# Solving Linear Equations


Dept of Mechanical & Industrial Engineering
University of Iowa

# Contents

- Preliminaries
  - Elementary Matrix Operations
  - Elementary Matrices
  - Echelon Matrices
  - Rank of Matrices

- Elimination Methods
  - Gauss Elimination
  - Gauss-Jordan Elimination

- Factorization Methods
  - Product Form of Inverse (PFI)
  - LU Factorization
  - $LDL^T$ Factorization
  - Cholesky $LL^T$ Factorization

## Elementary Row Operations

- Multiply any row of the matrix by a (positive or negative) scalar
- Add to any row a scalar multiple of another row
- Interchange two rows of the matrix

*(Strictly speaking, the third is not "elementary", because it can be accomplished by a sequence of the other two row operations!)*

# Elementary Column Operations

- Multiply any column by a (positive or negative) scalar
- Add to any column a scalar multiple of another column
- Interchange two columns of the matrix

# Equivalence of Matrices

Matrix **A** is ***equivalent*** to matrix **B** (**A** ~ **B**) if **B** is the result of a sequence of elementary row &/or column operations on **A**.

If only row operations are used, then **A** is ***row-equivalent*** to B

If only column operations are used, then **A** is ***column-equivalent*** to B

# Echelon Matrix

--an mxn matrix with the properties

- each of the first k ($0 \le k \le m$) rows has some nonzero entries, and the remaining m-k rows consist only of zeroes
- the first nonzero entry in each of the first k rows is a "1"
- in each of the first k rows, the number of zeroes preceding the leading "1" is smaller than it is in the next row

# ECHELON MATRIX

*Example*

$$\begin{bmatrix} 1 & 5 & 0 & 3 & -1 & 2 & 8 \\ 0 & 0 & 1 & -1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k=3 \;\; = \textit{rank}$$

***Note:** every matrix is row-equivalent to some echelon matrix.*

## Theorem

If A is equivalent to B, then the rank of A equals the rank of B.

*RANK: size of the largest (square) nonsingular submatrix*

# Elementary Matrices

An ***elementary matrix*** E is the result of performing an elementary operation on an identity matrix.

*Example*

*(Elementary row operation: add −2 times first row to third row)*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

# Multiplication by an Elementary Matrix

*pre-multiplication by elementary matrix*

If E is an mxm elementary matrix and A is an mxn matrix, then EA equals the result of performing the same elementary ***row*** operation on matrix A.

*Example:*

*add –2 times first row to third row*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 4 \\ 5 & 1 & 3 & -1 \\ 4 & 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 5 & 1 & 3 & -1 \\ 0 & 5 & 1 & -6 \end{bmatrix}$$

If $E$ is an $m \times m$ elementary matrix and $A$ is an $m \times n$ matrix, then $AE$ equals the result of performing the same elementary ***column*** operation on matrix $A$.

*Example:*

*add −2 times third* ***column*** *to first* ***column***

$$\begin{bmatrix} 2 & -1 & 0 \\ 5 & 1 & 3 \\ 4 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 3 \\ 2 & 3 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

*result of subtracting twice third column from first*

***post-multiplication by elementary matrix***

# Calculation of Matrix Inverse

To compute $A^{-1}$, augment the matrix A on the right by the appropriate identity matrix $[\, A \mid I \,]$ , and perform elementary row operations on this matrix to obtain $[\, I \mid P \,]$ . Then $P = A^{-1}$

# Calculation of Matrix Inverse

*Example:*

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -5 & 3 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

and so

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

**Pivot**

Pivot operation on row **r**, column **s**

i.e., element $A_r^s$ of mxn matrix A:

A sequence of elementary row operations:

- For i=1,2,...m but $i \neq r$:

  add $-A_i^s / A_r^s$ times row r to row i

- Multiply row r by the scalar $1 / A_r^s$

*Effect: column s will consist of zeroes, with the exception of a "1" in row r.*

*Warning: this is not the only sequence of elementary row operations having this effect!*

## Pivot

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & \textcircled{3} \end{bmatrix}$$

*A pivot!*

$$\begin{bmatrix} 1 & 3/5 & 0 \\ -1 & -4/3 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 1/3\, R_3$

$R_2 \leftarrow R_2 - 1/3\, R_3$

$R_3 \leftarrow 1/3\, R_3$

***Not** a pivot!*

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & -4/3 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$R_1 \leftarrow R_1 - R_2$

$R_2 \leftarrow R_2 - 1/3\, R_3$

$R_3 \leftarrow 1/3\, R_3$

# Pivot Matrix

A pivot matrix corresponding to a pivot on row r, column s of a matrix A is the result of performing the same elementary row operations on the mxm identity matrix.

A pivot matrix is the product of elementary matrices!

## Pivot Matrix

*Differs from the mxm identity matrix only in column r*

$$\begin{bmatrix} 1 & 0 & \cdots & -\frac{A_1^s}{A_r^s} & \cdots & 0 & 0 \\ 0 & 1 & \cdots & -\frac{A_2^s}{A_r^s} & \cdots & 0 & 0 \\ & & \ddots & & & & \\ 0 & 0 & \cdots & \frac{1}{A_r^s} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -\frac{A_{m-1}^s}{A_r^s} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -\frac{A_m^s}{A_r^s} & \cdots & 0 & 1 \end{bmatrix}$$

# Pivot Matrix

To store a pivot matrix, we need not store the entire matrix, but only

- the number (r) of the pivot row
- column #r of the pivot matrix (the ***eta*** vector)

$$\eta = \left[ -\frac{A_1^s}{A_r^s}, -\frac{A_2^s}{A_r^s}, \cdots \frac{1}{A_r^s}, \cdots -\frac{A_m^s}{A_r^s} \right]$$

*This is sufficient information to reconstruct the pivot matrix.*

# Product Form of the Inverse

If matrix A is nonsingular, then a sequence of pivots down the diagonal of A (with possible row interchanges to avoid zero pivot elements) will reduce A to the identity matrix. This is equivalent to pre-multiplying A by a sequence of pivot matrices:

$$(P_m \cdots (P_3(P_2(P_1A)))\cdots) = I$$

$$\Longrightarrow \quad (P_m \cdots P_3 P_2 P_1) A = I$$

$$\Longrightarrow \quad A^{-1} = P_m \cdots P_3 P_2 P_1$$

# Product Form of the Inverse

In the Revised Simplex Method, computation of values in the tableau is done, not by pivoting in the tableau, but by either pre-multiplication or post-multiplication by the inverse matrix:

- Computation of simplex multipliers

$$\pi = c^B (A^B)^{-1}$$

*used in selecting pivot column*

- Computation of substitution rates

$$\alpha = (A^B)^{-1} A^s$$

*used in performing the pivot*

## Computing Simplex Multipliers

Solve $\pi A^B = c^B$ for $\pi$ :

$$\pi = c^B (A^B)^{-1}$$
$$= c^B (P_k P_{k-1} \cdots P_3 P_2 P_1)$$
$$= (((\cdots(c^B P_k) P_{k-1} \cdots P_3) P_2) P_1)$$

*"Backward Transformation", or BTRAN*

The pivot matrices are processed in the ***reverse*** of the order in which they were generated, i.e., $P_k P_{k-1} \ldots P_3 P_2 P_1$

## BTRAN

For each pivot matrix P,
we need to calculate $\pi = \nu P$

*column r*

$$\pi = \begin{bmatrix} \nu_1 & \nu_2 & \cdots & \nu_{m-1} & \nu_m \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \eta_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_r & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_{m-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \eta_m & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \nu_1 & \nu_2 & \cdots & \left(\sum_i \nu_i \eta_i\right) & \cdots & \nu_{m-1} & \nu_m \end{bmatrix}$$

*entry r*

## BTRAN

$$\pi_j = \begin{cases} \nu_j & \text{for } j \neq r \\ \sum_i \nu_i \eta_i & \text{for } j = r \end{cases}$$

Step 0: Set $\boldsymbol{\nu} = \mathbf{c}^B$ and k = # of ETA vectors

Step 1: Using BTRAN formula above, compute with ETA vector #k

Step 2: If k>1, let $\boldsymbol{\nu} = \boldsymbol{\pi}$ and k = k−1, and go to step 1; else proceed to step 3.

Step 3: The final value of $\boldsymbol{\pi}$ is the solution of $\boldsymbol{\pi}\, \mathbf{A}^B = \mathbf{c}^B$

FTRAN

Solve $A^B \alpha = A^s$ for substitution rates $\alpha$

$$\alpha = (A^B)^{-1} A^s$$

$$= (P_k P_{k-1} \cdots P_3 P_2 P_1) A^s$$

$$= (P_k (P_{k-1} \cdots P_3 (P_2 (P_1 A^s)) \cdots))$$

*"Forward Transformation", or FTRAN*

The pivot matrices are processed in the same order that they were generated,

i.e., $P_1, P_2, P_3, \cdots \quad P_{k-1}, P_k$

# FTRAN

*column r* ↓

$$\alpha = \begin{bmatrix} 1 & 0 & \cdots & \eta_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \eta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_r & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_{m-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \eta_m & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_m \end{bmatrix} = \begin{bmatrix} \nu_1 + \eta_1 \nu_r \\ \nu_2 + \eta_2 \nu_r \\ \vdots \\ \eta_r \nu_r \\ \nu_m + \eta_m \nu_r \end{bmatrix}$$

That is,

$$\alpha_i = \begin{cases} \nu_i + \eta_i \nu_r & \textbf{for } i \neq r \\ \eta_r \nu_r & \textbf{for } i = r \end{cases}$$

## FTRAN

$$\alpha_i = \begin{cases} \nu_i + \eta_i \nu_r & \textbf{for } \mathbf{i \neq r} \\ \eta_r \nu_r & \textbf{for } \mathbf{i = r} \end{cases}$$

Step 0: Set $\boldsymbol{\nu} = A^s$ (e.g., column of original tableau), and k=1.

Step 1: Using the FTRAN formula above, compute $\boldsymbol{\alpha}$

Step 2: If k < # of ETA vectors, then let $\boldsymbol{\nu} = \boldsymbol{\alpha}$ and k=k+1, and go to step 1; else proceed to step 3.

Step 3: The final value of $\boldsymbol{\nu}$ is the solution $\boldsymbol{\alpha}$ of the equation $A^B \alpha = A^s$

# Gauss Elimination

-- a method for solving Ax=b by performing a sequence of elementary row operations on the augmented matrix $[A \mid b]$ to reduce it to an echelon matrix. The solution is then obtained by "back-substitution".

**Example:**

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + 2x_3 = 2 \\ -x_1 - x_2 + x_3 = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 2 \\ -1 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Longrightarrow \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 + x_3 = -2 \\ x_3 = 3 \end{cases}$$

Backsubstitution:

$$\left.\begin{cases} x_1 = 4 - x_2 - x_3 \\ \left.\begin{array}{l} x_2 = -2 - x_3 \\ x_3 = 3 \end{array}\right\} \Longrightarrow x_2 = -5 \end{cases}\right\} \Longrightarrow x_1 = 6$$

# Gauss-Jordan Elimination

--similar to Gauss elimination, except that the coefficient matrix is diagonalized by further elementary row operations, eliminating non-zeroes above as well as below the diagonal. Eliminates the need for "back-substitution".

**Example:**

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + 2x_3 = 2 \\ -x_1 - x_2 + x_3 = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 2 \\ -1 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That is,

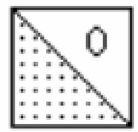
$$\begin{cases} x_1 = 6 \\ x_2 = -5 \\ x_3 = 3 \end{cases}$$

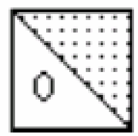
*Compared to "Gauss Elimination Plus Back Substitution", Gauss-Jordan Elimination requires more computation--*
*especially if the equations are to be solved for several right-hand-side vectors!*

# Gauss Elimination as Matrix Factorization

$$A = P\,L\,U$$

P is a permutation matrix (which performs the interchange of rows for partial pivoting)

L is a lower triangular matrix, 


U is an upper triangular matrix 

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 + R_1]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_2]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

*Upper-triangular matrix*

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

*Lower triangular matrices*

$$\underbrace{E_2 E_1}_{\hat{L}} A = U$$

*Lower triangular matrix*

$$\hat{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad \hat{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = L$$

$$\hat{L}\, A = U \quad \Longrightarrow \quad A = \hat{L}^{-1} U = L\, U$$

*Matrix A is factored into a product of lower & upper triangular matrices!*

Suppose that we need to solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ -x_1 - x_2 + x_3 = 5 \\ x_2 + 3x_3 = -1 \end{cases}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

To solve $Ax=b$, i.e., $L(Ux)=b$:

- solve $Ly=b$ for y *(forward substitution)*

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \Longrightarrow \begin{cases} y_1 = 2 \\ y_2 = 5 + y_1 = 7 \\ y_3 = -1 - y_2 = -8 \end{cases}$$

- solve $Ux=y$ for x *(backward substitution)*

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 7 \\ -8 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = 2 - 2x_2 - x_3 = -36 \\ x_2 = 7 - 2x_3 = 23 \\ x_3 = -8 \end{cases}$$

# Cholesky Factorization

Suppose that A is a ***symmetric*** & ***positive definite*** matrix.

Then the ***Cholesky factorization*** of A is

$$A = \hat{L}\ \hat{L}^T$$

where $\hat{L}$ is a ***lower triangular*** matrix.

*Computation:*

Suppose that we have the factorization

$$A = L\ D\ L^T$$

Then if $D_i^i \geq 0$, we can define a new diagonal matrix $\hat{D}$ where

$$\hat{D}_i^i \equiv \sqrt{D_i^i}$$

Then $A = L\ D\ L^T = L\ \hat{D}\ \hat{D}\ L^T = \left(L\hat{D}\right)\ \left(L\hat{D}\right)^T = \hat{L}\ \hat{L}^T$ where $\hat{L} = L\hat{D}$

## Example:

We wish to find the Cholesky factorization of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

## *Cholesky factorization…*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2} R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & 1 & 0 & 1 & \frac{3}{2} \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -\frac{1}{2} & -1 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

*Inverse:* $R_3 \leftarrow R_3 + \frac{1}{2} R_1$

*Inverse:* $R_3 \leftarrow R_3 + R_2$

$L^{-1}$ *(lower triangular)* $\quad$ $U$ *(upper triangular)*

The lower triangular matrix $L$ is found by performing (on the identity matrix) the inverse of the row operations used to reduce the A matrix:

$$\left.\begin{array}{l} R_3 \leftarrow R_3 + \tfrac{1}{2} R_1 \\ R_3 \leftarrow R_3 + R_2 \end{array}\right\} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tfrac{1}{2} & 1 & 1 \end{bmatrix}$$

We now have the LU factorization of matrix A:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tfrac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \tfrac{1}{2} \end{bmatrix}$$

Define the diagonal matrix D:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Note that

$$\hat{U} = D^{-1}U = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

And so,

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Define the diagonal matrix $\hat{D}$ where $\hat{D}_i^i \equiv \sqrt{D_i^i}$ :

$$\hat{D} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Then compute $\hat{L} = L\hat{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix}$

So the Cholesky factorization is

$$A = \hat{L}\hat{L}^T = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$