Simplex Algorithm: an Introduction

PAR, Inc. is a small manufacturer of golf equipment and supplies, including a

- STANDARD golf bag, and a
- DELUXE golf bag.

Each bag produced requires 4 operations, with the following processing times (hrs):

	cut & dye	sew	finish	inspect & pack
STANDARD	$\frac{7}{10}$	$\frac{1}{2}$	1	$\frac{1}{10}$
DELUXE	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{4}$

After studying departmental workload projections, the plant manager estimates that the following time will be available for production of golf bags during the next quarter:

Dept.	Man-hrs.
Cut-&-Dye	630
Sewing	600
Finishing	708
Inspect-&-Pack	135

PAR's distributor is convinced that everything which PAR makes can be easily sold, with a resulting profit of

$10 per STANDARD bag and
$ 9 per DELUXE bag.

PAR wishes to determine the number of each type bag which will maximize the profit.

Definition of Variables

X_1 = # STANDARD bags produced next qtr.

X_2 = # DELUXE bags produced next qtr.

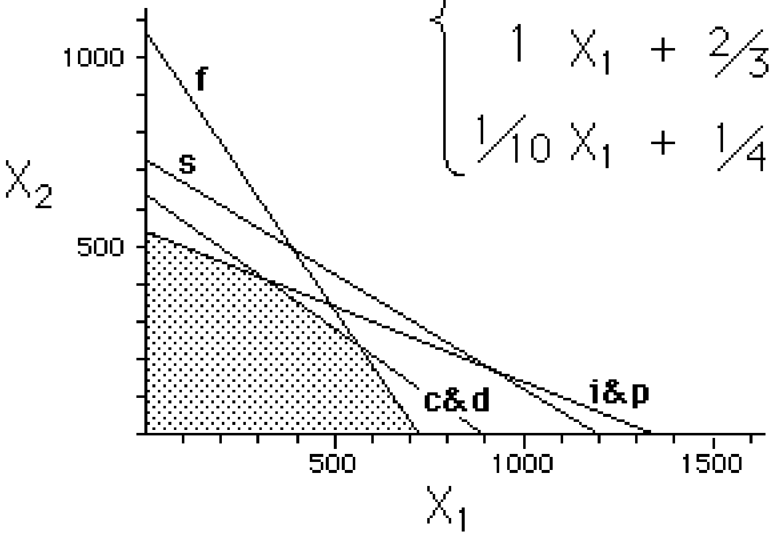
LP: Maximize $10\,X_1 + 9\,X_2$

subject to

$$\begin{cases} \tfrac{7}{10}X_1 + 1\,X_2 \le 630 \\ \tfrac{1}{2}X_1 + \tfrac{5}{6}X_2 \le 600 \\ 1\,X_1 + \tfrac{2}{3}X_2 \le 708 \\ \tfrac{1}{10}X_1 + \tfrac{1}{4}X_2 \le 135 \\ X_1 \ge 0 \quad X_2 \ge 0 \end{cases}$$

a product-mix LP model

$$
\left\{
\begin{array}{rcrcl}
\tfrac{7}{10} X_1 & + & 1\, X_2 & \le & 630 \\
\tfrac{1}{2} X_1 & + & \tfrac{5}{6} X_2 & \le & 600 \\
1\, X_1 & + & \tfrac{2}{3} X_2 & \le & 708 \\
\tfrac{1}{10} X_1 & + & \tfrac{1}{4} X_2 & \le & 135
\end{array}
\right.
\qquad
\begin{array}{l}
\textit{DEPT.} \\
\textit{cut\&dye} \\
\textit{sewing} \\
\textit{finishing} \\
\textit{inspect\& pack}
\end{array}
$$



The shaded region satisfies all four inequality constraints!

Converting to "standard" LP model

Define "slack" variables

S_1 = unused hours in Cut-&-Dye Dept.

S_2 = unused hours in Sewing Dept.

S_3 = unused hours in Finishing Dept.

S_4 = unused hours in Inspect-&-Pack Dept.

and

Z = profit

By the introduction of the "slack" variables, the inequalities (with the exception of the non-negativity restrictions) become equations:

$$\text{Maximize} \quad 10X_1 + 9X_2 = Z$$

$$\text{subject to} \begin{cases} \frac{7}{10}X_1 + 1X_2 + S_1 = 630 \\ \frac{1}{2}X_1 + \frac{5}{6}X_2 + S_2 = 600 \\ 1X_1 + \frac{2}{3}X_2 + S_3 = 708 \\ \frac{1}{10}X_1 + \frac{1}{4}X_2 + S_4 = 135 \\ X_1 \geq 0 \quad X_2 \geq 0 \end{cases}$$

Tableau

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4		rhs
1	10	9	0	0	0	0	=	0
0	$\frac{7}{10}$	1	1	0	0	0	=	630
0	$\frac{1}{2}$	$\frac{5}{6}$	0	1	0	0	=	600
0	1	$\frac{2}{3}$	0	0	1	0	=	708
0	$\frac{1}{10}$	$\frac{1}{4}$	0	0	0	1	=	135

Notice that the system of equations represented by the tableau has essentially been "solved" for the variables Z, S_1, S_2, S_3, and S_4 in terms of the variables X_1 and X_2:

"basic" variables

$$\begin{cases} Z = 0 + 10X_1 + 9X_2 \\ S_1 = 630 - \tfrac{7}{10}X_1 - 1X_2 \\ S_2 = 600 - \tfrac{1}{2}X_1 - \tfrac{5}{6}X_2 \\ S_3 = 708 - 1X_1 - \tfrac{2}{3}X_2 \\ S_4 = 135 - \tfrac{1}{10}X_1 - \tfrac{1}{4}X_2 \end{cases}$$

X_1 and X_2 are parameters, or "nonbasic" variables

"complete" solution

$$\begin{cases} Z = 0 + 10\,X_1 + 9\,X_2 \\ S_1 = 630 - \tfrac{7}{10}\,X_1 - 1\,X_2 \\ S_2 = 600 - \tfrac{1}{2}\,X_1 - \tfrac{5}{6}\,X_2 \\ S_3 = 708 - 1\,X_1 - \tfrac{2}{3}\,X_2 \\ S_4 = 135 - \tfrac{1}{10}\,X_1 - \tfrac{1}{4}\,X_2 \end{cases}$$

If we assign arbitrary values to X_1 & X_2, we get "**particular**" solutions,

e.g., $X_1 = 100$ *standard bags*,

$X_2 = 120$ *deluxe bags*

$$\begin{cases} Z = 2080 \ \$ \\ S_1 = 440 \ \textit{hrs.} \\ S_2 = 450 \ \textit{hrs.} \\ S_3 = 528 \ \textit{hrs.} \\ S_4 = 95 \ \textit{hrs.} \end{cases}$$

"complete" solution

$$
\left\{
\begin{array}{l}
Z = 0 + 10\,X_1 + 9\,X_2 \\
S_1 = 630 - \tfrac{7}{10}\,X_1 - 1\,X_2 \\
S_2 = 600 - \tfrac{1}{2}\,X_1 - \tfrac{5}{6}\,X_2 \\
S_3 = 708 - 1\,X_1 - \tfrac{2}{3}\,X_2 \\
S_4 = 135 - \tfrac{1}{10}\,X_1 - \tfrac{1}{4}\,X_2
\end{array}
\right.
$$

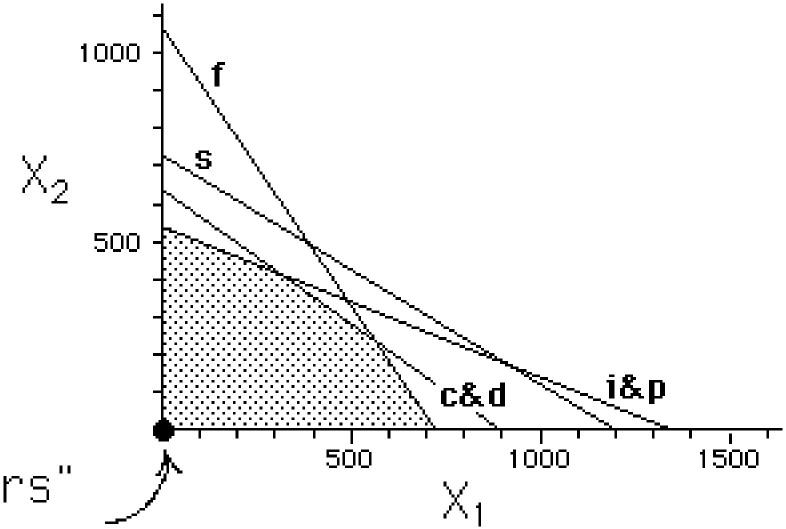
If we let the "nonbasic" variables X_1 & X_2 be zero, then we obtain a "**basic**" solution:

basic solution

$$
\left\{
\begin{array}{l}
Z = 0 \ \$ \\
S_1 = 630 \ \textit{hrs.} \\
S_2 = 600 \ \textit{hrs.} \\
S_3 = 708 \ \textit{hrs.} \\
S_4 = 135 \ \textit{hrs.}
\end{array}
\right.
$$

$$\boxed{\textit{basic solution}}\ \begin{cases} Z = 0 & \$ \\ S_1 = 630 & \textit{hrs.} \\ S_2 = 600 & \textit{hrs.} \\ S_3 = 708 & \textit{hrs.} \\ S_4 = 135 & \textit{hrs.} \end{cases}$$

(This basic solution is the plan to produce ***neither*** the STANDARD ***nor*** the DELUXE golf bags, resulting in all available production time being unused.)



This basic solution
is one of the "corners"
of the feasible region,
which is a polyhedron.

Looking at the PROFIT equation,

$$Z = 0 + 10X_1 + 9\,X_2$$

we see that this basic solution is not optimal, since an increase in ***either*** X_1 ***or*** X_2 results in an ***increase*** in the profit Z.

Let's arbitrarily select X_1 (i.e., production of the STANDARD golf bag) to be increased. Each unit of increase in X_1 results in a \$10 increase in Z (profit).

As X_1 is increased, the values of the basic variables S_1, S_2, S_3, and S_4 are also altered.

$$\begin{cases} S_1 = 630 - \frac{7}{10} X_1 - \ldots \\ S_2 = 600 - \frac{1}{2} X_1 - \ldots \\ S_3 = 708 - 1\, X_1 - \ldots \\ S_4 = 135 - \frac{1}{10} X_1 - \ldots \end{cases}$$

An **in**crease of 1 unit of X_1 $\Longrightarrow$

- $\frac{7}{10}$ unit **de**crease in S_1
- $\frac{1}{2}$ unit **de**crease in S_2
- 1 unit **de**crease in S_3
- $\frac{1}{10}$ unit **de**crease in S_4

"substitution rates"

An **in**crease of 1 unit of X_1 $\Longrightarrow$

$\frac{7}{10}$ unit **de**crease in S_1
$\frac{1}{2}$ unit **de**crease in S_2
1 unit **de**crease in S_3
$\frac{1}{10}$ unit **de**crease in S_4

How much may X_1 *be increased?*

A further increase in X_1 is "blocked" when one of the (currently) basic variables reaches its lower bound (zero). To continue increasing X_1 would cause a violation in the nonnegativity of the basic variable.

An **in**crease of 1 unit of X_1 $\Longrightarrow$

$$\begin{cases} 7/10 \text{ unit } \textbf{de}\text{crease in } S_1 \\ 1/2 \text{ unit } \textbf{de}\text{crease in } S_2 \\ 1 \text{ unit } \textbf{de}\text{crease in } S_3 \\ 1/10 \text{ unit } \textbf{de}\text{crease in } S_4 \end{cases}$$

current values

$$\begin{cases} S_1 = 630 \\ S_2 = 600 \\ S_3 = 708 \\ S_4 = 135 \end{cases}$$

$$\begin{cases} S_1 = 630 - 7/10\, X_1 \\ S_2 = 600 - 1/2\, X_1 \\ S_3 = 708 - 1\, X_1 \\ S_4 = 135 - 1/10\, X_1 \end{cases}$$

$$\begin{cases} S_1 = 630 - \frac{7}{10} X_1 \geq 0 \\ S_2 = 600 - \frac{1}{2} X_1 \geq 0 \\ S_3 = 708 - 1 X_1 \geq 0 \\ S_4 = 135 - \frac{1}{10} X_1 \geq 0 \end{cases} \Longrightarrow \begin{cases} \frac{7}{10} X_1 \leq 630 \\ \frac{1}{2} X_1 \leq 600 \\ 1 X_1 \leq 708 \\ \frac{1}{10} X_1 \leq 135 \end{cases}$$

$$\Longrightarrow \begin{cases} X_1 \leq \dfrac{630}{\frac{7}{10}} \\ X_1 \leq \dfrac{600}{\frac{1}{2}} \\ X_1 \leq \dfrac{708}{1} \\ X_1 \leq \dfrac{135}{\frac{1}{10}} \end{cases} \Longrightarrow \begin{cases} X_1 \leq 900 \\ X_1 \leq 1200 \\ X_1 \leq 708 \\ X_1 \leq 1350 \end{cases}$$

least upper bound

As we increase X_1 from zero, the first "block" occurs at

$$\left\{\begin{array}{l} X_1 \leq 900 \\ X_1 \leq 1200 \\ X_1 \leq 708 \\ X_1 \leq 1350 \end{array}\right.$$

least upper bound

$\mathbf{min}\{900, 1200, 708, 1350\}$

$= 708$, where S_3 becomes zero.

We now wish to "re-solve" the system of equations so that X_1 *is a basic variable and* S_3 *is nonbasic (and therefore zero).*

Current tableau

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	10	9	0	0	0	0	0
0	$7/10$	1	1	0	0	0	630
0	$1/2$	$5/6$	0	1	0	0	600
0	1	$2/3$	0	0	1	0	708
0	$1/10$	$1/4$	0	0	0	1	135

"Pivot" on the element in the column of the new basic variable and the blocking row.

PIVOT

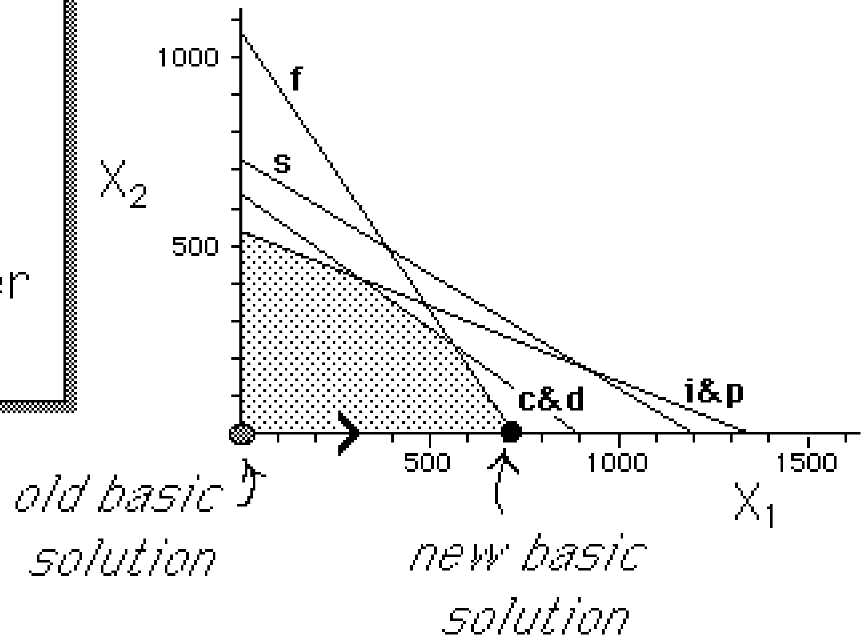
Subtract $10 \times$ROW4 from ROW1
Subtract $(7/10)$ROW4 from ROW2
Subtract $(1/2)$ROW4 from ROW3
Subtract $(1/10)$ROW4 from ROW5

New tableau resulting from the pivot

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	$\frac{7}{3}$	0	0	-10	0	-7080
0	0	$\frac{8}{15}$	1	0	$-\frac{7}{10}$	0	134.4
0	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	246
0	1	$\frac{2}{3}$	0	0	1	0	708
0	0	$\frac{11}{60}$	0	0	$-\frac{1}{10}$	1	64.2

Basic Variables

A pivot corresponds to a move along an edge from one corner to another!



"complete" solution

$$\begin{cases} Z = 7080 + \frac{7}{3} X_2 - 10 S_3 \\ S_1 = 134.4 - \frac{8}{15} X_2 + \frac{7}{10} S_3 \\ S_2 = 246 - \frac{1}{2} X_2 + \frac{1}{2} S_3 \\ X_1 = 708 - \frac{2}{3} X_2 - 1 S_3 \\ S_4 = 64.2 - \frac{11}{60} X_2 + \frac{1}{10} S_3 \end{cases}$$

This is another representation of the ***same*** "complete" solution of the system of equations.

For example, if we let $X_2=120$ and $S_3 = 528$, we get the same "particular" solution which was mentioned earlier.

"complete" solution

$$\begin{cases} Z = 7080 + \frac{7}{3} X_2 - 10 S_3 \\ S_1 = 134.4 - \frac{8}{15} X_2 + \frac{7}{10} S_3 \\ S_2 = 246 - \frac{1}{2} X_2 + \frac{1}{2} S_3 \\ X_1 = 708 - \frac{2}{3} X_2 - 1 S_3 \\ S_4 = 64.2 - \frac{11}{60} X_2 + \frac{1}{10} S_3 \end{cases}$$

The ***basic*** solution corresponding to this choice of basic variables is ***different***, however:

$X_2 = 0$ and $S_3 = 0$ yield $Z = 7080$ ¢ and

$$\begin{cases} S_1 = 134.4 & \textit{hrs.} \\ S_2 = 246 & \textit{hrs.} \\ X_1 = 708 & \textit{bags} \\ S_4 = 64.2 & \textit{hrs.} \end{cases}$$

Note that the current basic solution is *still* not optimal, however, since increasing X_2 will further increase the profit:

$$Z = 7080 + \tfrac{7}{3} X_2 - 10\, S_3$$

"relative profit"

The coefficient of a variable in the equation for the profit, Z, is called the "relative profit"

The variable X_2 is the *only* nonbasic variable with a positive relative profit, so we will select it to be increased.

$$\begin{cases} S_1 = 134.4 - \frac{8}{15} X_2 + \ldots \\ S_2 = 246 - \frac{1}{2} X_2 + \ldots \\ X_1 = 708 - \frac{2}{3} X_2 - \ldots \\ S_4 = 64.2 - \frac{11}{60} X_2 + \ldots \end{cases}$$

substitution rates

$$\begin{bmatrix} \frac{8}{15} \\ \frac{1}{2} \\ \frac{2}{3} \\ \frac{11}{60} \end{bmatrix}$$

As before, we will increase the nonbasic variable until one of the basic variables reaches its lower bound (zero), which "blocks" any further increase in X_2.

Nonnegativity of the basic variables provides bounds on X_2:

$$\begin{cases} S_1 = 134.4 - \frac{8}{15} X_2 \geq 0 \\ S_2 = 246 - \frac{1}{2} X_2 \geq 0 \\ X_1 = 708 - \frac{2}{3} X_2 \geq 0 \\ S_4 = 64.2 - \frac{11}{60} X_2 \geq 0 \end{cases} \Longrightarrow \begin{cases} X_2 \leq \dfrac{134.4}{8/15} = 252 \\ X_2 \leq \dfrac{246}{1/2} = 492 \\ X_2 \leq \dfrac{708}{2/3} = 1062 \\ X_2 \leq \dfrac{64.2}{11/60} = 350.18 \end{cases}$$

As soon as X_2 reaches the smallest of these bounds (in this case 252), any further increase is blocked, since it would force a basic variable (in this case S_1) to become negative!

Minimum Ratio Test

The increase of a nonbasic variable is blocked when it reaches the minimum of the ratios of right-hand-sides to ***positive*** substitution rates in the constraint rows.

The variable which is basic in the row with the minimum ratio will be replaced by the increased variable.

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	$\frac{7}{3}$	0	0	-10	0	-7080
0	0	$\frac{8}{15}$	1	0	$-\frac{7}{10}$	0	134.4
0	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	246
0	1	$\frac{2}{3}$	0	0	1	0	708
0	0	$\frac{11}{60}$	0	0	$-\frac{1}{10}$	1	64.2

pivot column

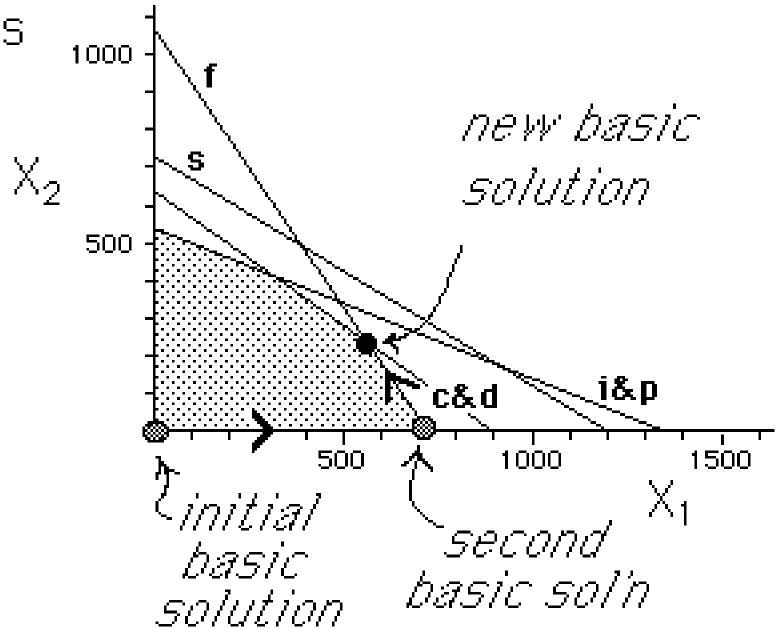
The pivot row is the row with the minimum ratio of rhs to (positive) substitution rate!

Result of the pivot

$-Z$	X_1	X_2	S_1	S_2	S_3	S_4	rhs
1	0	0	$-35/8$	0	$-111/16$	0	-7668
0	0	1	$15/8$	0	$-21/16$	0	252
0	0	0	$-15/16$	1	$5/32$	0	120
0	1	0	$-10/8$	0	$15/8$	0	540
0	0	0	$-11/32$	0	$9/64$	1	18

A pivot corresponds to a move along an edge from one corner to an adjacent corner:

At this new basic solution, the nonbasic variables S_1 & S_3 are zero, i.e., the first and third constraints are "tight"



$\boxed{\text{"complete" solution}}$

$$\begin{cases} Z = 7668 - \frac{35}{8} S_1 - \frac{111}{16} S_3 \\ X_2 = 252 - \frac{15}{8} S_1 + \frac{21}{16} S_3 \\ S_2 = 120 + \frac{15}{16} S_1 - \frac{5}{32} S_3 \\ X_1 = 540 + \frac{10}{8} S_1 - \frac{15}{8} S_3 \\ S_4 = 18 + \frac{11}{32} S_1 - \frac{9}{64} S_3 \end{cases}$$

The basic solution corresponding to this choice of basis is to produce 540 STANDARD golf bags and 252 DELUXE golf bags, with 120 and 18 hours unused in the sewing and the inspect&pack depts., respectively.

Looking at the equation for PROFIT, we see that the "relative profits" of the nonbasic variables are both negative:

$$Z = 7668 - \tfrac{35}{8} S_1 - \tfrac{111}{16} S_3$$

This means that ***any*** positive values assigned to the variables S_1 and S_3 will result in a profit of ***less*** than $7668.

Therefore, the current basic solution ***must be optimal!***