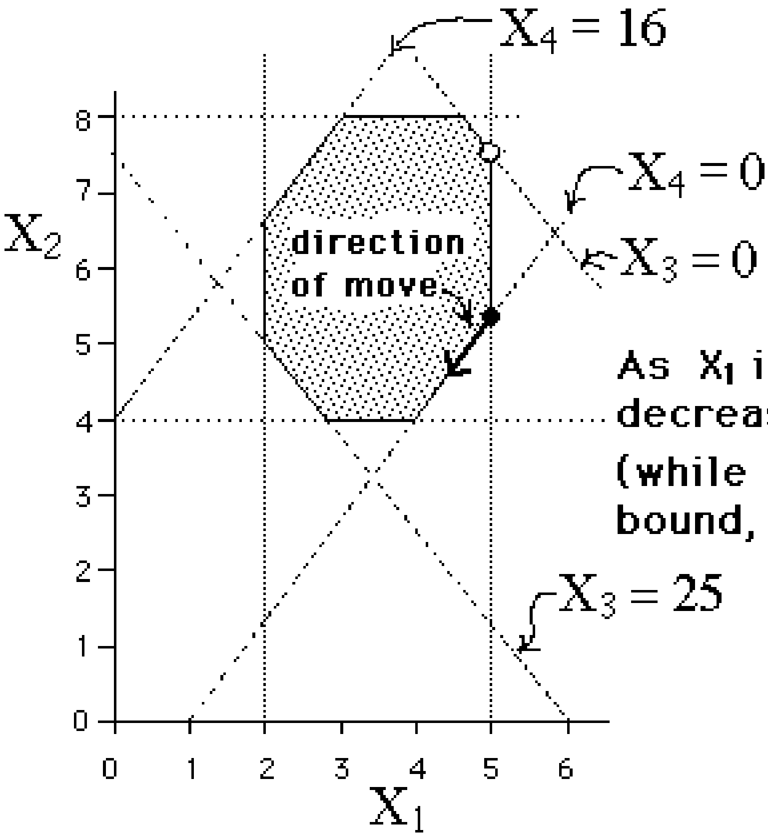
```
Entering variable is X[1] from set U
Substitution Rates= ¯1.33333 10.3333
```

The negative substitution rate indicates that the first basic variable (X_2) will also decrease as X_1 is decreased, while the positive substitution rate indicates that the second basic variable (X_3) will increase as X_1 is decreased.

```
________________________________
Increasing variables:         3
Block at value:           1.581
________________________________
Decreasing variables:         2
Block at value:           1.000
________________________________
Block at X[2] at value 1
```

$$\frac{U_3 - x_3}{\alpha_2} = \frac{25 - 8\frac{2}{3}}{10\frac{1}{3}}$$

$$\frac{x_2 - L_2}{\alpha_1} = \frac{5\frac{1}{3} - 4}{\frac{4}{3}}$$



B = {2,3}
L = {4}
U = {1}

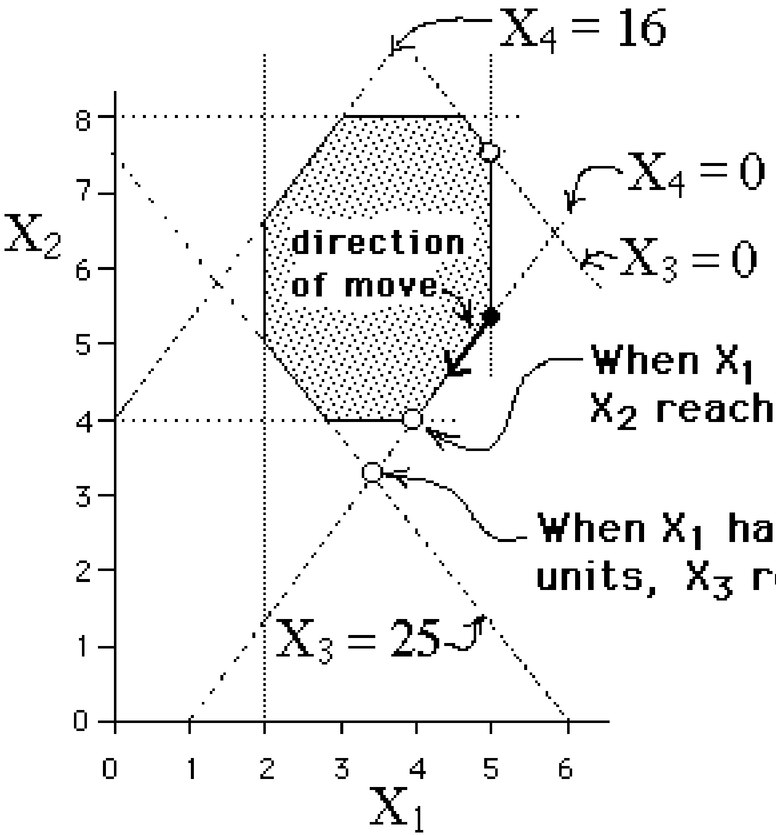
As X_1 is decreased, X_2 is also decreased, while X_3 is increased. (while X_4 is kept at its lower bound, zero)

Partition

$B = \{2,3\}$

$L = \{4\}$

$U = \{1\}$



When X_1 has decreased by 1 unit, X_2 reaches its lower bound.

When X_1 has decreased by 1.581 units, X_3 reaches its upper bound.

Current partition:

B= 1 3 / L= 4 2 / U=

Basis inverse matrix = $\begin{bmatrix} 0 & 0.25 \\ 1 & -1.25 \end{bmatrix}$

$B = \{1, 3\}$
$L = \{4, 2\}$
$U = \emptyset$

Basic solution= 4 4 19 0 with Z = 172

Simplex multipliers= 0 4.5

Reduced costs= 0 38.5 0 ¯4.5

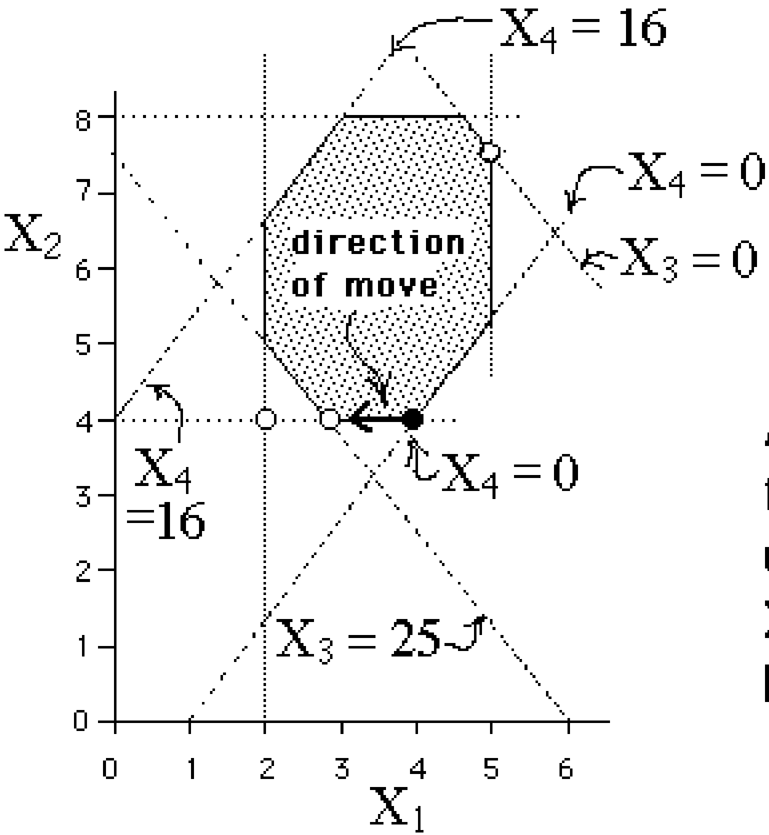
Entering variable is X[4] from set L

Substitution Rates= 0.25 ¯1.25

Increasing variables: 3
Block at value: 4.800

Decreasing variables: 1
Block at value: 8.000

Block at X[3] at value 4.8



Partition

B = { 1, 3 }
L = { 4, 2 }
U = ∅

As X_4 is increased, first X_3 reaches its upper bound, and then X_1 reaches its lower bound.

Current partition:
B= 1 4 / L= 2 / U= 3

$B = \{1, 4\}$
$L = \{2\}, U = \{3\}$

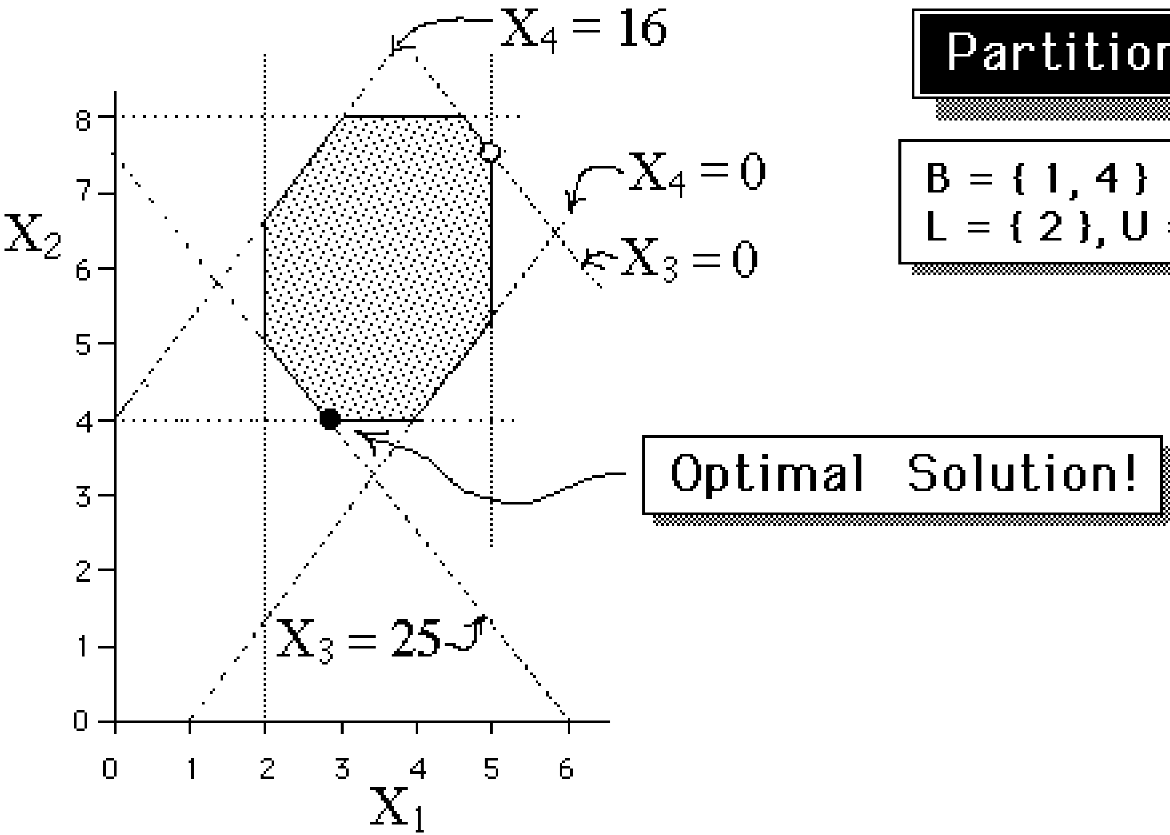
Basis inverse matrix = $\begin{bmatrix} 0.2 & 0 \\ -0.8 & 1 \end{bmatrix}$

Basic solution= 2.8 4 25 4.8 with Z = 150.4

Simplex multipliers= 3.6 0
Reduced costs= 0 10.6 −3.6 0

The positive reduced cost indicates that lowering X_2 would improve the solution... but X_2 is already at its lower bound.

The negative reduced cost indicates that increasing X_3 would improve the solution... but X_3 is already at its upper bound.



Partition

B = { 1, 4 }

L = { 2 }, U = { 3 }

Since no change in the nonbasic variables will yield an improved solution, the current solution is optimal!

B = { 1, 4 }
L = { 2 }, U = { 3 }

Optimal Solution

i	1	2	3	4
X[i]	2.800	4.000	25.000	4.800

Objective Z= 150.4

EXAMPLE

$$\text{Max } 2x_1 + 4x_2 + 5x_3 + 3x_4$$

subject to

$$x_1 + 3x_2 + 6x_3 + 2x_4 \leq 24$$

$$5x_1 + 4x_2 + 4x_4 \leq 20$$

Simple upper bounds

$$\begin{cases} 0 \leq x_1 \leq 3 \\ 0 \leq x_2 \leq 4 \\ 0 \leq x_3 \leq 3 \\ 0 \leq x_4 \leq 3 \end{cases}$$

Iteration 1

Current partition:

B= 5 6 / L= 1 2 3 4 / U= *empty*

Basis inverse matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Basic solution= 0 0 0 0 24 20 with Z = 0

Simplex multipliers= 0 0

Relative profits= 2 4 5 3 0 0

Entering variable is X[3] from set L $\theta = \boxed{\theta_0 = 3 - 0}$

Substitution Rates= 6 0

Decreasing variables: 5
Block at value: 4.000 ← $\boxed{\theta_1}$

Variable does NOT enter basis,
but moves to opposite bound

Iteration 2

Current partition:
B= 5 6 / L= 1 2 4 / U= 3

Basis inverse matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

basis inverse matrix is unchanged!

Basic solution= 0 0 3 0 6 20 with Z = 15

Simplex multipliers= 0 0

Relative profits= 2 4 5 3 0 0

$\theta_0 = 4 - 0$

Entering variable is X[2] from set L
Substitution Rates= 3 4

Decreasing variables:	5	6
Block at value:	2.000	5.000

Block at X[5] at value 2

$\theta = \theta_1$

variable 2 replaces 5, and 5 enters L!

Iteration 3

Current partition:

B= 2 6 / L= 1 4 5 / U= 3

Basis inverse matrix =

$$\begin{bmatrix} 0.333333 & 0 \\ ^{-}1.33333 & 1 \end{bmatrix}$$

Basic solution= 0 2 3 0 0 12 with Z = 23

Simplex multipliers= 1.33333 0
Relative profits= 0.666667 0 ¯3 0.333333 ¯1.33333 0

Entering variable is X[3] from set U
Substitution Rates= 2 ¯8

$\theta_0 = 3 - 0$

Increasing variables: 2
Block at value: 1.000

$\theta_1 = \theta$

Decreasing variables: 6
Block at value: 1.500

θ_2

Block at X[2] at value 1

Variable 2 is replaced by variable 3 in B, and variable 2 enters set U

Iteration 4

```
Current partition:
B= 3 6 /  L= 1 4 5 /  U= 2
```

Basis inverse matrix =

$$\begin{bmatrix} 0.166667 & 0 \\ 0 & 1 \end{bmatrix}$$

```
Basic solution= 0 4 2 0 0 4 with Z = 26

Simplex multipliers= 0.833333 0

Relative profits= 1.16667 1.5 0 1.33333 ¯0.833333 0

Entering variable is X[4] from set L
Substitution Rates= 0.333333 4
```

$\theta_0 = 3 - 0$

```
Decreasing variables:        3       6
Block at value:          6.000   1.000
```

```
Block at X[6] at value 1
```

$\theta_2 = \infty$ θ_1

Variable 6 is replaced in B by variable 4, and enters set L

Iteration 5

Current partition:
B= 3 4 / L= 1 5 6 / U= 2

Basis inverse matrix =

$$\begin{bmatrix} 0.166667 & ^{-}0.0833333 \\ 0 & 0.25 \end{bmatrix}$$

Basic solution= 0 4 1.66667 1 0 0 with Z = 27.3333

Simplex multipliers= 0.833333 0.333333

Relative profits= ¯0.5 0.166667 0 0 ¯0.833333 ¯0.333333

variable 1 is in L and cannot be decreased

variable 2 is in U and cannot be increased

variable 5 is in L and cannot be decreased

variable 6 is in L and cannot be decreased

Optimal partition: B = { 3, 4 }, L = { 1, 5, 6 }, U = { 2 }

Optimal Solution

i	1	2	3	4	5	6
X[i]	0.000	4.000	1.667	1.000	0.000	0.000

Objective Z= 27.3333