Example: *Revised Simplex Method*

Consider the LP:

$$\text{Minimize } z = 3x_1 + 2x_2 + 6x_3$$

$$\text{subject to} \begin{cases} 4x_1 + 8x_2 - x_3 \le 5 \\ 7x_1 - 2x_2 + 2x_3 \ge 4 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

By introducing *slack* and *surplus* variables,

the problem is rewritten with *equality* constraints as

Minimize cx *subject to* $Ax=b$, $x \geq 0$

where

$C = [3, 2, 6, 0, 0]$,

$b = [5, 4]$ and

$$A = \begin{bmatrix} 4 & 8 & -1 & 1 & 0 \\ 7 & -2 & 2 & 0 & -1 \end{bmatrix}.$$

Although x_4 (the slack variable in 1st constraint) can be used as a basic variable in the first row,

the choice of a basic variable in 2nd constraint is not obvious,

requiring solution of a "Phase One" problem with *artificial* variables introduced.

Suppose that "Phase One" has found the initial basis $B = \{1,2\}$ *for the constraints, i.e., basic variables* x_1 *and* x_2.

We begin the first iteration of the revised simplex method (RSM) by computing the **basis inverse matrix**:

$$\text{B}=\{1,2\} \Rightarrow A^B = \begin{bmatrix} 4 & 8 \\ 7 & -2 \end{bmatrix} \Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix}$$

and the values of the **current basic variables**,

$$\begin{aligned} x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \left(A^B\right)^{-1} b \\ &= \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.65625 \\ 0.296875 \end{bmatrix} \end{aligned}$$

Next we compute the **simplex multiplier vector** π, to be used in "pricing" the *nonbasic* columns:

$$\pi = c_B \left(A^B \right)^{-1}$$

$$= \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 0.03125 & 0.125 \\ 0.109375 & -0.0625 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3125 & 0.25 \end{bmatrix}$$

Use the simplex multiplier vector π to compute the reduced cost of the *nonbasic* variables (x_3,x_4, & x_5), starting with x_3:

$$\overline{c}_3 = c_3 - \pi A^3 = 0 - [0.3125, 0.25]\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 5.1875 > 0$$

Since this reduced cost is *POSITIVE*, increasing x_3 would increase the cost, and so x_3 is rejected as a pivot variable.

(If we had been maximizing rather than minimizing, of course, then increasing x_3 would benefit the objective!)

We now proceed to the next nonbasic variable, x_4.

Use the simplex multiplier vector π to compute the reduced cost of the *nonbasic variable* x_4:

$$\overline{c}_4 = c_4 - \pi A^4 = 0 - [0.3125, 0.25]\begin{bmatrix}1\\0\end{bmatrix} = -0.3125$$

Increasing x_4 will improve (i.e. lower) the solution, since its reduced cost is negative!

Rather than continuing to "price" the remaining nonbasic variables (in this case, only x_5),
we will proceed by entering x_4 into the basis!

The vector of **substitution rates** of x_4 for the basic variables is

$$\alpha = \left(A^B\right)^{-1} A^j = \begin{bmatrix} 0.03125 & 0.125 \\ 0.10937 & -0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.03125 \\ 0.10937 \end{bmatrix}$$

Perform the **minimum ratio test** to determine which variable leaves the basis.

$$\min\left\{\frac{x_B}{\alpha_B} : \alpha_B > 0\right\} = \min\left\{\frac{0.6562}{0.0312}, \frac{0.29687}{0.10937}\right\}$$

= min {21, **2.7143**} = 2.7143

Since the second ratio is minimum, the second basic variable (i.e., x_2) is replaced by the entering variable (i.e., x_4), and
the new basis is B = {1, 4}.

Updating the basis inverse matrix with a pivot:

$$\left[\left(A^B\right)^{-1} \mid \alpha\right] = \begin{bmatrix} 0.03125 & 0.125 & 0.03125 \\ 0.10937 & -0.0625 & \boxed{0.109375} \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0.1428 & 0 \\ 1 & -0.5714 & 1 \end{bmatrix} \Rightarrow \left(A^B\right)^{-1} = \begin{bmatrix} 0 & 0.1428 \\ 1 & -0.5714 \end{bmatrix}$$

Compute, for this new basis,

$$c_B = [3,0], A^B = \begin{bmatrix} 4 & 1 \\ 7 & 0 \end{bmatrix}, \left(A^B\right)^{-1} = \begin{bmatrix} 0 & 0.142857 \\ 1 & -0.571429 \end{bmatrix},$$

$$x_B = \left(A^B\right)^{-1} b = [0.571429 \quad 2.71429], \pi = c_B \left(A^B\right)^{-1} = [0 \quad 0.428571]$$

The *reduced costs* of the nonbasic variables {2, 3, 5} are

now: $\overline{c}_2 = 2.85714, \overline{c}_3 = 5.14286, \overline{c}_5 = 0.428571$

Since the reduced costs are all nonnegative,

the current solution is **optimal**!