TRANSPORTATION PROBLEM

author

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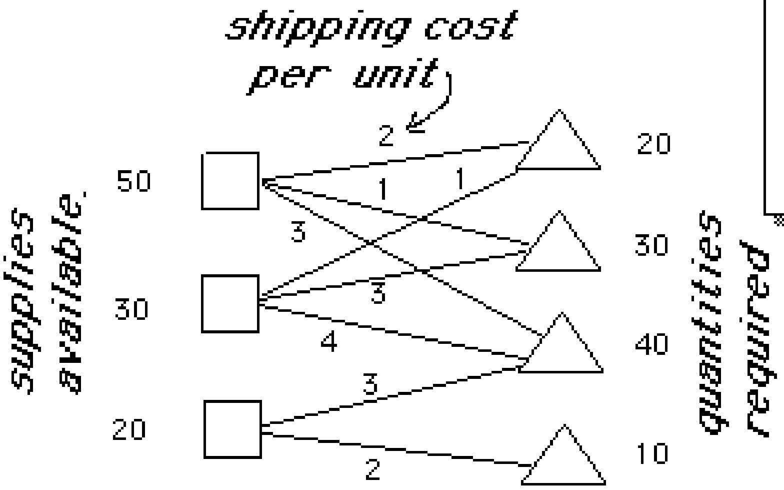
Transportation Problem



Find the least-cost shipments which will satisfy the given requirements, using the available supplies

Directory

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LP Formulation of the Transportation Problem

Let i index the sources, and j the destinations
m = # of sources, n = # destinations

Given:

S_i = quantity of goods available at source i

D_j = quantity of goods required at destination j

C_{ij} = unit cost of shipping goods from source i to destination j

Find:

X_{ij} = quantity of goods to be shipped from source i to destination j ⇦ ⇨

Minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$

subject to

$$\sum_{j=1}^{n} X_{ij} \leq S_i \quad \text{for } i=1, \ldots m$$

no more is shipped from source i than is available

$$\sum_{i=1}^{m} X_{ij} \geq D_j, \quad j=1, \ldots n$$

requirement at dstn. j is met

$$X_{ij} \geq 0, \quad \text{all } i \;\&\; j$$

This is an LP with: m×n variables

m+n constraints

(not including nonnegativity)

The standard, "balanced", transportation problem has total supply = total demand

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

so that all constraints will be "tight" at a feasible solution, i.e.,

$$\sum_{j=1}^{n} X_{ij} = S_i \quad \text{for } i=1, \ldots m$$

$$\sum_{i=1}^{m} X_{ij} = D_j \quad \text{for } j=1, \ldots n$$

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Conversion to Standard Form

When total supply exceeds total demand: $\sum_{i=1}^{m} S_i > \sum_{j=1}^{n} D_j$

Create a "dummy" destination (n+1) whose "demand" is equal to the surplus supply:

$$D_{n+1} = \sum_{i=1}^{m} S_i - \sum_{j=1}^{n} D_j$$

and let the cost of "shipping" to this destination be $C_{i,n+1} = 0$

(X_{ij} will equal the unshipped supply at source i)

Conversion to Standard Form

When total demand exceeds total supply $\sum_{i=1}^{m} S_i < \sum_{j=1}^{n} D_j$

In this case, the problem is infeasible, i.e., not all demand can be satisfied.

One can create a "dummy" source (m+1) whose available supply is the shortfall, i.e.,

$$S_{m+1} = \sum_{j=1}^{n} D_j - \sum_{i=1}^{m} S_i$$

and define the cost of "shipping" to be

$C_{m+1,\,j}$ = unit shortage cost at destination j

The LP tableau

An example with 3 sources, 6 destinations

X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}	X_{31}	X_{32}	X_{33}	X_{34}	X_{35}	X_{36}		
.95	1.05	.80	.15	1.00	0	.35	1.80	1.40	.80	.30	0	.90	1.80	1.60	.70	.85	0		MIN
1	1	1	1	1	1													=	12
						1	1	1	1	1	1							=	7
												1	1	1	1	1	1	=	15
1						1						1						=	5
	1						1						1					=	4
		1						1						1				=	4
			1						1						1			=	11
				1						1						1		=	8
					1						1						1	=	2

Even though the transportation problem is an LP problem and is solved by the Simplex Method for LP, we do not use the usual LP tableau.

The TP tableau

Sources \ Dstns	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	.80	.15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	5	4	4	11	8	2	sum= 34

Shipments will be written here

Unit shipping cost

To perform the Simplex Method, we need to:

- obtain an initial basic feasible solution
- "price" the nonbasic variables & select an entering variable
- select the basic variable which will leave the basis

How are these steps performed using the transportation tableau?

The Simplex method requires a BASIC FEASIBLE SOLUTION (bfs) to begin.
(# of basic variables is $m+n-1$)

3 commonly used methods:

- ⇨ Northwest Corner Method
- ⇨ Least-Cost Method
- ⇨ Vogel's Approximation Method

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Obtaining an initial b.f.s.

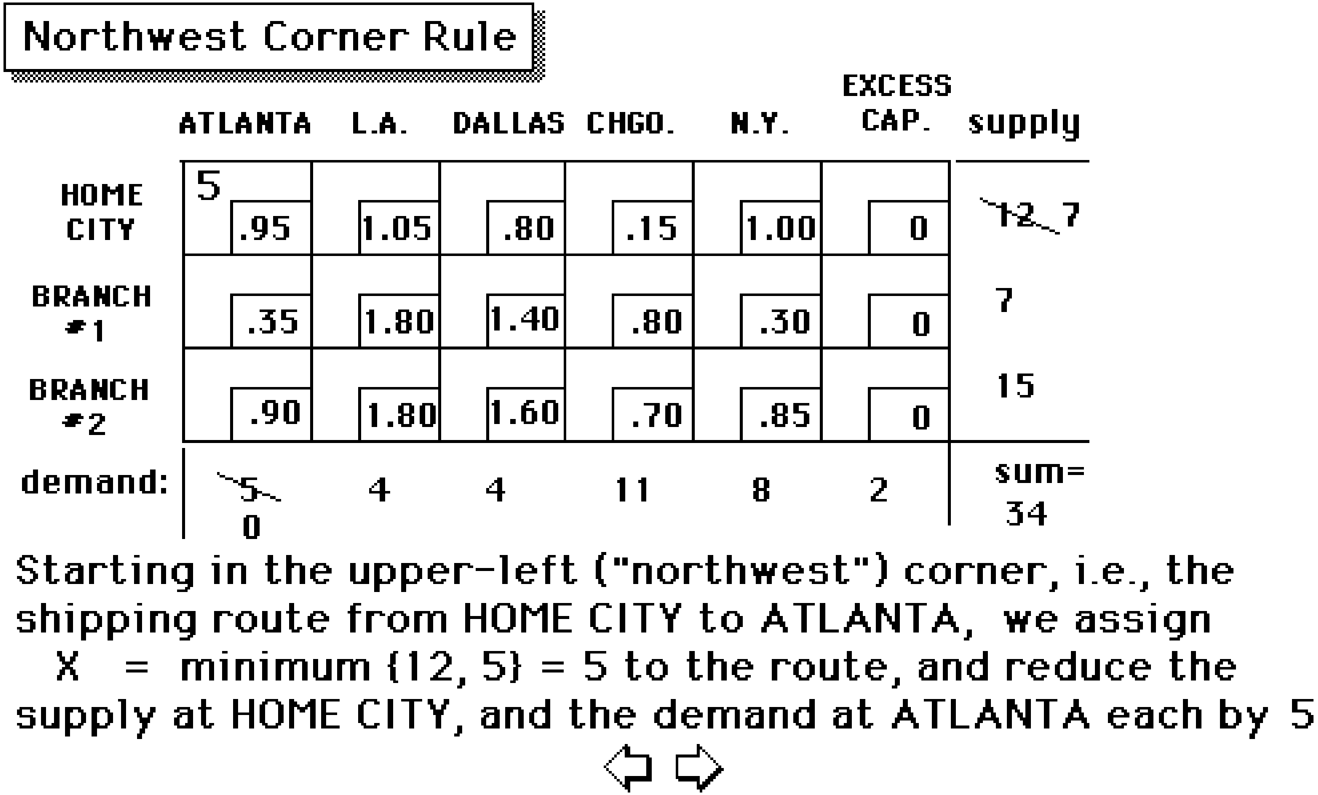
"Northwest Corner Rule"

Step 1: Assign to the upper left corner of the TP tableau the minimum of the supply in that row & the demand in that column:

$$X_{ij} = \text{minimum}\{S_i\ , D_j\ \}$$

Step 2: Reduce the supply & demand for that row & column by X_{ij}

Step 3: Delete any row &/or column with zero supply or demand, and return to step 1.



Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	1.05	.80	.15	1.00	0	~~12~~ 7
BRANCH #1	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	~~5~~ 0	4	4	11	8	2	sum= 34

Starting in the upper-left ("northwest") corner, i.e., the shipping route from HOME CITY to ATLANTA, we assign X = minimum {12, 5} = 5 to the route, and reduce the supply at HOME CITY, and the demand at ATLANTA each by 5

Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	.80	.15	1.00	0	~~12~~ ~~7~~ 3
BRANCH #1	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	~~5~~ 0	~~4~~ 0	4	11	8	2	sum= 34

Assign X_{12}= min{ 7, 4} = 4 to the shipping route from HOME CITY to L.A.

Reduce supply for HOME CITY & demand for L.A. by 4

Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	3 .80	.15	1.00	0	~~12~~ ~~7~~ ~~3~~ 0
BRANCH #1	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	~~5~~ 0	~~4~~ 0	~~4~~ 1	11	8	2	sum= 34

Assign X_{13}= min { 3, 4} = 3 to the shipping route from HOME CITY to DALLAS

Reduce supply at HOME CITY & demand at DALLAS by 3

Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	3 .80	.15	1.00	0	~~12~~ ~~7~~ ~~3~~ 0
BRANCH #1	.35	1.80	1 1.40	.80	.30	0	~~7~~ 6
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	~~5~~ 0	~~4~~ 0	~~4~~ ~~1~~ 0	11	8	2	sum= 34

Assign X_{23}= min { 7, 1} = 1 to the shipping route
from BRANCH #1 to DALLAS

Reduce supply at BRANCH #1 & demand at DALLAS by 1

Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	3 .80	.15	1.00	0	~~12~~ ~~7~~ ~~3~~ 0
BRANCH #1	.35	1.80	1 1.40	6 .80	.30	0	~~7~~ ~~6~~ 0
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	~~5~~ 0	~~4~~ 0	~~4~~ ~~1~~ 0	~~11~~ 5	8	2	sum= 34

Assign X_{24}= min {6, 11} = 6 to the shipping route from BRANCH #1 to CHGO.

Reduce supply at BRANCH #1 & demand at CHGO. by 6

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Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	3 .80	.15	1.00	0	~~12~~ ~~7~~ ~~3~~ 0
BRANCH #1	.35	1.80	1 1.40	6 .80	.30	0	~~7~~ ~~6~~ 0
BRANCH #2	.90	1.80	1.60	5 .70	8 .85	2 0	15
demand:	~~5~~ 0	~~4~~ 0	~~4~~ ~~1~~ 0	~~11~~ 5	8	2	sum= 34

Continuing, we get X_{34}=5, X_{35}=8, and X_{36}= 2

Northwest Corner Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	5 .95	4 1.05	3 .80	.15	1.00	0	12
BRANCH #1	.35	1.80	1 1.40	6 .80	.30	0	7
BRANCH #2	.90	1.80	1.60	5 .70	8 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

These 8 shipments are feasible & basic

Total cost of this shipping plan is $27.85

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Obtaining an initial b.f.s.

"Least-Cost Rule"

Whereas the NW-corner rule ignored costs completely, this rule selects the least-cost shipping route for the next assignment. (Otherwise, similar to the NW-corner rule.)

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	.80	11 .15	1.00	0	~~12~~ 1
BRANCH #1	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	5	4	4	~~11~~ 0	8	2	sum= 34

Ignoring the EXCESS CAPACITY destination, the least-cost shipment is from HOME CITY to CHGO.

Assign min{12, 11} to this shipping route, and reduce the supply & demand.

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	.80	11 .15	1.00	0	~~12~~ 1
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	~~7~~ 0
BRANCH #2	.90	1.80	1.60	.70	.85	0	15
demand:	5	4	4	~~11~~ 0	~~8~~ 1	2	sum= 34

Assign minimum {7, 8} = 7 to the shipping route from BRANCH #1 to N.Y., and reduce the supply & demand for this route.

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	.80	11 .15	1.00	0	~~12~~ 1
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	~~7~~ 0
BRANCH #2	.90	1.80	1.60	.70	1 .85	0	~~15~~ 14
demand:	5	4	4	~~11~~ 0	~~8~~ ~~1~~ 0	2	sum= 34

Assign minimum {15, 1} = 1 to the shipping route from BRANCH #2 to N.Y., and reduce supply & demand

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	1 .80	11 .15	1.00	0	~~12~~ ~~1~~ 0
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	~~7~~ 0
BRANCH #2	.90	1.80	1.60	.70	1 .85	0	~~15~~ ~~14~~ 9
demand:	5	4	~~4~~ 3	~~11~~ 0	~~8~~ ~~1~~ 0	2	sum= 34

Assign minimum{ 1,4 } = 1 to the shipping route from HOME CITY to DALLAS, and reduce the supply & demand.

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	1 .80	11 .15	1.00	0	~~12~~ ~~1~~ 0
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	~~7~~ 0
BRANCH #2	5 .90	1.80	1.60	.70	1 .85	0	~~15~~ ~~14~~ 9
demand:	~~5~~ 0	4	~~4~~ 3	~~11~~ 0	~~8~~ ~~1~~ 0	2	sum= 34

Next, we assign minimum{14, 5} = 5 to the shipping route from BRANCH #2 to ATLANTA, and reduce supply & demand.

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	.80	11 .15	1.00	0	~~12~~ ~~1~~ 0
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	~~7~~ 0
BRANCH #2	5 .90	1.80	1.60	.70	1 .85	0	~~15~~ ~~14~~ 9
demand:	~~5~~ 0	4	~~4~~ 3	~~11~~ 0	~~8~~ ~~1~~ 0	2	sum= 34

We continue, assigning the amounts required by each of L.A., DALLAS, and "EXCESS CAP." to the shipping route from BRANCH #2

Least-Cost Rule

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	1 .80	11 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	4 1.80	3 1.60	.70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

These 8 shipments are feasible, and a basic solution, with cost $21.90

Obtaining an initial b.f.s.

Vogel's Approximation Method (VAM)

For each row, compute a "penalty" equal to the difference between the two smallest costs in that row.

(If we do NOT select the least-cost cell in this row for assigning a shipment, we will pay at least this much more per unit!)

Likewise, compute a "penalty" for each column, equal to the difference between the two smallest costs in that column.

Vogel's Approximation Method (VAM)

Find the maximum penalty (which may be on either a row or column), and the least-cost cell within that row or column.

As in NW-corner Method, assign as great a shipment as possible to this cell, reduce the supply & demand for the row & column, and repeat (recomputing the penalties)

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VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
		0.55	0.75	0.60	0.55	0.55	0	
HOME CITY	0.15	.95	1.05	.80	.15	1.00	0	12
BRANCH #1	0.30	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	0.70	.90	1.80	1.60	.70	.85	0	15
demand:		5	4	4	11	8	2	sum= 34

The penalty for DALLAS is
1.40 - 0.80 = 0.60

i.e., if DALLAS does not receive
its shipment from HOME CITY,
the cost will be at least 0.60/unit greater

VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
		0.55	0.75	0.60	0.55	0.55	0	
HOME CITY	0.15	.95	4 1.05	.80	.15	1.00	0	~~12~~ 8
BRANCH #1	0.30	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	0.70	.90	1.80	1.60	.70	.85	0	15
demand:		5	~~4~~ 0	4	11	8	2	sum= 34

The maximum penalty is that for L.A.

So we select the least-cost cell for L.A. (HOME CITY- L.A.)

VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
		0.55		0.60	0.55	0.55	0	
HOME CITY	0.15	.95	4 1.05	.80	.15	1.00	0	~~12~~ 8
BRANCH #1	0.30	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	0.70	.90	1.80	1.60	.70	.85	2 0	~~15~~ 13
demand:		5	~~4~~ 0	4	11	8	~~2~~ 0	sum= 34

The maximum penalty now is that of BRANCH #2.

We select the least-cost cell in that row (BRANCH#2 - EXCESS CAP)

VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
		0.55		0.60	0.55	0.55		
HOME CITY	0.65	.95	4 1.05	.80	8 .15	1.00	0	~~12~~ ~~8~~ 0
BRANCH #1	0.05	.35	1.80	1.40	.80	.30	0	7
BRANCH #2	0.15	.90	1.80	1.60	.70	.85	2 0	~~15~~ 13
demand:		5	~~4~~ 0	4	~~11~~ 3	8	~~2~~ 0	sum= 34

After updating the penalties, we select the HOME CITY row, and the least-cost cell in that row (HOME CITY - CHGO.)

VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
		$\boxed{0.55}$		0.20	0.10	0.55		
HOME CITY		.95	4 1.05	.80	8 .15	1.00	0	~~12~~ ~~8~~ 0
BRANCH #1	0.05	$\boxed{5 \quad .35}$	1.80	1.40	.80	.30	0	~~7~~ 2
BRANCH #2	0.15	.90	1.80	1.60	.70	.85	2 0	~~15~~ 13
demand:		~~5~~ 0	~~4~~ 0	4	~~11~~ 3	8	~~2~~ 0	sum= 34

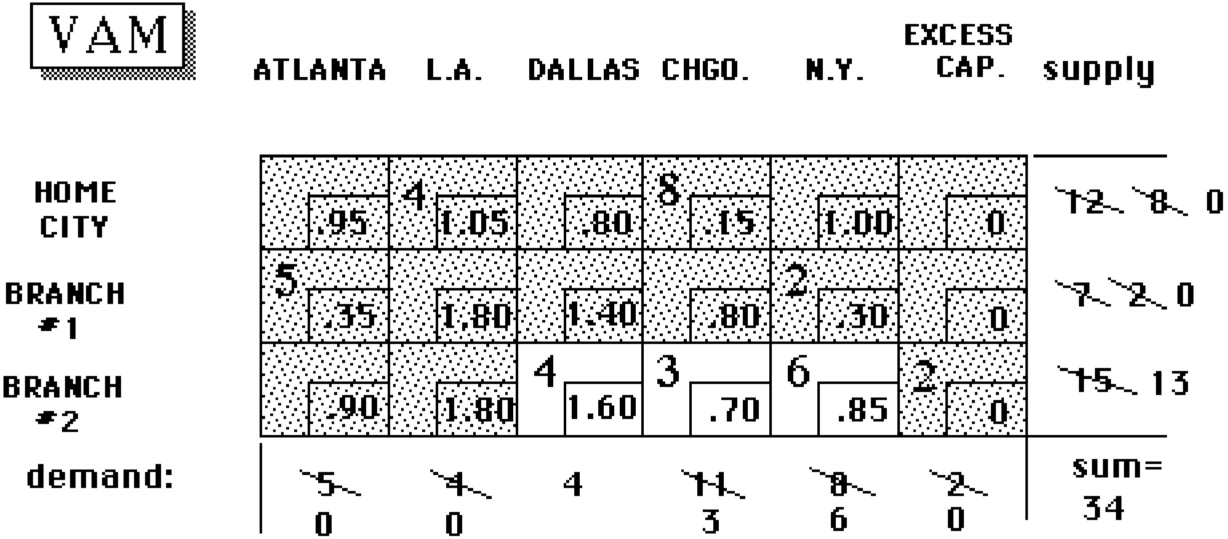
Again we update the penalties, and choose the largest penalty, that of ATLANTA, and the least-cost cell in that column (BRANCH#1 - ATLANTA)

VAM

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
				0.20	0.10	0.55		
HOME CITY		.95	4 1.05	.80	8 .15	1.00	0	~~12~~ ~~8~~ 0
BRANCH #1	0.50	5 .35	1.80	1.40	.80	2 .30	0	~~7~~ ~~2~~ 0
BRANCH #2	0.15	.90	1.80	1.60	.70	.85	2 0	~~15~~ 13
demand:		~~5~~ 0	~~4~~ 0	4	~~11~~ 3	~~8~~ 6	~~2~~ 0	sum= 34

The maximum penalty is now that of BRANCH #1, and we select the least-cost cell in that row (BRANCH#1 - N.Y.)

VAM



	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	4 1.05	.80	8 .15	1.00	0	~~12~~ ~~8~~ 0
BRANCH #1	5 .35	1.80	1.40	.80	2 .30	0	~~7~~ ~~2~~ 0
BRANCH #2	.90	1.80	4 1.60	3 .70	6 .85	2 0	~~15~~ 13
demand:	~~5~~ 0	~~4~~ 0	4	~~11~~ 3	~~8~~ 6	~~2~~ 0	sum= 34

Since only one source remains, we can complete the solution!

VAM

	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	4 1.05	.80	8 .15	1.00	0	12
BRANCH #1	5 .35	1.80	1.40	.80	2 .30	0	7
BRANCH #2	.90	1.80	4 1.60	3 .70	6 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

The total shipping cost for this solution is $21.35

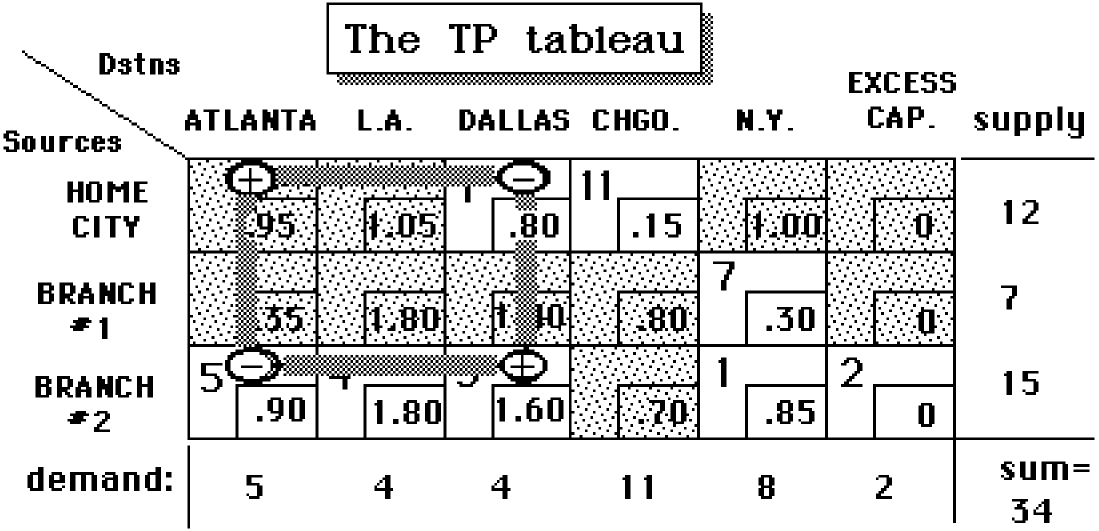
Computing Reduced Costs

To begin a simplex iteration, we must select a variable (shipment) to enter the solution.

This variable should have a **negative reduced cost.**

$$\text{reduced cost of a route} = \left\{ \begin{array}{l} \text{change in cost function} \\ \text{if one unit is shipped} \\ \text{along that route} \end{array} \right.$$

The TP tableau



Sources \ Dstns	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	1 .80	11 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	4 1.80	3 1.60	.70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

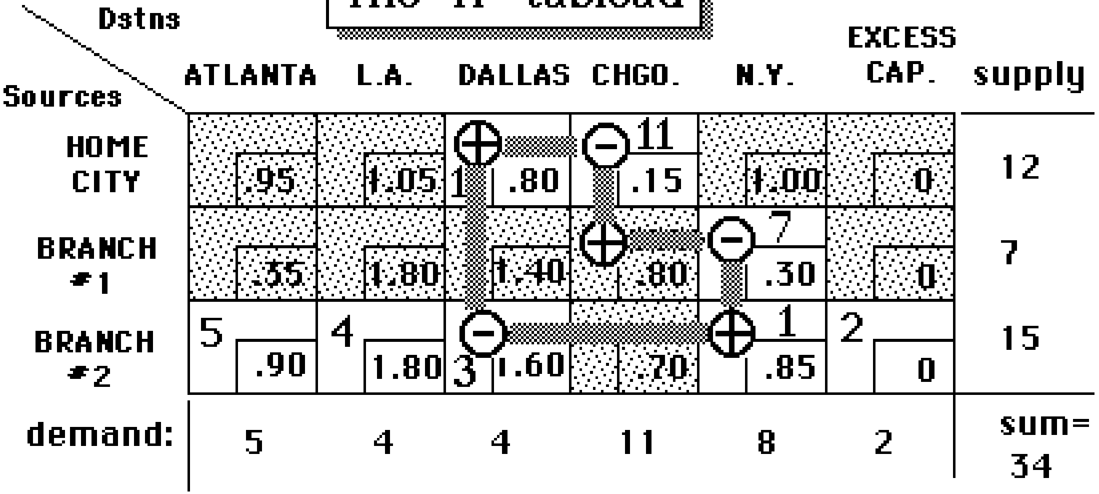
Suppose that we ship one unit from HOME CITY to ATLANTA.

Change in total cost = +0.95 − 0.80 + 1.60 − 0.90

= +0.85

(Reduced Cost)

The TP tableau

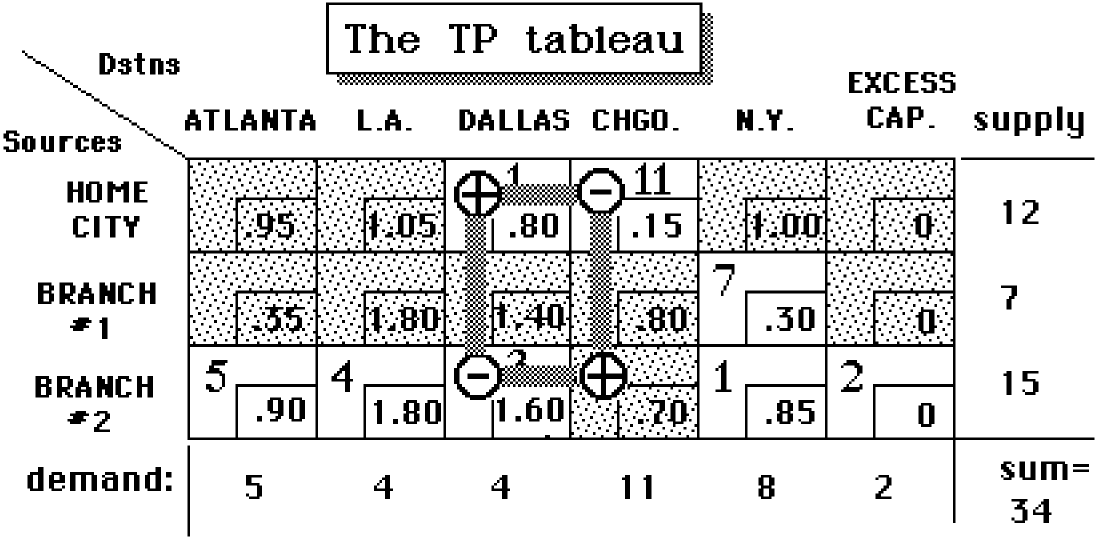


Sources \ Dstns	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	⊕ 1 .80	⊖ 11 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	⊕ .80	⊖ 7 .30	0	7
BRANCH #2	5 .90	4 1.80	⊖ 3 1.60	.70	⊕ 1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

If we ship ONE unit from BRANCH#1 to CHGO., the required adjustments are somewhat more complex. The reduced cost is 0.80−0.30+0.85−1.60+0.80−0.15

= + 0.40 ⇦ ⇨

The TP tableau



Sources \ Dstns	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	1 .80	11 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	4 1.80	2 1.60	.70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

In this tableau, only the (BRANCH#2 - CHGO.) route has a negative reduced cost (=+0.70-0.15+ 0.80-1.60 = - 0.25)

That is, every unit we ship along this route reduces our total cost by $0.25.

An easier method for computing reduced costs:

Let u_i = dual variable (simplex multiplier) for the supply constraint: $\sum_{j=1}^{n} X_{ij} = S_i$

v_j = dual variable (simplex multiplier) for the demand constraint: $\sum_{i=1}^{m} X_{ij} = D_j$

Then the reduced cost, as in the revised simplex method, is computed by: $\overline{C}_{ij} = C_{ij} - (u_i + v_j)$

⇦ ⇨

Then the reduced cost, as in the revised simplex method, is computed by:

$$\overset{\text{reduced cost}}{\overline{C}_{ij}} = \underset{\text{cost of } x_{ij}}{C_{ij}} - [u_1, u_2, \ldots u_m, v_1, \ldots v_n] \underset{\text{Column of tableau for } x_{ij}}{\begin{array}{c|l} \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \\ \\ \\ \\ \leftarrow \text{row } m+j \\ \\ \\ \\ \end{array} \end{array}}$$

$$= C_{ij} - (u_i + v_j)$$

which is much simpler than identifying the cycles!

Computing the simplex multipliers

Recall that the simplex multipliers are the values of the dual variables.

We will next write the dual constraints, and use "Complementary Slackness" to compute the dual variables.

The Primal LP

X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}	X_{31}	X_{32}	X_{33}	X_{34}	X_{35}	X_{36}		
.95	1.05	.80	.15	1.00	0	.35	1.80	1.40	.80	.30	0	.90	1.80	1.60	.70	.85	0		MIN
1	1	1	1	1	1													=	12
						1	1	1	1	1	1							=	7
												1	1	1	1	1	1	=	15
1						1						1						=	5
	1						1						1					=	4
		1						1						1				=	4
			1						1						1			=	11
				1						1						1		=	8
					1						1						1	=	2

The Dual LP

Max $12u_1+7u_2+ \ldots + 2v_6$

subject to

$$u_1 + v_1 \leq 0.95$$
$$u_1 + v_2 \leq 1.05$$
$$u_1 + v_3 \leq 0.80$$
$$u_1 + v_4 \leq 0.15$$
$$u_1 + v_5 \leq 1.00$$
$$\vdots$$
$$u_3 + v_5 \leq 0.85$$
$$u_3 + v_6 \leq 0$$

(u_i & v_j unrestricted in sign)

The dual constraints are

$$u_i + v_j \le C_{ij} \text{ for all } i \;\&\; j$$

Complementary Slackness implies that

$$X_{ij} > 0 \implies u_i + v_j = C_{ij}$$

This provides us with $\underbrace{(m+n-1)}_{\#\text{of basic variables}}$ equations from which we can compute the $(m+n)$ unknowns.

(Because the system of equations is "overdetermined", we can assign an arbitrary value, e.g., zero, to one of the dual variables.)

u_i \ v_j		**ATLANTA**	**L.A.**	**DALLAS**	**CHGO.**	**N.Y.**	**EXCESS CAP.**
HOME CITY	**0**	.95	1.05	1 .80	11 .15	1.00	0
BRANCH #1		.35	1.80	1.40	.80	7 .30	0
BRANCH #2		5 .90	4 1.80	3 1.60	.70	1 .85	2 0

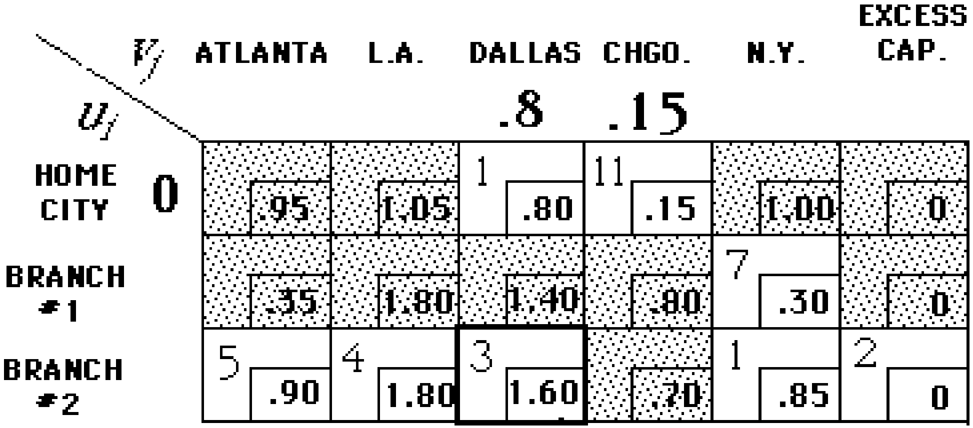
Let's arbitrarily set $u_1 = 0$

Then complementary slackness implies that

$$\boxed{\begin{aligned} u_1 + v_3 &= 0.80 \\ \Rightarrow v_3 &= 0.80 \end{aligned}}$$

and

$$\boxed{\begin{aligned} u_1 + v_4 &= 0.15 \\ \Rightarrow v_4 &= 0.15 \end{aligned}}$$



v_j u_i		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
				.8	.15		
HOME CITY	0	.95	1.05	1 .80	11 .15	1.00	0
BRANCH #1		.35	1.80	1.40	.80	7 .30	0
BRANCH #2		5 .90	4 1.80	3 1.60	.70	1 .85	2 0

Now we can use Complementary Slackness to obtain

$$u_3 + v_3 = 1.60$$
$$u_3 + .8 = 1.60$$
$$\Rightarrow u_3 = .8$$

u_i \ v_j		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
				.8	.15		
HOME CITY	0	.95	1.05	1 .80	11 .15	1.00	0
BRANCH #1		.35	1.80	1.40	.80	7 .30	0
BRANCH #2	.8	5 .90	4 1.80	3 1.60	.70	1 .85	2 0

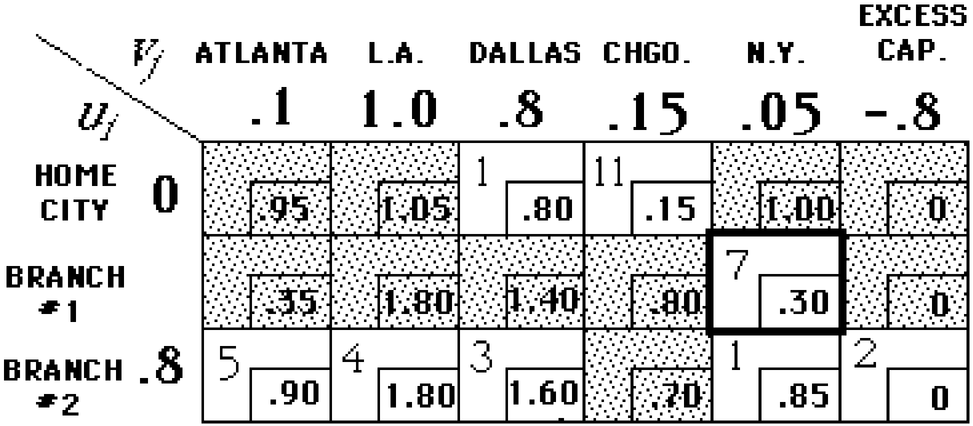
Now we can use complementary slackness to obtain v_1, v_2, v_5, and v_6

$u_3 + v_1 = 0.90$
$\Rightarrow v_1 = 0.10$

$u_3 + v_2 = 1.80$
$\Rightarrow v_2 = 1.00$

$u_3 + v_5 = 0.85$
$\Rightarrow v_5 = 0.05$

$u_3 + v_6 = 0$
$\Rightarrow v_6 = -0.8$



u_i \ v_j	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
	.1	1.0	.8	.15	.05	-.8
HOME CITY 0	.95	1.05	1 .80	11 .15	1.00	0
BRANCH #1	.35	1.80	1.40	.80	7 .30	0
BRANCH #2 .8	5 .90	4 1.80	3 1.60	.70	1 .85	2 0

Finally, we can use v_4 to compute u_2:

$$\boxed{\begin{array}{l} u_2 + v_4 = 0.30 \\ \Rightarrow u_2 = 0.25 \end{array}}$$

u_i \ v_j		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
		.1	1.0	.8	.15	.05	-.8
HOME CITY	0	.95	1.05	1 .80	11 .15	1.00	0
BRANCH #1	.25	.35	1.80	1.40	.80	7 .30	0
BRANCH #2	.8	5 .90	4 1.80	3 1.60	.70	1 .85	2 0

Now let's use the simplex multipliers to compute the reduced costs, using the formula: $\overline{C}_{ij} = C_{ij} - (u_i + v_j)$

$$\boxed{\begin{aligned}\overline{C}_{11} &= 0.95 - (0+.1)\\ &= 0.85\end{aligned}}$$

$$\boxed{\begin{aligned}\overline{C}_{24} &= 0.80 - (0.25+0.15)\\ &= 0.40\end{aligned}}$$

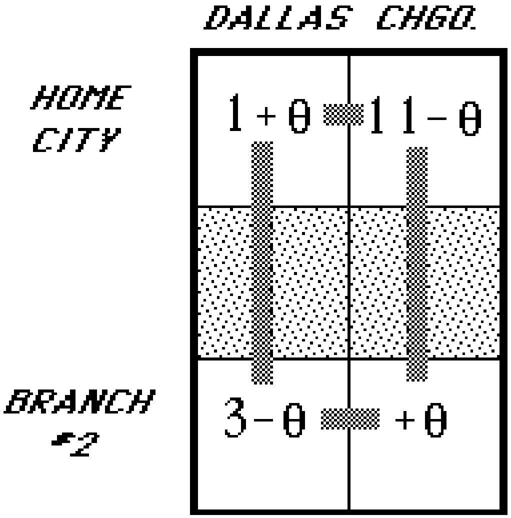
$$\boxed{\begin{aligned}\overline{C}_{34} &= 0.70 - (0.8+0.15)\\ &= -0.25\end{aligned}}$$

These are in agreement with the earlier computations!

Selecting variable to leave the basis

Once we have selected the variable to enter the basis, we must select the variable to leave the basis.

(In the simplex method, this is usually decided by the "MINIMUM RATIO TEST")



	DALLAS	CHGO.
HOME CITY	$1+\theta$	$11-\theta$
BRANCH #2	$3-\theta$	$+\theta$

Since each unit shipped along the BRANCH#2-CHGO. route reduces our cost by $0.25, so we wish to ship as much as possible.

What is the upper limit on θ ?

As soon as $\theta = 3$, the shipment from BRANCH#2-DALLAS becomes zero, preventing any further increase in θ

The TP tableau

Dstns Sources	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	1.05	4 .80	8 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	4 1.80	1.60	3 .70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

The new solution has a total shipping cost of \$21.15, a savings of \$0.75 $(= 3 \times 0.25)$

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
	v_j / u_i	.35	1.35	.8	.15	.3	-.55
HOME CITY	0	.95	1.05	4 .80	8 .15	1.00	0
BRANCH #1	0	.35	1.80	1.40	.80	7 .30	0
BRANCH #2	.55	5 .90	4 1.80	1.60	3 .70	1 .85	2 0

$$X_{ij} > 0$$
$$\Rightarrow u_i + v_j = C_{ij}$$

To proceed with the next iteration, we first compute the dual variables.

For example, start with $u_1 = 0$:

$$u_1 = 0 \Rightarrow \begin{cases} v_3 = 0.8 \\ v_4 = 0.15 \end{cases} \Rightarrow u_3 = 0.55 \Rightarrow \begin{cases} v_1 = 0.35 \\ v_2 = 1.25 \\ v_5 = 0.3 \\ v_6 = -0.55 \end{cases} \Rightarrow u_2 = 0$$

		ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.
	v_j	.35	1.35	.8	.15	.3	-.55
	u_i						
HOME CITY	0	.95	1.05	4 .80	8 .15	1.00	0
BRANCH #1	0	.35	1.80	1.40	.80	7 .30	0
BRANCH #2	.55	5 .90	4 1.80	1.60	3 .70	1 .85	2 0

The reduced costs may now be computed:

$$\overline{C}_{11} = 0.95-(0+0.35) = +0.60$$

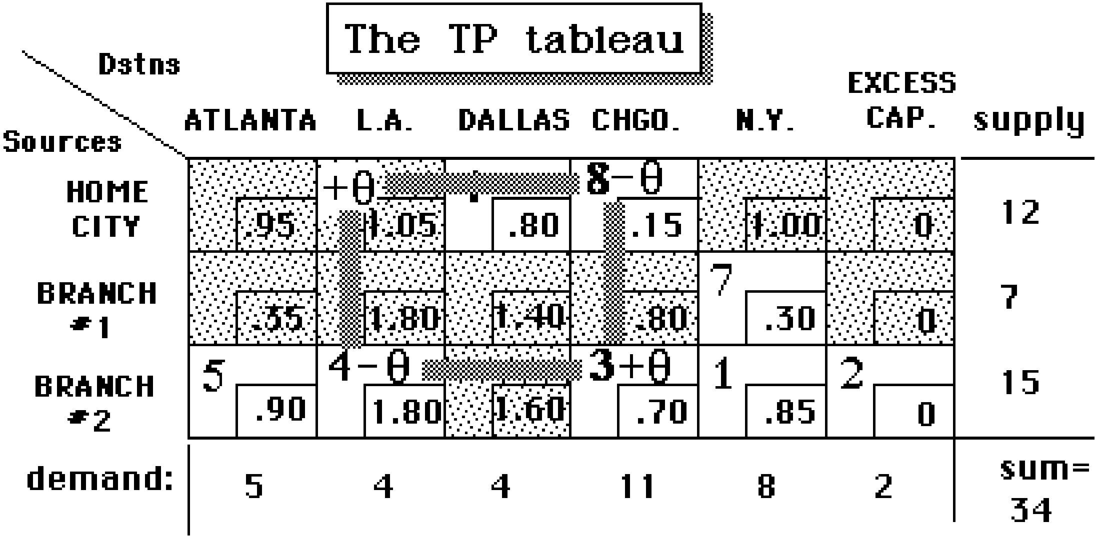
$$\overline{C}_{12} = 1.05-(0+1.25) = -0.20$$

etc.

$$\overline{C}_{ij} = C_{ij} - (u_i + v_j)$$

Since $\overline{C}_{12} < 0$, we may enter X_{12} into the solution.

The TP tableau



Dstns Sources	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	$+\theta$ 1.05	.80	$8-\theta$ .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	$4-\theta$ 1.80	1.60	$3+\theta$ .70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

We identify the cycle formed by adding the new shipment, and determine the adjustments required. The maximum allowed increase in θ is $\min\{8,4\}=4$

The TP tableau

Dstns / Sources	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	4 1.05	4 .80	4 .15	1.00	0	12
BRANCH #1	.35	1.80	1.40	.80	7 .30	0	7
BRANCH #2	5 .90	1.80	1.60	7 .70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

The new basic solution, with the cost reduced by $X_{12} \times \overline{C}_{12} = 4 \times 0.20 = 0.80$

The TP tableau

u_i \ v_j	0.9	1.60	1.35	0.7	0.85	0	supply
-0.55	.95	4 1.05	4 .80	4 .15	1.00	0	12
-0.55	.35	1.80	1.40	.80	7 .30	0	7
0	5 .90	1.80	1.60	7 .70	1 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

By first assigning $u_3 = 0$, the dual variables shown above are computed.

v_j u_i	0.9	1.60	1.35	0.7	0.85	0
-0.55	.95	4 1.05	4 .80	4 .15	1.00	0
-0.55	.35	1.80	1.40	.80	7 .30	0
0	5 .90	1.80	1.60	7 .70	1 .85	2 0

The reduced costs of the nonbasic variables are:

i	1	1	1	2	2	2	2	2	3	3
j	1	5	6	1	2	3	4	6	2	3
$\overline{C}_{ij}$	+0.6	+0.7	+0.55	0	+0.75	+0.6	+0.65	+0.55	+0.2	+0.25

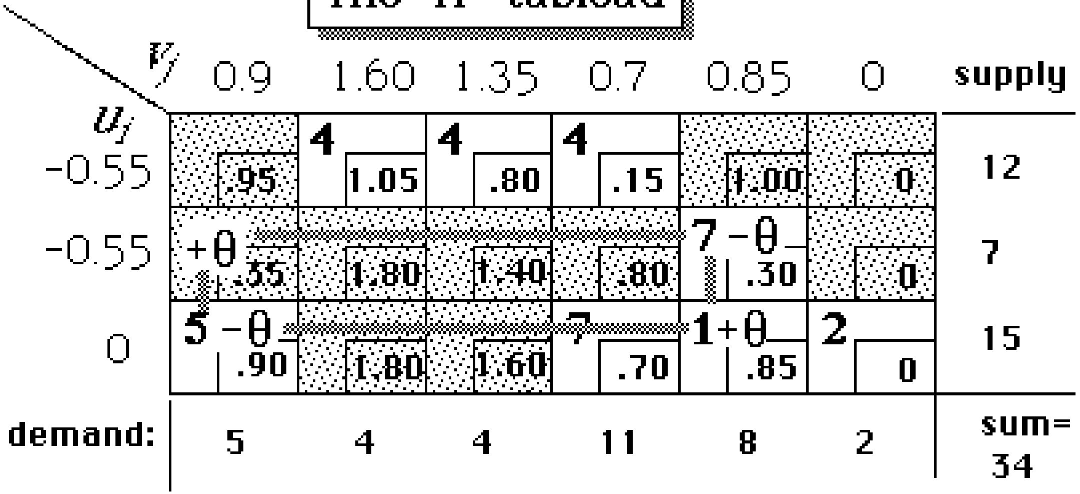
Since the reduced costs are nonnegative, the above solution is optimal!

u_i \ v_j	0.9	1.60	1.35	0.7	0.85	0
-0.55	.95	**4** 1.05	**4** .80	**4** .15	1.00	0
-0.55	.35	1.80	1.40	.80	**7** .30	0
0	**5** .90	1.80	1.60	**7** .70	**1** .85	**2** 0

Because one nonbasic variable has a zero reduced cost, there is an alternate optimal solution!

i	1	1	1	2	2	2	2	2	3	3
j	1	5	6	1	2	3	4	6	2	3
$\overline{C}_{ij}$	+0.6	+0.7	+0.55	0	+0.75	+0.6	+0.65	+0.55	+0.2	+0.25

The TP tableau

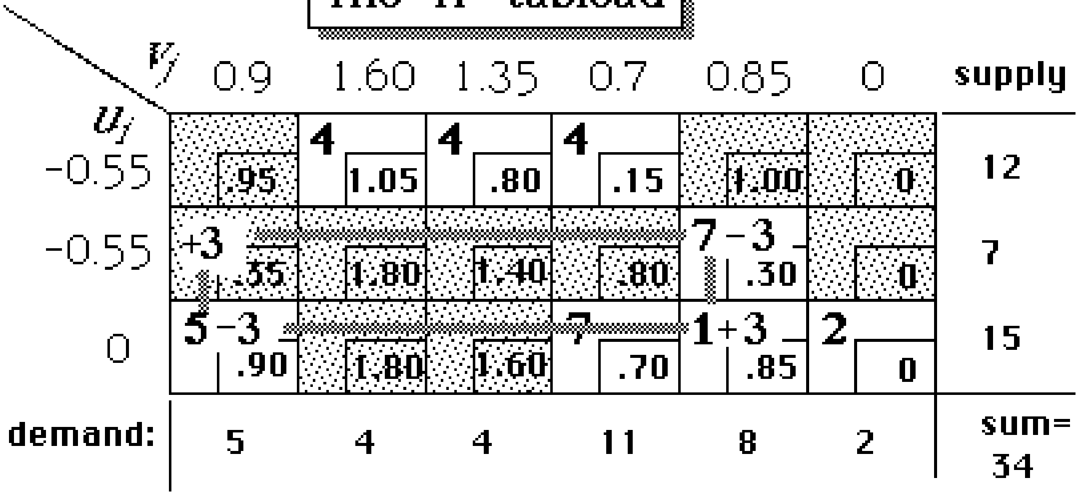


u_i \ v_j	0.9	1.60	1.35	0.7	0.85	0	supply
-0.55	.95	4 1.05	4 .80	4 .15	1.00	0	12
-0.55	+θ .35	1.80	1.40	.80	7 −θ .30	0	7
0	5 −θ .90	1.80	1.60	7 .70	1+θ .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

Increasing X_{21} by θ results in no change in the total cost, since the reduced cost is zero.

Any increase up to 5 will be feasible, and therefore optimal!

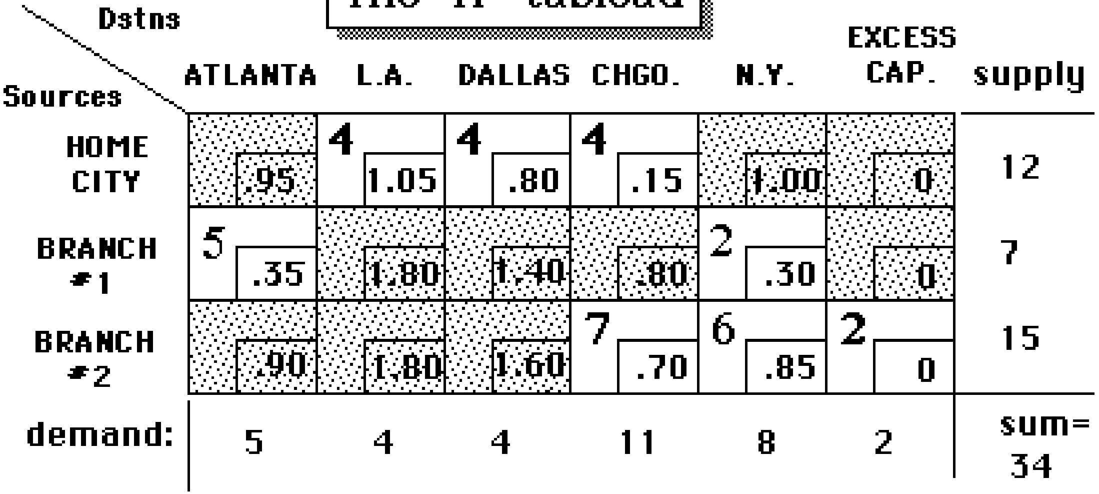
The TP tableau



u_i \ v_j	0.9	1.60	1.35	0.7	0.85	0	supply
-0.55	.95	4 1.05	4 .80	4 .15	1.00	0	12
-0.55	+3 .35	1.80	1.40	.80	7 −3 .30	0	7
0	5 −3 .90	1.80	1.60	7 .70	1+3 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

For example, an increase of 3 is optimal (although this gives us 9 positive shipments, which exceeds the number of basic variables, and is therefore optimal but not basic!)

The TP tableau



Dstns Sources	ATLANTA	L.A.	DALLAS	CHGO.	N.Y.	EXCESS CAP.	supply
HOME CITY	.95	4 1.05	4 .80	4 .15	1.00	0	12
BRANCH #1	5 .35	1.80	1.40	.80	2 .30	0	7
BRANCH #2	.90	1.80	1.60	7 .70	6 .85	2 0	15
demand:	5	4	4	11	8	2	sum= 34

If $\theta = 5$, then X_{31} becomes zero and can leave the basis, giving the basic optimal solution shown above.

A Complication: Degeneracy

A degenerate feasible solution is one in which a basic variable is zero.

When this occurs, the next basis change may not result in an improvement in the total cost!

Example:

				supply
	5	3	8	4
	14	4	5	4
	4	2	8	8
demand	2	4	10	

Initial bfs found by NW-corner rule:

$u_i \backslash v_j$	5	3	4	**supply**
0	**2** 5	**2** 3	8	4
1	14	**2** 4	**2** 5	4
4	4	2	**8** 8	8
demand	2	4	10	

Cost= 98

Reduced costs:

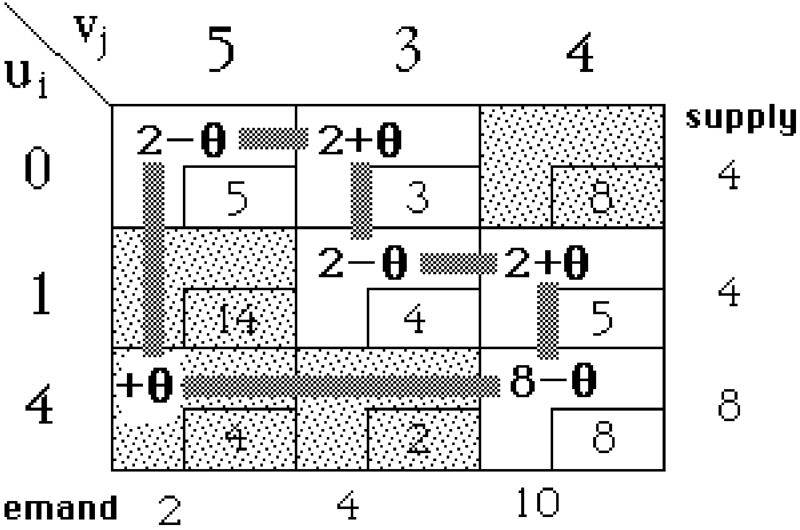
$\overline{C}_{13} = 8-(0+4) > 0$

$\overline{C}_{21} = 14 - (1+5) > 0$

$\overline{C}_{31} = 4 - (4+5) < 0$

$\overline{C}_{32} = 2 - (4+3) < 0$

Either X_{31} or X_{32} may enter the solution.
Let's arbitrarily select X_{31}



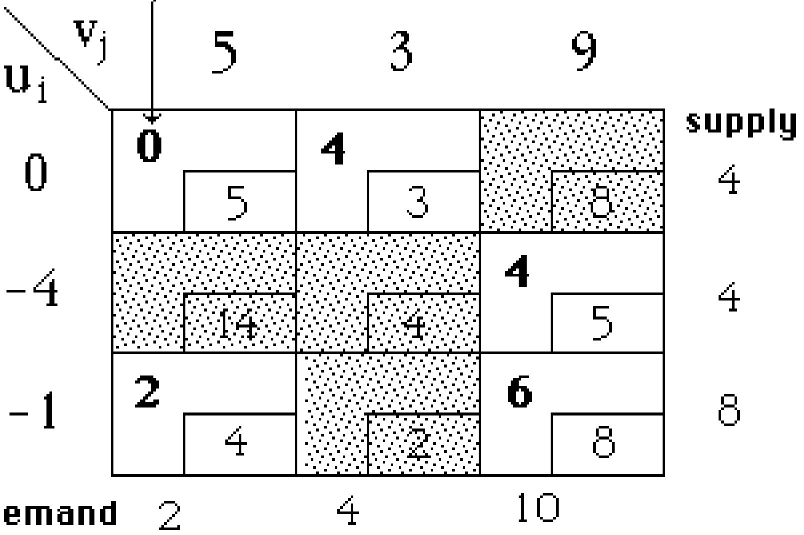
u_i \ v_j	5	3	4	**supply**
0	$2-\theta$ 5	$2+\theta$ 3	8	4
1	14	$2-\theta$ 4	$2+\theta$ 5	4
4	$+\theta$ 4	2	$8-\theta$ 8	8
demand	2	4	10	

As θ is increased, two of the basic variables reach zero simultaneously!

Only one basic variable can be replaced by X_{31}, while the other remains in the basis, even though it's value is zero.

New bfs is **degenerate**

Suppose we keep X_{11} in the basis (at value 0)



u_i \ v_j	5	3	9	supply
0	**0** 5	**4** 3	8	4
-4	14	4	**4** 5	4
-1	**2** 4	2	**6** 8	8
demand	2	4	10	

Reduced costs:

$\overline{C}_{13} = 8-(0+9) < 0$

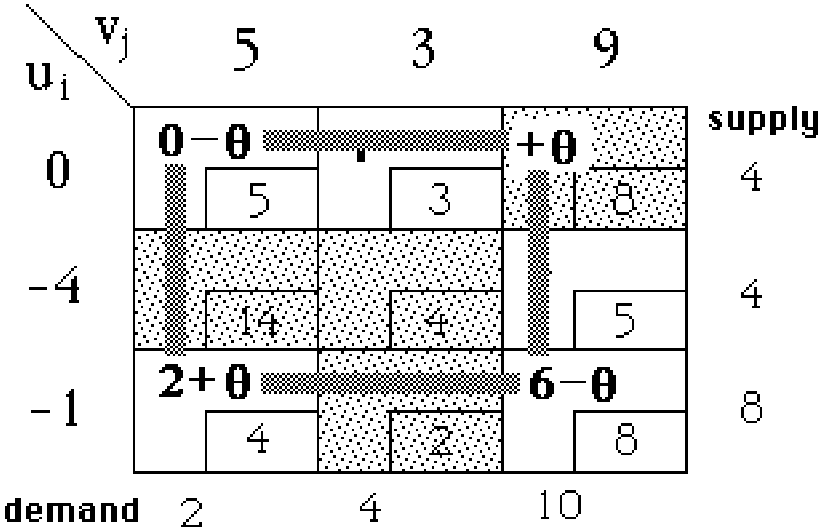
$\overline{C}_{21} = 14-(-4+5) > 0$

$\overline{C}_{22} = 4-(-4+3) > 0$

$\overline{C}_{32} = 2-(-1+3) = 0$

Cost is 88

Only X_{13} has a negative reduced cost, so it enters the basis next.

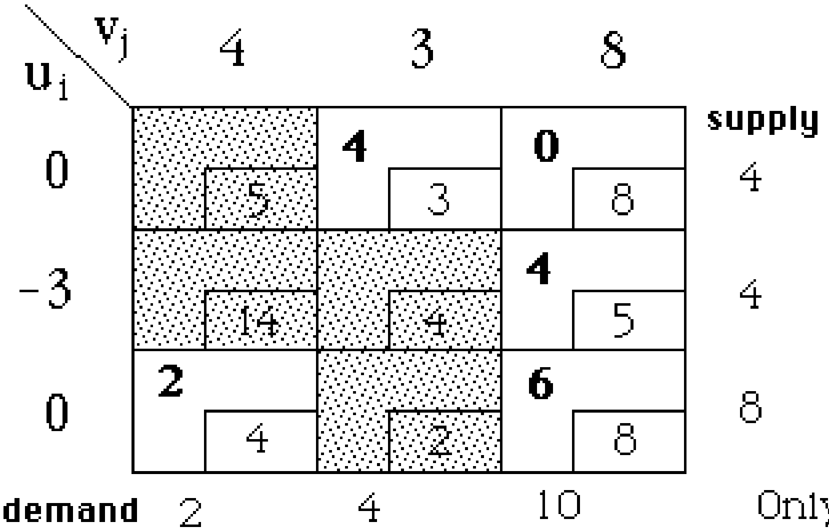


u_i \ v_j	5	3	9	supply
0	$0-\theta$ 5	3	$+\theta$ 8	4
−4	14	4	5	4
−1	$2+\theta$ 4	2	$6-\theta$ 8	8
demand	2	4	10	

As we try to increase X_{13}, we see that we are immediately "blocked" at $\theta = 0$!

If $\theta > 0$, then X_{11} becomes negative (& the solution is infeasible).

Even though we cannot increase X_{13}, we do change the basis.



u_i \ v_j	4	3	8	**supply**
0	5	**4** 3	**0** 8	4
-3	14	4	**4** 5	4
0	**2** 4	2	**6** 8	8
demand	2	4	10	

Cost remains at 88

Reduced costs:

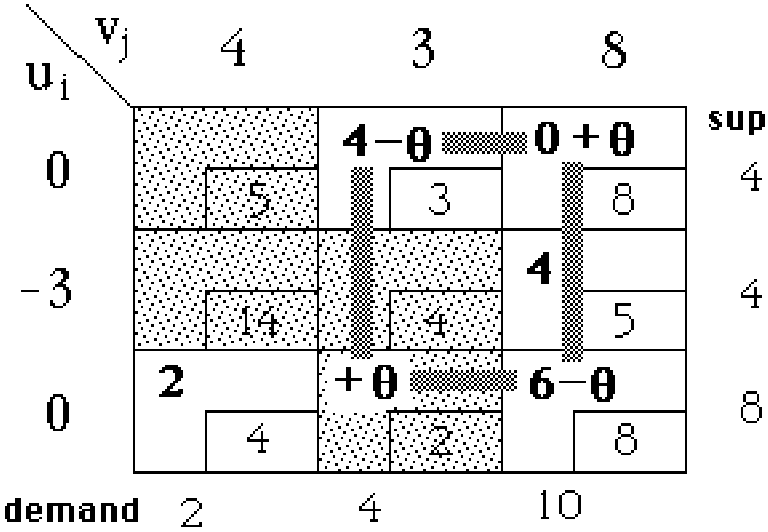
$C_{11} = 5-(0+4) > 0$

$C_{21} = 14-(-3+4) > 0$

$C_{22} = 4-(-3+3) > 0$

$C_{32} = 2-(0+3) < 0$

Only X_{32} has a negative reduced cost, so we will enter it into the basis.



u_i \ v_j	4	3	8	supply
0	5	**4−θ** 3	**0+θ** 8	4
−3	14	4	**4** 5	4
0	**2** 4	**+θ** 2	**6−θ** 8	8
demand	2	4	10	

At this iteration, even though the solution is degenerate, we are able to increase the variable entering the basis.

The next bfs is NOT degenerate!

u_i \ v_j	4	2	8	**supply**
0	5	3	**4** 8	4
-3	14	4	**4** 5	4
0	**2** 4	**4** 2	**2** 8	8
demand	2	4	10	

Reduced costs:

$C_{11} = 5-(0+4) > 0$

$C_{12} = 3-(0+2) > 0$

$C_{21} = 14-(-3+4) > 0$

$C_{22} = 4-(-3+2) > 0$

Cost is 84

This solution is optimal!

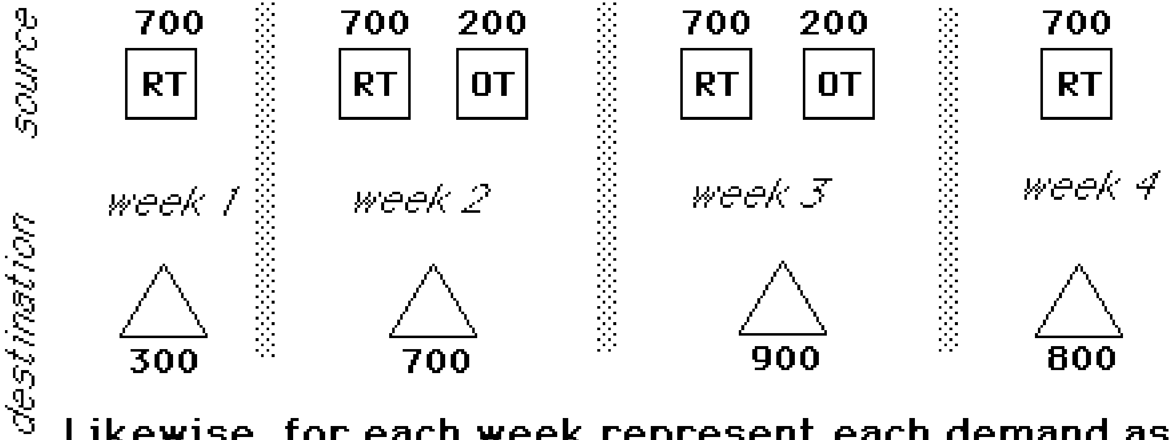
A Production Planning Problem

- demands for next 4 weeks (which must be satisfied) are: 300, 700, 900, and 800
- regular production capacity is 700/week
- overtime is available in the SECOND & THIRD weeks, adding 200 to the production capacity
- production costs are $10/unit during weeks #1&2, increasing to $15/unit during weeks #3&4; overtime adds $5/unit to the cost.
- excess production may be stored at a cost of $3/unit per week.

How should production be scheduled to minimize costs?

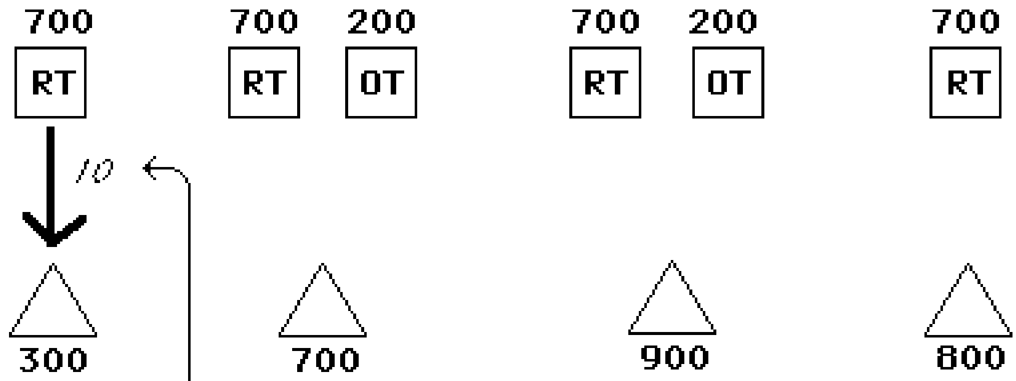
A TRANSPORTATION model of production planning:

For each week, represent each of regular and overtime capacities as a source:



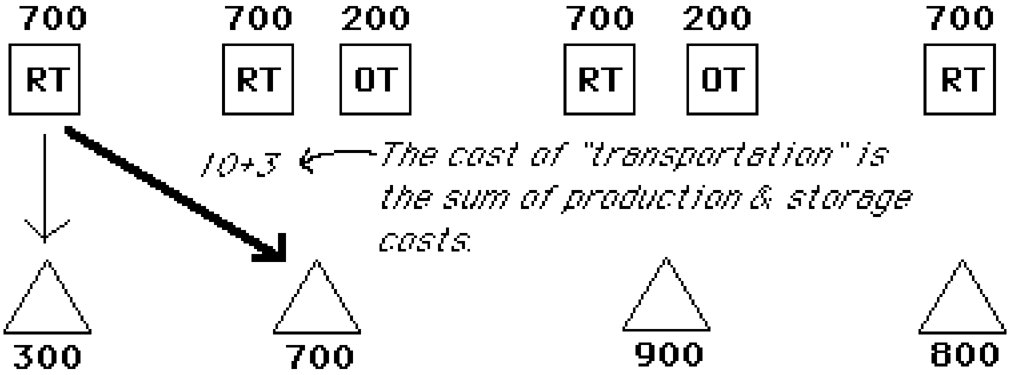
Likewise, for each week represent each demand as a destination.

Units which are produced in week #1 and which satisfy demand in week #1 are modeled as a flow from the source node to the destination node:



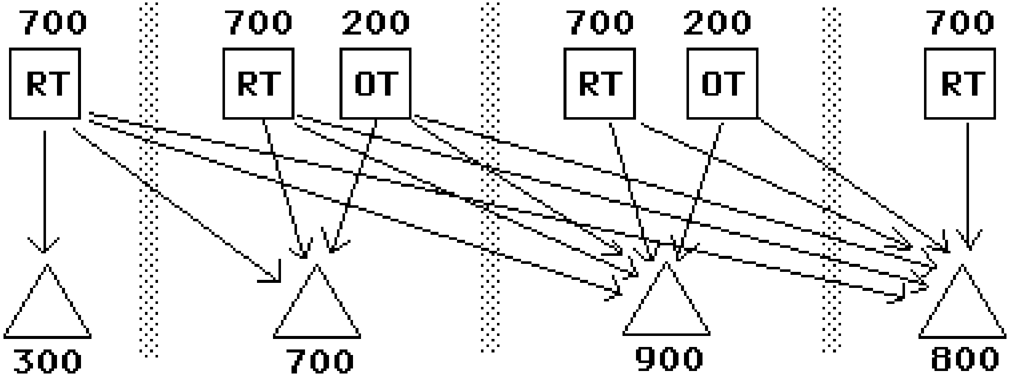
The cost of "transportation" is the production cost

Units which are produced in week #1 and which are used to satisfy the demand in week #2 are modeled by a flow from the week #1 source to the week #2 destination:

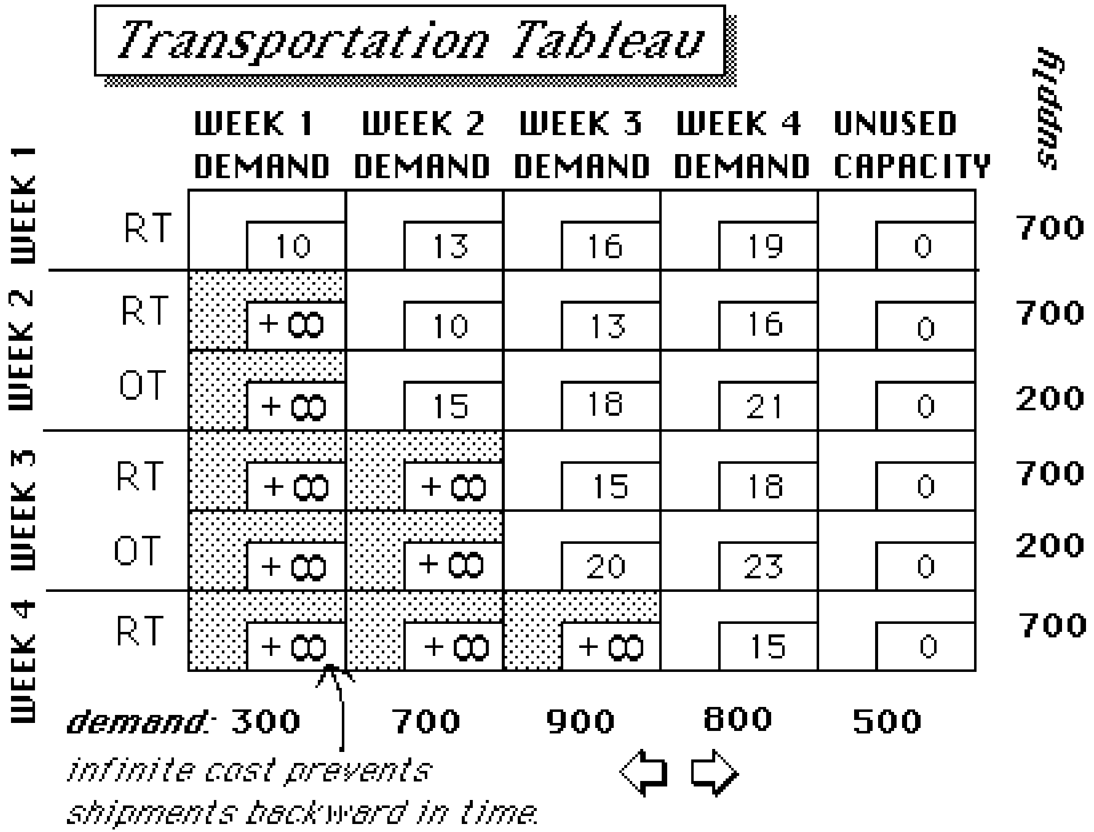


The cost of "transportation" is the sum of production & storage costs.

Flows in this model do not represent changes in geographical location!



Note that flows above are never "backward" in time.



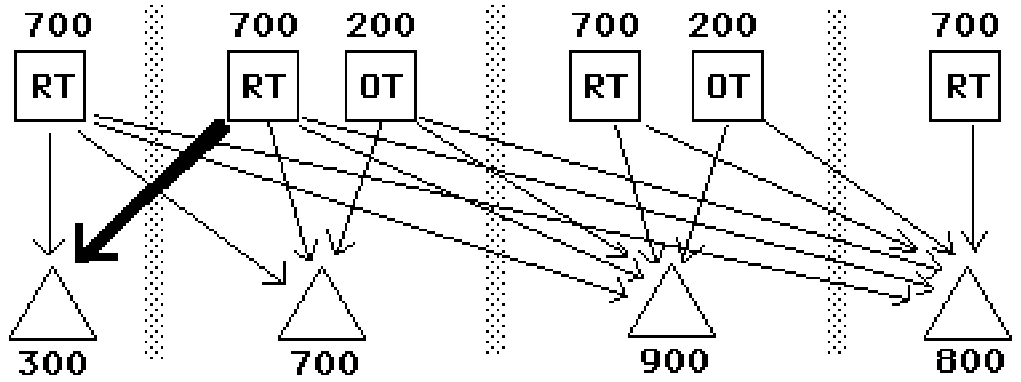
Transportation Tableau

		WEEK 1 DEMAND	WEEK 2 DEMAND	WEEK 3 DEMAND	WEEK 4 DEMAND	UNUSED CAPACITY	*supply*
WEEK 1	RT	10	13	16	19	0	700
WEEK 2	RT	+∞	10	13	16	0	700
	OT	+∞	15	18	21	0	200
WEEK 3	RT	+∞	+∞	15	18	0	700
	OT	+∞	+∞	20	23	0	200
WEEK 4	RT	+∞	+∞	+∞	15	0	700
demand:		300	700	900	800	500	

infinite cost prevents shipments backward in time.

What meaning could a shipment backward in time have?

Suppose we produce a unit in week 2 with which to satisfy week 1's demand:



That is, week 1 demand has been "backordered"
The cost of such a "shipment" should include backorder costs