Revised Simplex Method

an example...

(includes both Phases I & II)

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Minimize $z=3x_1+5x_2+4x_3+7x_4+5x_5+4x_6$

subject to

$$\begin{cases} 2x_1 - x_2 + x_4 + 3x_6 = 10 \\ x_1 + 3x_3 - x_4 + 3x_5 + 2x_6 \geq 12 \\ 4x_2 + 2x_3 + 3x_4 + x_5 \geq 15 \end{cases}$$

and $x_j \geq 0 \quad \forall j = 1, \ldots 6$

Because of the lack of a slack variable in each constraint, we must use ***Phase I*** *to find an* ***initial feasible basis****.*

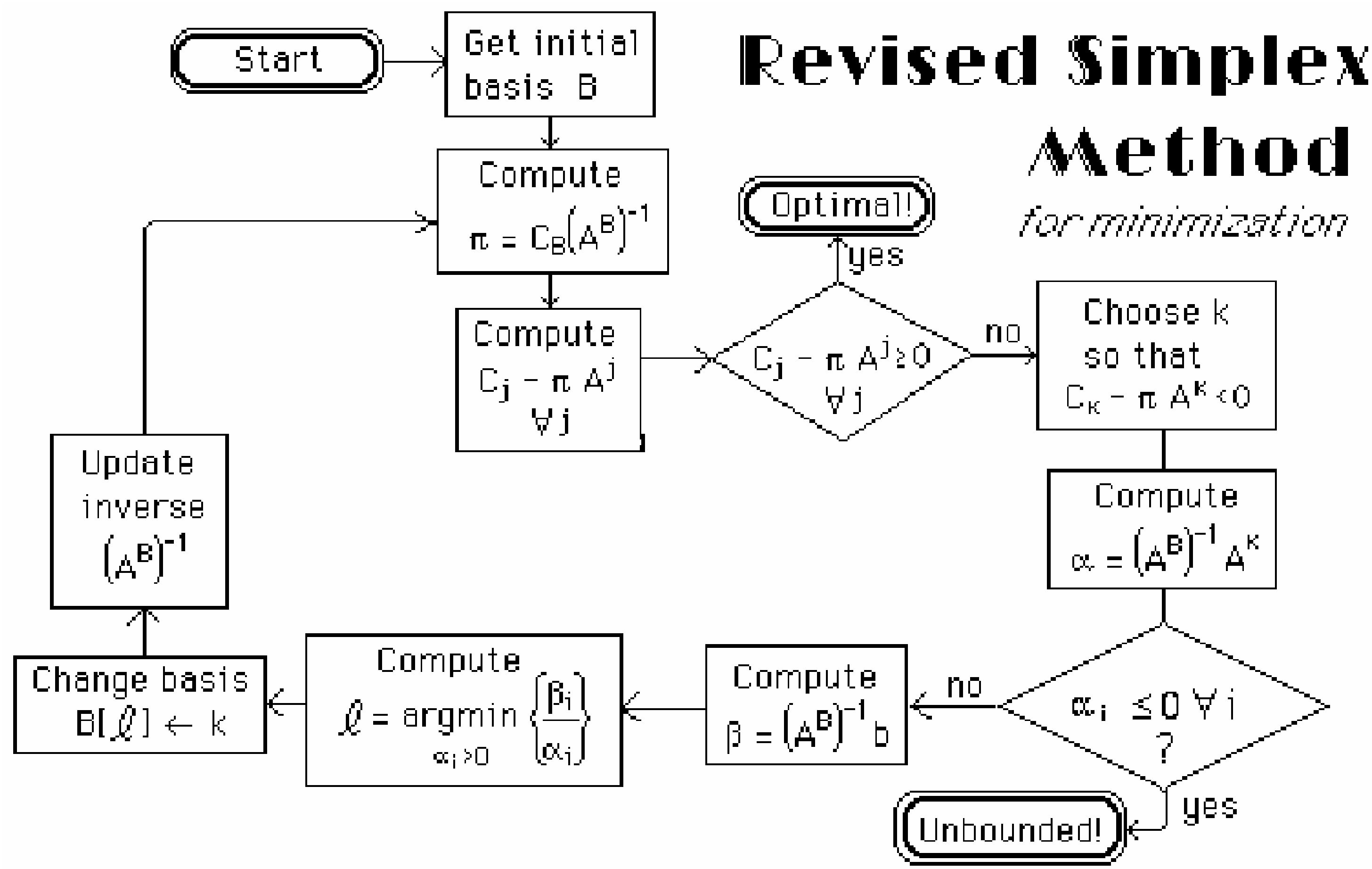
Add variables X_9, X_{10}, X_{11} (artificial variables), and a Phase I ***objective*** *of minimizing the* ***sum*** *of these three variables.*

Phase One

1	2	3	4	5	6	7	8	9	0	1	b	
0	0	0	0	0	0	0	0	1	1	1	0	*phase one objective*
3	5	4	7	5	4	0	0	0	0	0	0	*phase two objective*
2	-1	0	1	0	3	0	0	**1**	0	0	10	
1	0	3	-1	3	2	-1	0	0	**1**	0	12	
0	4	2	3	1	0	0	-1	0	0	**1**	15	

Values of basic (artificial) variables are:

i	Xi
9	10
10	12
11	15



Revised Simplex Method
for minimization
Start
Get initial basis B
Compute π = C_B(A^B)^-1
Compute C_j − π A^j ∀ j
C_j − π A^j ≥ 0 ∀ j
yes
Optimal!
no
Choose k so that C_k − π A^k < 0
Compute α = (A^B)^-1 A^k
α_i ≤ 0 ∀ i ?
yes
Unbounded!
no
Compute β = (A^B)^-1 b
Compute ℓ = argmin_{α_i > 0} {β_i / α_i}
Change basis B[ℓ] ← k
Update inverse (A^B)^-1

Iteration 1

Current partition: *(**B** = basis, **N** = non-basis)*

B= {9 10 11}, N= {1 2 3 4 5 6 7 8}

Basis inverse is

```
1 0 0
0 1 0
0 0 1
```

$$\Rightarrow \pi = c_B\left(A^B\right)^{-1} = [1,1,1]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = [1,1,1]$$

Simplex multipliers (dual solution):

i	π
1	1
2	1
3	1

Coefficient matrix A

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Compute each reduced cost: $c_j - \pi A^j$, e.g. $\underline{c}_1$= 0-[1,1,1]•[2 1 0]=**-3**

Reduced (Artificial) **Costs**

j	Cj	set	
1	-3	N	
2	-3	N	
3	-5	N	← *enter, since* $\underline{C}_j < 0 \Rightarrow$ *decrease in objective!*
4	-3	N	
5	-4	N	
6	-5	N	
7	1	N	
8	1	N	

Entering variable is X[3] from set N

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
9	0	10	0	-	inf	
10	0	12	3	↓	4.0	← *minimum ratio*
11	0	15	2	↓	7.5	

α = substitution rate,
d = direction of change of basic variable,
ratio = rhs/α for all α>0

Change of basis

```
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  Basic variable X[10] leaves basis

  New partition: B={9 3 11}, N={1 2 4 5 6 7 8 10}
```

Updating basis inverse matrix:

affix column of substitution rates alongside the old inverse, and pivot:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \boxed{3} \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & \boxed{1} \\ 0 & -2/3 & 1 & 0 \end{bmatrix}$$

old inverse *new inverse*

Iteration 2

Current partition:

B= {9 3 11}, N= {1 2 4 5 6 7 8 10}

Basis inverse is

```
1  0         0
0  0.333333  0
0 -0.666667  1
```

$$\Rightarrow \pi = c_B \left(A^B\right)^{-1} = [1,0,1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} = [1, -\tfrac{2}{3}, 1]$$

Simplex multipliers (dual solution):

i	π
1	1
2	-0.666667
3	1

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Reduced cost of X_1, for example, is $\underline{C}_1$=

$C_1 - \pi A^1 = 0 - [1, -0.667, 1] \bullet [2, 1, 0] = -2 + 0.667 = -1.333$

Reduced (Artificial) **Costs**

j	Cj	set	
1	-1.33333	N	
2	-3	N	
4	-4.66667	N	←*enter*
5	1	N	
6	-1.66667	N	
7	-0.666667	N	
8	1	N	
10	1.66667	N	

any of variables 1, 4, 6, or 7 would improve the phase I objective

Entering variable is X[4] from set N

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
9	0	10	1.000000	↓	10.00000	
3	0	4	-0.333333	↑	--	
11	0	7	3.666667	↓	1.90909	←*min ratio*

Change of basis

Basic variable X[11] leaves basis

New partition: B= {9 3 **4**}, N= {1 2 5 6 7 8 10 **11**}

Updating basis inverse matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1/3 & 0 & -1/3 \\ 0 & -2/3 & 1 & \boxed{11/3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2/11 & -3/11 & 0 \\ 0 & 3/11 & 1/11 & 0 \\ 0 & -2/11 & 3/11 & 1 \end{bmatrix}$$

old inverse new inverse

$$\pi = c_B \left(A^B\right)^{-1} = [1,0,0]\begin{bmatrix} 1 & 2/11 & -3/11 \\ 0 & 3/11 & 1/11 \\ 0 & -2/11 & 3/11 \end{bmatrix} = \left[1, 2/11, -3/11\right]$$

Iteration 3

Current partition:

B= {9 3 4},   N= {1 2 5 6 7 8 10 11}

Basis inverse is

```
1  0.181818 -0.272727
0  0.272727  0.090909
0 -0.181818  0.272727
```

Simplex multipliers (dual solution):

$$\pi = c_B\left(A^B\right)^{-1} = [1,0,0]\begin{bmatrix} 1 & 2/11 & -3/11 \\ 0 & 3/11 & 1/11 \\ 0 & -2/11 & 3/11 \end{bmatrix} = \left[1, 2/11, -3/11\right]$$

i	π
1	1
2	0.181818
3	-0.272727

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Reduced cost of X1, for example, is

$$\underline{c}_1 = c_1 - \pi A^1 = 0 - \begin{bmatrix}1 & 0.181818 & -0.272727\end{bmatrix} \bullet \begin{bmatrix}2 & 1 & 0\end{bmatrix} = -2.181818$$

Reduced (Artificial) **Costs**

j	Cj	set	
1	-2.18182	N	
2	2.09091	N	
5	-0.272727	N	
6	-3.36364	N	← *enter*
7	0.181818	N	
8	-0.272727	N	
10	0.818182	N	
11	1.27273	N	

any of variables 1, 5, 6, &8 would improve the phase I objective

Entering variable is X[6] from set N

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
9	0	8.09091	3.36363	↓	2.40541	← *min ratio*
3	0	4.63636	0.54545	↓	8.50000	
4	0	1.90909	-0.36363	↑	--	

Change of basis

Basic variable X[9] leaves basis
New partition: B= {6 3 4}, N= {1 2 5 7 8 10 11 9}

All artificial variables (9, 10, 11) are now nonbasic.
*****Feasibility has been achieved! End Phase One*****

Optimal Phase One Solution

i	X[i]	Cz
1	0	0
2	0	0
3	3.32432	0
4	2.78378	0
5	0	0
6	2.40541	0
7	0	0
8	0	0

Phase one objective = sum of artificial variables = 0

Begin Phase II

Phase Two

Iteration 1

Current partition:

B= {6 3 4},   N= {1 2 5 7 8}

Basis inverse is

0.297297	0.0540541	-0.0810811
-0.162162	0.243243	0.135135
0.108108	-0.162162	0.243243

Simplex multipliers (dual solution)

i	π
1	1.2973
2	0.0540541
3	1.91892

2	-1	0	1	0	3	0	0	1	0	0
1	0	3	-1	3	2	-1	0	0	1	0
0	4	2	3	1	0	0	-1	0	0	1

Reduced costs

j	Cj	set	
1	0.351351	N	*only variable 2 would improve the phase II objective*
2	-1.37838	N	←*enter*
5	2.91892	N	
7	0.054054	N	
8	1.91892	N	

Entering variable is X[2] from set N

Determine variable leaving basis (pivot row):

B	L	X	α	d	ratio	
6	0	2.40541	-0.6216	↑	--	
3	0	3.32432	0.7027	↓	4.730	
4	0	2.78378	0.8648	↓	3.218	←*min ratio*

Change of basis

Basic variable X[4] leaves basis
New partition: B= {6 3 2},   N= {1 5 7 8 4}

Optimal Solution

Objective function: Z= 37.9688

Primal Solution

(reduced costs are all nonnegative)

i	L	X[i]	set	reduced cost
1	0	0	N	0.4375
2	0	3.21875	B	0
3	0	1.0625	B	0
4	0	0	N	1.59375
5	0	0	N	2.53125
6	0	4.40625	B	0
7	0	0	N	0.3125
8	0	0	N	1.53125

Dual Solution

i	π
1	1.125
2	0.3125
3	1.53125