

# Revised Simplex Method for LP



author

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## **Basis Set**

The set of indices, i.e., column numbers, of the basic variables forms the **basis**, denoted **B**

For example,

if  $X_2$ ,  $X_7$ , and  $X_3$  are basic variables, then

$$\mathbf{B} = \{2, 7, 3\}$$

## Basic Variables

The basic variables corresponding to the basis  $B = \{3,6,1\}$  form a subvector of  $X$ , denoted by

$$X_B = \{ X_3, X_6, X_1 \}$$

The subvector of the cost vector  $C$  corresponding to this basis is denoted

$$C_B = \{ C_3, C_6, C_1 \}$$

## Basis Matrix

The columns of the constraint coefficient matrix  $A$  indexed by the basis  $B$  is the **basis matrix** denoted  $A^B$

For example, if  $B=\{3,6,1\}$  and the original tableau is

-Z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

then

$$A^B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

## **Basis Inverse**

The inverse of the basis matrix will be called the **basis inverse**

For example, if  $B = \{3, 6, 1\}$  and  $A^B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

then the basis inverse is

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

In the simplex method, we pivot in the tableau in order to change the basis:

-Z  $X_1$   $X_2$   $X_3$   $X_4$   $X_5$   $X_6$   $X_7$  b

1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

*original tableau*

*tableau with basic variables -Z,  $X_3$ ,  $X_6$ ,  $X_1$*

-Z	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$	b
1	0	3.22	0	2.56	0.333	0	-0.556	-3.33
0	0	0.444	1	0.111	-0.333	0	-0.111	1.33
0	0	0.778	0	0.444	0.667	1	-1.44	0.333
0	1	-0.333	0	1.67	0	0	0.333	1

If we have the basis inverse matrix and the original tableau, we can compute the current tableau:

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 3 & 2 & -1 & 0 & 0 \\ 5 & 0 & 2 & 9 & 0 & 1 & 0 \\ 3 & -1 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$(A^B)^{-1}A = \begin{bmatrix} 0 & 0.444 & 1 & 0.111 & -0.333 & 0 & -0.111 \\ 0 & 0.778 & 0 & 0.444 & 0.667 & 1 & -1.44 \\ 1 & -0.333 & 0 & 1.67 & 0 & 0 & 0.333 \end{bmatrix}$$

*which is that section of the current tableau which contained A originally!*

If the variable  $X_k$  has been chosen to enter the basis, then the "minimum ratio test" must be performed to choose the pivot row.

The "**substitution rates**" to be used in the denominator of these ratios may therefore be obtained by  $(A^B)^{-1}A^k$ .

e.g.,

$$\alpha = (A^B)^{-1}A^4 = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 4/9 \\ 5/3 \end{bmatrix}$$



We can use the basis inverse matrix to compute the values of the basic variables:

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$(A^B)^{-1} b = \begin{bmatrix} 1.33 \\ 0.333 \\ 1 \end{bmatrix}$$

*which is the RHS of the current tableau, i.e., the values of the basic variables:*

$$\begin{cases} X_3 = 4/3 \\ X_6 = 1/3 \\ X_1 = 1 \end{cases}$$

*Note the order, which corresponds to  $B = \{3, 6, 1\}$*

The **reduced cost** of variable  $X_k$  is

$$\bar{C}_k = C_k - \sum_{i=1}^m C_{B_i} \alpha_i$$

where  $\alpha_i$  is the substitution rate of  $X_k$  for the basic variable in row  $i$ , i.e., of  $X_{B_i}$

*That is, if  $X_k$  is increased by 1 unit,  $C_k$  is added to the cost, while the basic variable in row  $i$  is reduced by the amount  $\alpha_i$ , thereby saving (if  $\alpha_i > 0$ ) a cost  $C_{B_i} \alpha_i$*

Since  $\alpha = (A^B)^{-1} A^k$ ,  $\bar{C}_k = C_k - C_B (A^B)^{-1} A^k$

If we define  $\pi = C_B (A^B)^{-1}$ , then

$$\begin{aligned}\bar{C}_k &= C_k - \pi A^k \\ &= C_k - \sum_{i=1}^m \pi_i A_i^k\end{aligned}$$

*original tableau:*

-Z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\pi = C_B (A^B)^{-1} = [1, 0, 2] \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} = [1/3, 0, 5/9]$$

-Z	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>	X <sub>7</sub>	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$\pi = \left[ \frac{1}{3}, 0, \frac{5}{9} \right]$$

Suppose we want to "price" the nonbasic variable X<sub>4</sub>:

$$\begin{aligned} \bar{C}_4 &= C_4 - \pi A^4 \\ &= 6 - \left( \frac{1}{3} \times 2 + 0 \times 9 + \frac{5}{9} \times 5 \right) = \frac{23}{9} \\ &= 2.555555 \end{aligned}$$

In the **Revised Simplex Method** , no pivoting is done in the LP tableau.

When an entry of the tableau is needed, it is computed, using the inverse of the basis matrix.

- relative profit (or reduced cost) of  $X_j$  is

$$C_j - \boldsymbol{\pi} A^j$$

where

$$\boldsymbol{\pi} = C_B (A^B)^{-1}$$

- substitution rates for the entering variable  $X_j$  (used in the minimum ratio test) are  $\boldsymbol{\alpha} = (A^B)^{-1} A^j$

## Revised Simplex Method

Step 0: determine an initial feasible basis  $B$

Step 1: compute the simplex multiplier vector

$$\pi = C_B (A^B)^{-1}$$

Step 2: price the nonbasic columns, i.e., compute the reduced cost of each nonbasic variable

Step 3: select the entering variable  $X_k$

Step 4: compute the substitution rates of  $X_k$

$$\alpha = (A^B)^{-1} A^k$$

## Revised Simplex Method

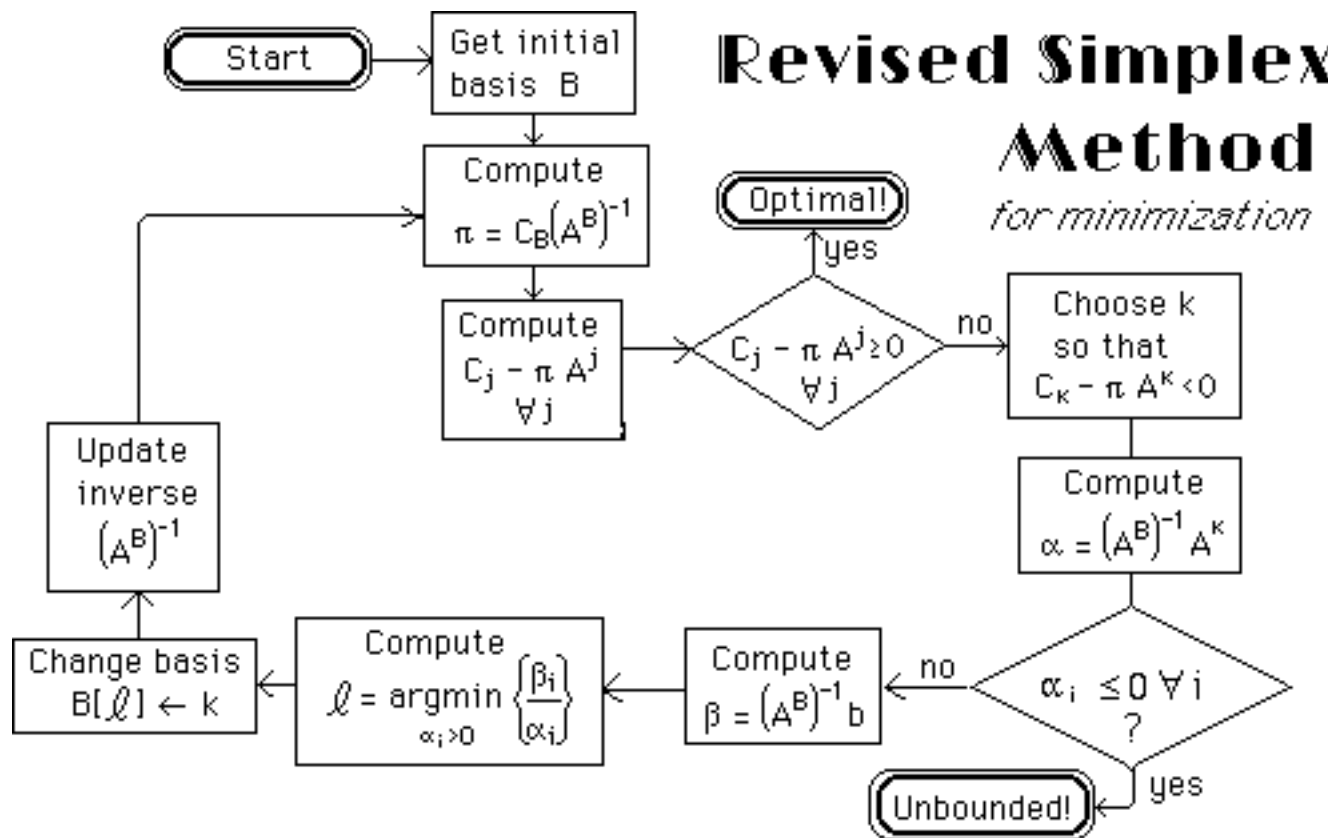
Step 5: determine the variable to leave the basis

$$\min_{a_i > 0} \left\{ \frac{\beta_i}{\alpha_i} \right\} = \frac{\beta_\ell}{\alpha_\ell} \quad \text{where } \beta = X_B = (A^B)^{-1} b$$

Step 6: replace the  $\ell^{\text{th}}$  index of B with k, and  
update the inverse matrix  $(A^B)^{-1}$

Return to step 1.





-Z	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\pi = C_B (A^B)^{-1} = \left[ 1/3, 0, 5/9 \right]$$

### Example:

$$\begin{aligned} \text{Pricing } X_7: \bar{C}_7 &= C_7 - \pi A^7 \\ &= 0 - \left( \frac{1}{3} \times 0 + 0 \times 0 + \frac{5}{9} \times 1 \right) = -5/9 \end{aligned}$$

So  $X_7$  will enter the basis

*original tableau*

-Z	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>	X <sub>7</sub>	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

*current tableau*

-Z	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>	X <sub>7</sub>	b
1	0	0				0	$-5/9$	
0	0		1			0		
0	0		0			1		
0	1		0			0		

*Since the reduced cost of the variable X<sub>7</sub> is negative, we will enter it into the basis.*

*To choose the pivot row will require the "substitution rates" in the column.*

*original tableau*

-Z	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>	X <sub>7</sub>	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Calculation of "substitution rates"  $\alpha = (A^B)^{-1} A^j$

$$\alpha = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/9 \\ -13/9 \\ 1/3 \end{bmatrix}$$

*current tableau*

$-Z$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$b$
1	0	0	0	0	0	$-5/9$		
0	0	1	0	0	0	$-1/9$		
0	0	0	0	1	0	$-13/9$		
0	1	0	0	0	0	$1/3$		

*Next, to perform the "minimum ratio test" for selecting the pivot row, we need the current right-hand-side:  $\beta = x_B = (A^B)^{-1} b$*

*original tableau*

-Z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$$B = \{3, 6, 1\}$$

$$(A^B)^{-1} = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\beta = x_B = (A^B)^{-1} b = \begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1 \end{bmatrix}$$

*current tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$	b
1	0	0				0	$-5/9$	
0	0		1			0	$-1/9$	$2/3$
0	0		0			1	$-13/9$	$1/3$
0	1		0			0	$1/3$	1

*Now we can select the pivot row, which is the last row (since there is only one positive substitution rate). That is,  $X_7$  enters the basis, replacing  $X_1$ , so that the new basis is  $B = \{3, 6, 7\}$*

*original tableau*

-Z	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>	X <sub>5</sub>	X <sub>6</sub>	X <sub>7</sub>	b
1	2	3	1	6	0	0	0	0
0	1	1	3	2	-1	0	0	5
0	5	0	2	9	0	1	0	8
0	3	-1	0	5	0	0	1	3

$B = \{3, 6, 7\}$

$$A^B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Rather than to compute the basis inverse matrix "from scratch", it is possible to update the old basis inverse matrix:*

$$\underbrace{\begin{bmatrix} 1/3 & 0 & -1/9 \\ -2/3 & 1 & -13/9 \\ 0 & 0 & 1/3 \end{bmatrix}}_{\text{old inverse}} \left\| \begin{array}{c} -1/9 \\ -13/9 \\ \textcircled{1/3} \end{array} \right. \text{pivot column} \longrightarrow \underbrace{\begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{new inverse}} \left\| \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right.$$



We are then ready to begin another iteration of the revised simplex method.

$$\boldsymbol{\pi} = \left[-\frac{1}{3}, 0, 0\right]$$

$$\bar{C}_2 = 3 - (\boldsymbol{\pi}_1 - \boldsymbol{\pi}_3) = \frac{8}{3}$$

$$\bar{C}_4 = 6 - (2\boldsymbol{\pi}_1 + 9\boldsymbol{\pi}_2 + 5\boldsymbol{\pi}_3) = \frac{16}{3}$$

$$\bar{C}_5 = 0 - (-\boldsymbol{\pi}_1) = \frac{1}{3}$$

Since the reduced costs of the nonbasic variables are all positive, the current basic solution is optimal!

## Advantages of Revised Simplex Method

- Because we pivot only in the basis inverse matrix, each pivot requires updating only  $m \times m$  numbers, not  $(m+1)(n+1)$  (where  $m = \# \text{rows}$ ,  $n = \# \text{columns}$ )

*For example, if  $m=100$  and  $n=1000$  (not unusually big in practice), the RSM updates only 10,000 numbers in the basis inverse, while the ordinary simplex method updates 100,000 numbers in the tableau.*

## Advantages of Revised Simplex Method

- Most real-world LP problems have an original tableau which is very sparse, e.g., only 1% to 5% of the numbers are nonzero. This allows efficiency of storage in memory. After several pivots in the tableau, however, the tableau usually loses most of this sparsity! RSM, using the original tableau, can still take advantage of sparsity.

## Advantages of Revised Simplex Method

- Because computation is done using only the basis inverse and the original data, the build-up of round-off errors in the computation is largely avoided.

*(LP software will periodically re-invert the basis inverse "from scratch" to avoid this buildup of round-off errors.)*

The Revised Simplex Method (or some variation or extension of it) is used in virtually all commonly-available software for linear programming.

*Other common extensions:*

- *storage of the basis inverse in "product-form" to save both storage space and computation*
- *factoring the basis inverse into triangular factors to make computation of substitution rates and simplex multipliers more efficient.*
- *simple and general upper bounds on variables*

## Exercise

*original  
tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	RHS
1	1	1	-4	0	0	0	0
0	1	1	2	1	0	0	9
0	1	1	-1	0	1	0	2
0	-1	1	1	0	0	1	4

$-Z$   $X_1$   $X_2$   $X_3$   $X_4$   $X_5$   $X_6$  RHS

1							
0			$\frac{1}{3}$	0	$-\frac{2}{3}$		
0			0	1	1		
0			$\frac{1}{3}$	0	$\frac{1}{3}$		

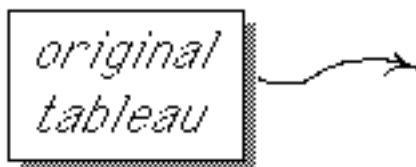
*current  
tableau*

**Complete the  
missing elements!**

Solution

**Solution:**


*original  
tableau*



$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	RHS
1	1	1	-4	0	0	0	0
0	1	1	2	1	0	0	9
0	1	1	-1	0	1	0	2
0	-1	1	1	0	0	1	4

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	RHS
1	0	4	0	1	0	2	17
0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
0	0	2	0	0	1	1	6
0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$

*current  
tableau*




## Exercise

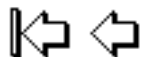
*original  
tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	RHS
1	<b>a</b>	-1	2	0	0	0
0	<b>b</b>	<b>c</b>	<b>d</b>	1	0	6
0	-1	3	<b>e</b>	0	1	1

*current  
tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	RHS
1	0	-7	<b>j</b>	<b>k</b>	<b>m</b>	-9
0	<b>g</b>	2	-1	$\frac{1}{2}$	0	<b>f</b>
0	<b>h</b>	<b>i</b>	1	$\frac{1}{2}$	1	4

Find the values of  
"a" through "m"



Solution



**Solution**

$a = 3$

$b = 2$

$c = 4$

$d = -2$

$e = 2$

$f = 3$

$g = 1$

$h = 0$

$i = 5$

$j = 5$

$k = -\frac{3}{2}$

$m = 0$

*original  
tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	RHS
1	<b>a</b>	-1	2	0	0	0
0	<b>b</b>	<b>c</b>	<b>d</b>	1	0	6
0	-1	3	<b>e</b>	0	1	1

*current  
tableau*

$-Z$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	RHS
1	0	-7	<b>j</b>	<b>k</b>	<b>m</b>	-9
0	<b>g</b>	2	-1	$\frac{1}{2}$	0	<b>f</b>
0	<b>h</b>	<b>i</b>	1	$\frac{1}{2}$	1	4

