Example: Revised Simplex Method

Consider the LP:

Minimize \( z = 3x_1 + 2x_2 + 6x_3 \)

subject to

\[
\begin{align*}
4x_1 + 8x_2 - x_3 & \leq 5 \\
7x_1 - 2x_2 + 2x_3 & \geq 4 \\
x_1 & \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]
By introducing \textit{slack} and \textit{surplus} variables, the problem is rewritten with \textit{equality} constraints as

\[\text{Minimize } cx \text{ subject to } Ax=b, \ x \geq 0\]

where

\[C = [3, 2, 6, 0, 0],\]
\[b = [5, 4] \text{ and }\]
\[A = \begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}.\]
Although $x_4$ (the slack variable in 1st constraint) can be used as a basic variable in the first row, the choice of a basic variable in 2nd constraint is not obvious, requiring solution of a “Phase One” problem with artificial variables introduced.

Suppose that “Phase One” has found the initial basis $B = \{1, 2\}$ for the constraints, i.e., basic variables $x_1$ and $x_2$. 
We begin the first iteration of the revised simplex method (RSM) by computing the **basis inverse matrix**:

\[ B=\{1,2\} \Rightarrow A^B = \begin{bmatrix} 4 & 8 \\ 7 & -2 \end{bmatrix} \Rightarrow (A^B)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix} \]

and the values of the **current basic variables**, 

\[ x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A^B)^{-1} b = \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.65625 \\ 0.296875 \end{bmatrix} \]
Next we compute the **simplex multiplier vector** \( \pi \), to be used in “pricing” the **nonbasic** columns:

\[
\pi = c_B (A^B)^{-1}
\]

\[
= \begin{bmatrix} 3 & 2 \\
0.03125 & 0.125 \\
0.109375 & -0.0625 \\
0.3125 & 0.25
\end{bmatrix}
\]
Use the simplex multiplier vector $\pi$ to compute the reduced cost of the nonbasic variables ($x_3, x_4,$ & $x_5$), starting with $x_3$:

$$\bar{c}_3 = c_3 - \pi A^3 = 0 - [0.3125, 0.25] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5.1875 > 0$$

Since this reduced cost is **positive**, increasing $x_3$ would increase the cost, and so $x_3$ is rejected as a pivot variable.

*(If we had been maximizing rather than minimizing, of course, then increasing $x_3$ would benefit the objective!)*

We now proceed to the next nonbasic variable, $x_4$. 
Use the simplex multiplier vector \( \pi \) to compute the reduced cost of the *nonbasic variable* \( x_4 \):

\[
\bar{c}_4 = c_4 - \pi A^4 = 0 - [0.3125, 0.25] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -0.3125
\]

Increasing \( x_4 \) will improve (i.e. lower) the solution, since its reduced cost is negative!
Rather than continuing to “price” the remaining nonbasic variables (in this case, only $x_5$), we will proceed by entering $x_4$ into the basis!

The vector of substitution rates of $x_4$ for the basic variables is

$$\alpha = (A^B)^{-1}A^i = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.03125 \\ 0.10937 \end{bmatrix}$$
Perform the **minimum ratio test** to determine which variable leaves the basis.

\[
\min \left\{ \frac{x_B}{\alpha_B} : \alpha_B > 0 \right\} = \min \left\{ \frac{0.6562}{0.0312}, \frac{0.29687}{0.10937} \right\}
\]

= \min \{21, 2.7143\} = 2.7143

Since the second ratio is minimum, the second basic variable (i.e., \(x_2\)) is replaced by the entering variable (i.e., \(x_4\)), and

the new basis is \(B = \{1, 4\}\).
Updating the basis inverse matrix with a pivot:

\[
\begin{bmatrix}
(A^B)^{-1} | \alpha
\end{bmatrix} = \begin{bmatrix}
0.03125 & 0.125 & 0.03125 \\
0.10937 & -0.0625 & 0.109375
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
0 & 0.1428 & 0 \\
1 & -0.5714 & 1
\end{bmatrix} \Rightarrow (A^B)^{-1} = \begin{bmatrix}
0 & 0.1428 \\
1 & -0.5714
\end{bmatrix}
\]
Compute, for this new basis,

\[
c_B = [3, 0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, (A^B)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix},
\]

\[
x_B = (A^B)^{-1} b = [0.571429 \quad 2.71429], \pi = c_B (A^B)^{-1} = [0 \quad 0.428571]
\]

The reduced costs of the nonbasic variables \{2, 3, 5\} are now:  \( \overline{c}_2 = 2.85714, \overline{c}_3 = 5.14286, \overline{c}_5 = 0.428571 \)

Since the reduced costs are all nonnegative,

the current solution is **optimal**!