Example:

Because of excessive pollution on the Momiss River, the state is going to build some pollution control stations.

Three sites are under consideration.

The state is interested in controlling the pollution levels of two pollutants: #1 & #2

The legislature requires that at least 80,000 lbs. of pollutant #1 and at least 50,000 lbs. of pollutant #2 be removed from the river annually.

author

↩

Data:

annual cost over lifetime of station

Site #	Cost of building station	Cost of treating 1 ton H_2O	Amt. removed per ton of water	
			Pollutant 1	Pollutant 2
1	\$100,000	\$20	0.4 lb.	0.3 lb.
2	\$ 60,000	\$30	0.25 lb.	0.2 lb.
3	\$ 40,000	\$40	0.2 lb.	0.25 lb.

Formulate an ILP to minimize the cost of meeting the state legislature's goals.

Model

Decision Variables

continuous:

X_i = tons of H_2O treated at station #i annually

binary:

$$Y_i = \begin{cases} 1 & \text{if a station is built at site \#i} \\ 0 & \text{otherwise} \end{cases}$$

Objective

$$\text{Minimize } \underbrace{100000Y_1 + 60000Y_2 + 40000Y_3}_{\textit{construction cost of stations}}$$

$$+ \underbrace{20X_1 + 30X_2 + 40X_3}_{\textit{water treatment costs}}$$

Constraints

The required amounts of pollutants are removed:

$$\begin{cases} 0.4\,X_1 + 0.25X_2 + 0.2X_3 \geq 80{,}000 & \textit{pollutant 1} \\ 0.3X_1 + 0.2X_2 + 0.25X_3 \geq 50{,}000 & \textit{pollutant 2} \end{cases}$$

Water cannot be treated at a station unless it has been built:

$$X_i \leq M\,Y_i \qquad \forall\, i$$

where M is a suitably "big" number

(If Y_i =0, this forces X_i =0)

Example:

Glueco produces 3 types of glue on 2 different production lines.

Each line can be utilized by up to 7 workers at a time. Workers on production line #1 are paid $500/week, and workers on line #2 are paid $900/week.

It costs $1000 to set up production line #1 for a week of production, and $2000 for line #2.

↵

Data:

	Weekly output per worker (Barrels)		
Line #	Glue #1	Glue #2	Glue #3
1	20	30	40
2	50	35	45
Weekly Rqmt:	120	150	200

Formulate an ILP to minimize the total cost of meeting weekly requirements.

Model

Decision Variables

continuous:

X_{ij} = man-weeks of time on line i producing glue j

integer:

Y_i = number of persons assigned to line i

binary:

$$Z_i = \begin{cases} 1 & \text{if line i is set up for production} \\ 0 & \text{otherwise} \end{cases}$$

Objective

$$\text{Minimize} \quad \underbrace{1000\, Z_1 + 2000\, Z_2}_{\textit{set-up costs}} + \underbrace{500\, Y_1 + 900\, Y_2}_{\textit{labor costs}}$$

Constraints

Requirements for glue are met:

$$20X_{11} + 50X_{21} \geq 120 \quad \textit{glue \#1}$$

$$30X_{12} + 35X_{22} \geq 150 \quad \textit{glue \#2}$$

$$40X_{13} + 45X_{23} \geq 200 \quad \textit{glue \#3}$$

Capacity of line:

$$X_{11} + X_{12} + X_{13} \leq Y_1 \quad \textit{line \#1}$$

$$X_{21} + X_{22} + X_{23} \leq Y_2 \quad \textit{line \#2}$$

Constraints

We cannot assign workers to a line unless it has been set up:

$$Y_i \leq 7 Z_i \;, i=1,2$$

Example:

The manager of the University's computer system wants to be able to access 5 different files. There are multiple copies of each file, scattered among 10 disks, as shown below:

File #	Disk 1	2	3	4	5	6	7	8	9	10
1	X	X		X	X			X	X	
2	X		X							
3		X			X		X			X
4			X			X		X		
5	X	X		X		X	X		X	X

If disk 3 or disk 5 is used, then disk 2 must be used.

Formulate an ILP to select the smallest set of disks which contain at least once copy of each file.

model

Decision Variables

$$X_i = \begin{cases} 1 & \text{if disk \#i is selected} \\ 0 & \text{otherwise} \end{cases}$$

Objective

Minimize $X_1 + X_2 + \ \dots \ + X_{10}$

Constraints

At least one copy of each file must be found among the selected disks:

$$\begin{array}{llllllllllll}
X_1 & + X_2 & & + X_4 & + X_5 & & & + X_8 & + X_9 & & \geq & 1 \\
X_1 & & + X_3 & & & & & & & & \geq & 1 \\
 & X_2 & & & + X_5 & & + X_7 & & & + X_{10} & \geq & 1 \\
 & & X_3 & & & + X_6 & & + X_8 & & & \geq & 1 \\
X_1 & + X_2 & & + X_4 & & + X_6 & + X_7 & & + X_9 & + X_{10} & \geq & 1
\end{array}$$

Example of a class of problems known as "set-covering problems"

Constraints

If disk 3 or disk 5 is used, then disk 2 must be used.

$$\begin{cases} X_3 \le X_2 \\ X_5 \le X_2 \end{cases}$$

or

$$X_3 + X_5 \le 2X_2$$

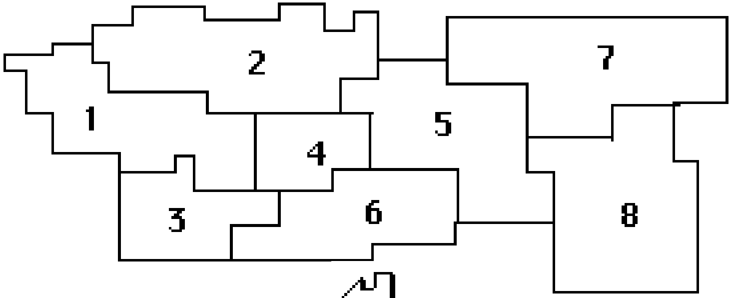
(The pair of constraints is preferred from a computational point of view.)

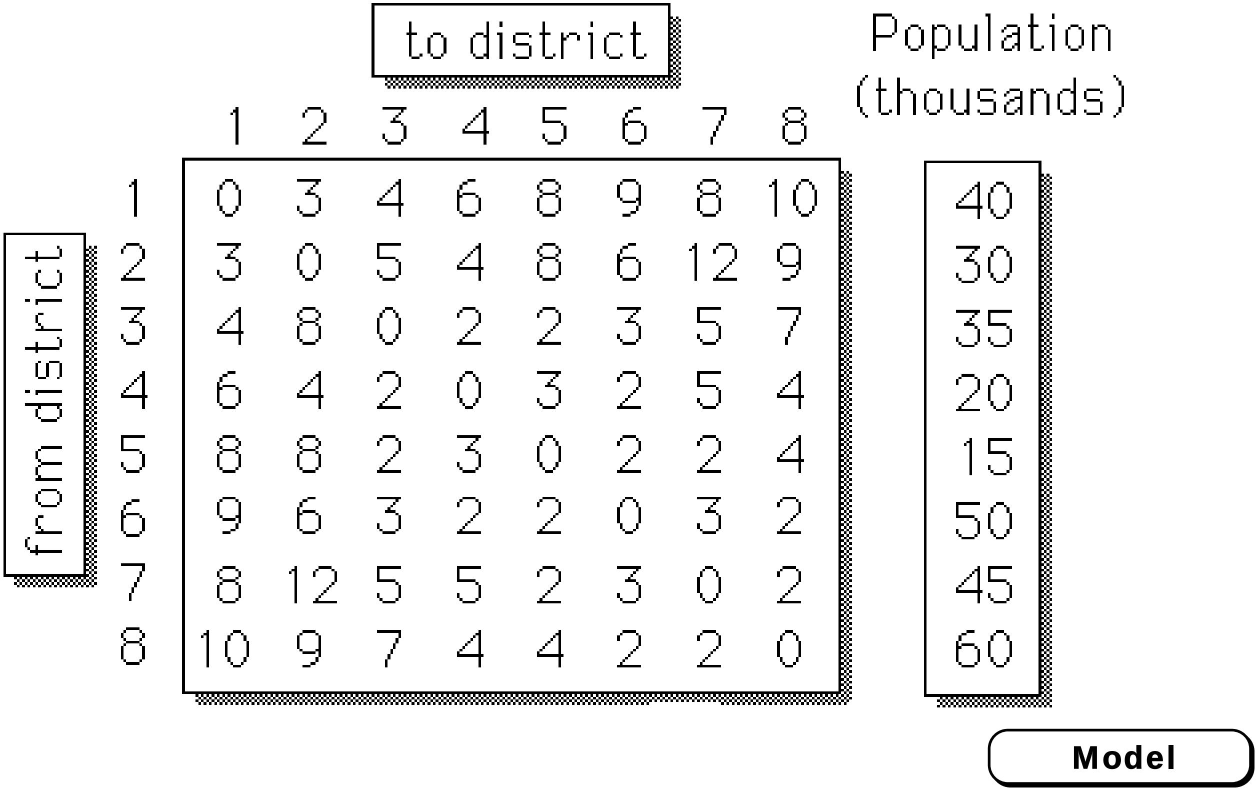
⏎

Example

Gotham City has been divided into 8 districts.

The city wishes to station 2 ambulances so as to maximize the number of people who live within 3 minutes of an ambulance.





to district

from district

	1	2	3	4	5	6	7	8	Population (thousands)
1	0	3	4	6	8	9	8	10	40
2	3	0	5	4	8	6	12	9	30
3	4	8	0	2	2	3	5	7	35
4	6	4	2	0	3	2	5	4	20
5	8	8	2	3	0	2	2	4	15
6	9	6	3	2	2	0	3	2	50
7	8	12	5	5	2	3	0	2	45
8	10	9	7	4	4	2	2	0	60

Decision Variables

The primary decision is whether to locate an ambulance in district i for i=1,2,...8:

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district i} \\ 0 & \text{otherwise} \end{cases}$$

Another set of decisions is whether to assign for service a district j to an ambulance location i

$$X_{ij} = \begin{cases} 1 & \text{if district j is to be served from district i} \\ 0 & \text{otherwise} \end{cases}$$

Objective

We wish to maximize the number of persons within 3 minutes' travel time of an ambulance

For each district j, we need to include in the objective an expression of the form

$$P_j \sum \{X_{ij} : t_{ij} \leq 3\}$$

e.g., for j=3:

$$35(X_{33}+X_{43}+X_{53}+X_{63})$$

since districts 3,4,5,&6 are within 3 minutes' travel time of district 3.

Assumes that not more than one ambulance is to serve district 3!

Objective

$$\text{Maximize } 40(X_{11}+X_{21}) + 30(X_{12}+X_{22})$$
$$+ 35(X_{33}+X_{43}+X_{53}+X_{63}) + 20(X_{34}+X_{44}+X_{54}+X_{64})$$
$$+ 15(X_{35}+X_{45}+X_{55}+X_{65}+X_{75})$$
$$+ 50(X_{36}+X_{46}+X_{56}+X_{66}+X_{76}+X_{86})$$
$$+ 45(X_{57}+X_{67}+X_{77}+X_{87}) + 60(X_{68}+X_{78}+X_{88})$$

Constraints

Every district must be assigned to an ambulance:

$$\sum_{i=1}^{8} X_{ij} = 1 \quad \forall\ j=1,2,\ldots 8$$

e.g.,

$$X_{11}+X_{21}+X_{31}+X_{41}+X_{51}+X_{61}+X_{71}+X_{81} = 1 \quad (j=1)$$

Constraints

District j can be served from district i only if there is an ambulance in district i:

$$X_{ij} \leq Y_i \quad \forall\, i \,\&\, j$$

or

$$\sum_{j=1}^{8} X_{ij} \leq 8Y_i \qquad \text{for } i=1,2,\ldots 8$$

e.g.,

$$X_{11}+X_{12}+X_{13}+X_{14}+X_{15}+X_{16}+X_{17}+X_{18} \leq 8Y_1$$

Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^{8} Y_i = 2,$$

i.e.,

$$Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8=2$$

Model Statistics

\# of binary variables:

Y: 8

X: 64

total: 72

Actually, upon reflection it appears to be clear that the integer restrictions on X could be replaced by upper bounds of 1, without an effect on the optimal solution.

\# of continuous variables: 0

\# of constraints: 17

Another Formulation

Define as before:

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district } i \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_j = \begin{cases} 1 & \text{if district } j \text{ is within 3 minutes of} \\ & \quad \text{an ambulance} \\ 0 & \text{otherwise} \end{cases}$$

Objective

$$\text{Maximize } 40X_1+30X_2+35X_3+20X_4+15X_5+50X_6 +45X_7+60X_8$$

Constraints

X_j must be zero unless at least one ambulance is located within 3 minutes of district j

$X_1 \leq Y_1 + Y_2$

$X_2 \leq Y_1 + Y_2$

$X_3 \leq Y_3 + Y_4 + Y_5 + Y_6$

$\vdots$

$X_8 \leq Y_6 + Y_7 + Y_8$

Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^{8} Y_i = 2,$$

i.e.,

$$Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8=2$$

Model Statistics

\# of binary variables:

Y: 8

X: 8

total: 16

Again, the integer restriction on X could be relaxed without changing the optimal solution

\# of continuous variables: 0

\# of constraints: 9

```
:MAX    40 X1 + 30 X2 + 35 X3 + 20 X4 + 15 X5 + 50 X6 +
45 X7  + 60 X8
 SUBJECT TO
        2)  X1 - Y1 - Y2   <=   0
        3)  X2 - Y1 - Y2   <=   0
        4)   X3 - Y3 - Y4 - Y5 - Y6   <=   0
        5)   X4 - Y3 - Y4 - Y5 - Y6 - Y7   <=   0
        6)   X5 - Y3 - Y4 - Y5 - Y6 - Y7   <=   0
        7)  X6 - Y3 - Y4 - Y5 - Y6 - Y7 - Y8 <=   0
        8)  X7 - Y5 - Y6 - Y7 - Y8   <=   0
        9)   X8 - Y6 - Y7 - Y8   <=   0
       10)   Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 =    2
 END
```

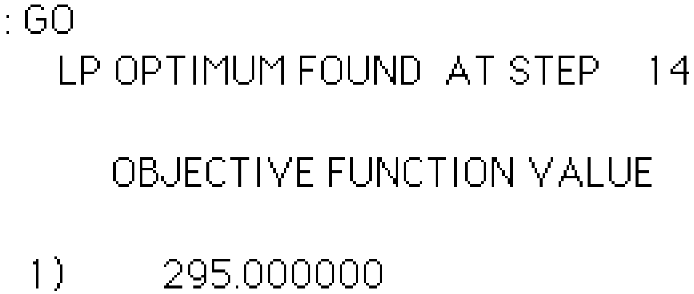
```
INTE    8
```

Use this to indicate that the first 8 variables encountered in the model (starting in row 1, then row 2, etc.) are integer.

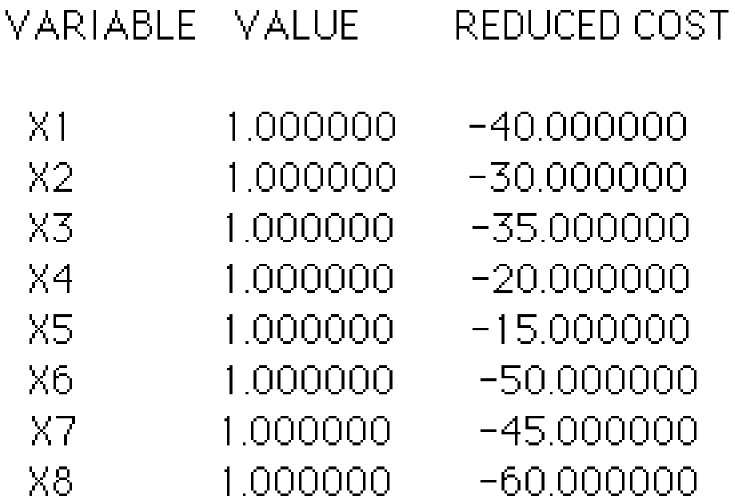
```
INTE     Y1
INTE     Y2
INTE     Y3
INTE     Y4
INTE     Y5
INTE     Y6
INTE     Y7
INTE     Y8
```

Or, you can indicate the integer restriction on each individual variable.

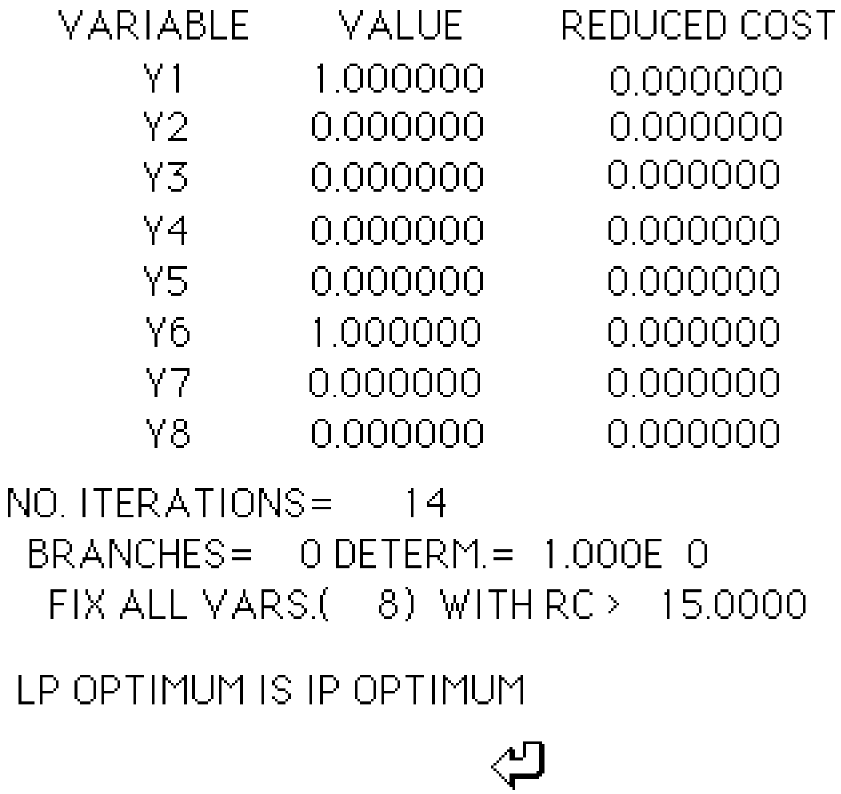
LINDO interprets INTEGER to mean BINARY integer! Otherwise, the command GINT is to be used.



```
: GO
   LP OPTIMUM FOUND  AT STEP    14

       OBJECTIVE FUNCTION VALUE

  1)       295.000000
```



```
VARIABLE   VALUE        REDUCED COST

  X1         1.000000    -40.000000
  X2         1.000000    -30.000000
  X3         1.000000    -35.000000
  X4         1.000000    -20.000000
  X5         1.000000    -15.000000
  X6         1.000000    -50.000000
  X7         1.000000    -45.000000
  X8         1.000000    -60.000000
```



```
        VARIABLE        VALUE          REDUCED COST
              Y1         1.000000          0.000000
              Y2         0.000000          0.000000
              Y3         0.000000          0.000000
              Y4         0.000000          0.000000
              Y5         0.000000          0.000000
              Y6         1.000000          0.000000
              Y7         0.000000          0.000000
              Y8         0.000000          0.000000

 NO. ITERATIONS=      14
  BRANCHES=    0 DETERM.=   1.000E  0
   FIX ALL VARS.(    8)  WITH RC >    15.0000

 LP OPTIMUM IS IP OPTIMUM
```

In this case, the LP solution happens to be integer, so that branch-and-bound is not necessary!