The
DUAL SIMPLEX
Algorithm

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The PRIMAL Simplex Method solves

$$\begin{array}{ll} \text{Minimize} & c^t x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

by starting with a basis B which is feasible in the above (primal) problem, i.e., a basis B for which the basic variables are positive: $x_B = (A^B)^{-1} b \geq 0$

and works toward satisfying the optimality criterion, namely

nonnegative reduced costs: $c - \pi^t A \geq 0$ where $\pi = c_B^t (A^B)^{-1}$

i.e., $\pi^t A \leq c$ or $A^t \pi \leq c^t$ (dual feasibility!)

The DUAL Simplex Method does just the opposite, starting with a basis satisfying the optimality criterion (dual feasibility)

$$c - \pi^t A \geq 0$$

and pivoting to attain primal feasibility!

$$x_B = (A^B)^{-1} b \geq 0$$

That is, at each iteration, the reduced costs in the objective function row of the tableau are nonnegative (if we are minimizing), while the right-hand-side may include some negative values!

EXAMPLE: Minimize $2X_1 + 3X_2 + 4X_3$
subject to

$$\begin{aligned} X_1 + 2X_2 + X_3 &\geq 3 \\ 2X_1 - X_2 + 3X_3 &\geq 4 \\ X_1 \geq 0,\ X_2 \geq 0,\ X_3 &\geq 0 \end{aligned}$$

First, we transform the inequalities to equalities, by defining surplus variables:

Minimize $2X_1 + 3X_2 + 4X_3$
subject to

$$\begin{aligned} X_1 + 2X_2 + X_3 - X_4 &= 3 \\ 2X_1 - X_2 + 3X_3 - X_5 &= 4 \\ X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0,\ X_5 &\geq 0 \end{aligned}$$

First, set up the simplex tableau.

¯Z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	1	2	1	¯1	0	3
0	2	¯1	3	0	¯1	4

Typically, we would need to define artificial variables and proceed with Phase One in order to find an initial feasible basis for the primal problem.

Instead, let's make the surplus variables basic:

−z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	1	2	1	−1	0	3
0	2	−1	3	0	−1	4

pivot here!

↓

−z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	−1	−2	−1	1	0	−3
0	−2	1	−3	0	1	−4

note that this solution is not feasible!

Even though this basic solution is not feasible, notice that the reduced costs in the objective row are all nonnegative!

¯Z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	¯1	¯2	¯1	1	0	¯3
0	¯2	1	¯3	0	1	¯4

So this basic solution is "dual feasible" !

This worked because the costs C are nonnegative. In other situations, we may have already solved an LP, and then need to solve

$$\begin{array}{ll} \text{Minimize} & Cx \\ \text{subject to} & Ax = \tilde{b} \neq b \\ & x \geq 0 \end{array}$$

How can we pivot so as to improve upon the feasibility in the primal constraints?

−z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	−1	−2	−1	1	0	−3
0	−2	1	−3	0	1	−4

Notice that if we pivot in a row with a NEGATIVE right-hand-side, and a negative element in that column for the pivot element, this row will have a POSITIVE right-hand-side in the resulting tableau!

−z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	¯1	¯2	¯1	1	0	¯3
0	¯2	1	¯3	0	1	¯4 ⇐
	⇧	⇧	⇧			

Let's arbitrarily select the bottom row for our pivot

Select a pivot location in either columns 1, 2, or 3, and notice the result!

PIVOT IN COLUMN 1

By pivoting on a negative element, we have improved upon primal feasibility (by making the right-hand-side positive):

¯Z	1	2	3	4	5	B
1	0	4	1	0	1	¯4
0	0	¯2.5	0.5	1	¯0.5	¯1
0	1	¯0.5	1.5	0	¯0.5	2

Also, note that we still have dual feasibility: the reduced costs in the objective row have remained nonnegative!

If you have not already done so, go back & try the other pivots to see what happens.

PIVOT IN COLUMN 2

When we pivot on a positive element, the right-hand-side remains negative:

−z	1	2	3	4	5	B
1	8	0	13	0	¯3	12
0	¯5	0	¯7	1	2	¯11
0	¯2	1	¯3	0	1	¯4

One of the reduced costs has become negative also, so in this case we have lost dual feasibility and made no gain in primal feasibility!

PIVOT IN COLUMN 3

Notice that by pivoting on a negative element, we have improved upon primal feasibility:

−Z	1	2	3	4	5	B
1	¯0.6667	4.333	0	0	1.333	¯5.333
0	¯0.3333	¯2.333	0	1	¯0.3333	¯1.667
0	0.6667	¯0.3333	1	0	¯0.3333	1.333

Unfortunately, we have lost dual feasibility: one of the reduced costs in the objective row has become negative!

OBSERVATIONS

Obviously, we need to pivot on a negative element in a row with a negative right-hand-side in order to improve on primal feasibility.

But we have seen that not just any negative pivot element will do... pivoting on some negative elements will result in a loss of dual feasibility, i.e., a negative reduced cost!

How do we select the proper pivot column?

Selection of Pivot Column in the Dual Simplex Method

To improve primal feasibility, while maintaining dual feasibility, perform a minimum ratio test:

Pivot in the column which minimizes the ratio

$$\frac{\bar{c}_j}{\left|\bar{a}_r^j\right|}$$ *for all negative candidate pivot elements* $\bar{a}_r^j$

⇨ **Derivation**

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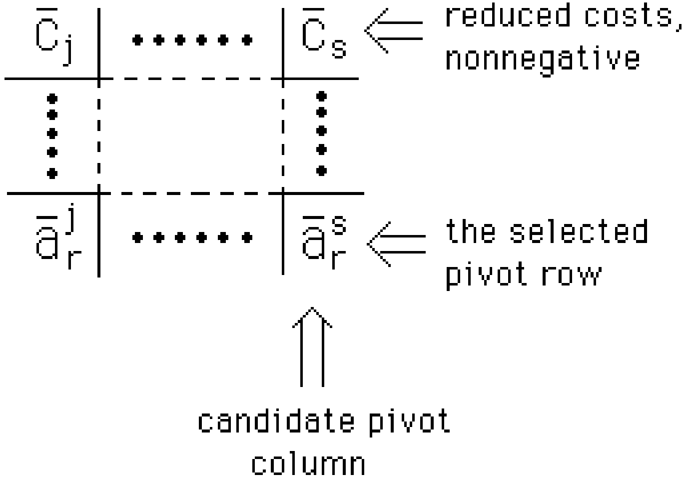
DERIVATION

Selection of Pivot Column in the Dual Simplex Method



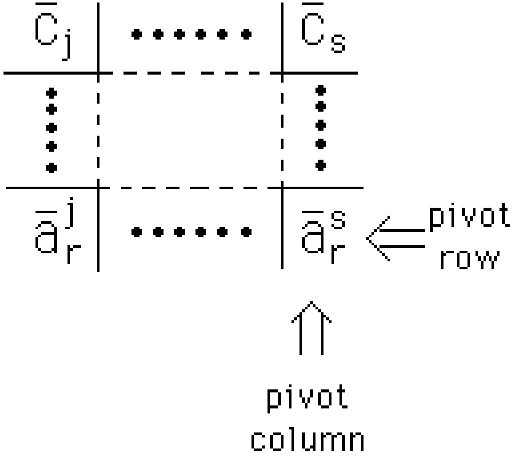
reduced costs, nonnegative

the selected pivot row

candidate pivot column

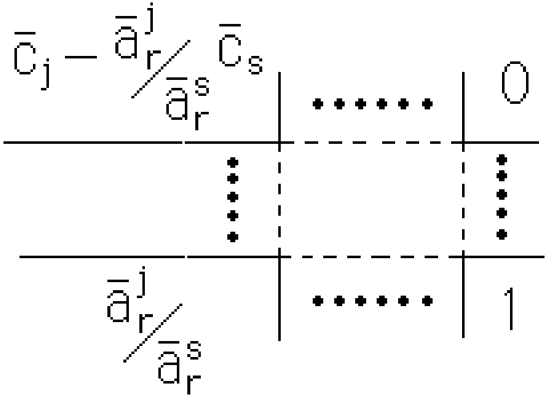
Suppose that we have already selected row ***r*** *for the pivot...*
If we choose column ***s*** *for the pivot, under what conditions will the reduced costs remain nonnegative?*

before the pivot



$$\begin{array}{c|c|c} \bar{c}_j & \cdots\cdots & \bar{c}_s \\ \hline \vdots & & \vdots \\ \hline \bar{a}_r^j & \cdots\cdots & \bar{a}_r^s \end{array}$$

after the pivot



$$\begin{array}{c|c|c} \bar{c}_j - \bar{a}_r^j / \bar{a}_r^s \, \bar{c}_s & \cdots\cdots & 0 \\ \hline \vdots & & \vdots \\ \hline \bar{a}_r^j / \bar{a}_r^s & \cdots\cdots & 1 \end{array}$$

Under what conditions will the reduced cost be non-negative?

In order to maintain dual feasibility, i.e., nonnegative reduced costs, we must choose the pivot column s such that

$\bar{c}_j - \bar{a}_r^j / \bar{a}_r^s \, \bar{c}_s$ *is nonnegative for all columns j!*

that is,

$$\bar{c}_j - \bar{a}_r^j / \bar{a}_r^s \, \bar{c}_s \geq 0 \Rightarrow \quad \bar{c}_j \geq \bar{a}_r^j / \bar{a}_r^s \, \bar{c}_s \quad \text{for all } j$$

Our pivot column (s) must satisfy, for all j,

$$\bar{c}_j \geq \frac{\bar{a}_r^j}{\bar{a}_r^s}\bar{c}_s \Rightarrow \begin{cases} \dfrac{\bar{c}_j}{\bar{a}_r^j} \geq \dfrac{\bar{c}_s}{\bar{a}_r^s} & \text{if} \quad \bar{a}_r^j > 0 \\ \dfrac{\bar{c}_j}{\bar{a}_r^j} \leq \dfrac{\bar{c}_s}{\bar{a}_r^s} & \text{if} \quad \bar{a}_r^j < 0 \end{cases}$$

Recall that

$$\left.\begin{array}{l} \bar{c}_j \text{ \& } \bar{c}_s \text{ are nonnegative} \\ \bar{a}_r^s \text{ is negative} \end{array}\right\} \Rightarrow \frac{\bar{c}_s}{\bar{a}_r^s} \leq 0$$

Consider a column j whose pivot row element $\bar{a}_r^j$ *is positive.*

Then, since the reduced cost in that column, $\bar{c}_j$ *, is nonnegative, the ratio* $\bar{c}_j / \bar{a}_r^j$ *is nonnegative, i.e.,*

$$\bar{c}_j / \bar{a}_r^j \geq 0 \geq \bar{c}_s / \bar{a}_r^s$$ *, so that*

$$\bar{c}_j / \bar{a}_r^j \geq \bar{c}_s / \bar{a}_r^s \quad \text{if} \quad \bar{a}_r^j > 0$$

Therefore, we need concern ourselves with columns j such that $\bar{a}_r^j < 0$ *!*

If the pivot-row element of column j is negative,
i.e., $\bar{a}_r^j < 0$

then since the reduced cost of column j is nonnegative,
i.e., $\bar{c}_j \geq 0$

clearly $\bar{c}_j / \bar{a}_r^j \leq 0$ *and we must choose our pivot*

column s such that $\bar{c}_j / \bar{a}_r^j \leq \bar{c}_s / \bar{a}_r^s$

How do we choose the pivot column s so as to satisfy

$$\bar{c}_j / \bar{a}_r^j \leq \bar{c}_s / \bar{a}_r^s \quad \text{for all } j \text{ such that } \bar{a}_r^j < 0 \ ?$$

Since column s is among those columns with negative values in the pivot row, it is clear that we must the column whose ratio $\bar{c}_j / \bar{a}_r^j$ *is largest, i.e.*

$$\bar{c}_s / \bar{a}_r^s = \text{maximum} \left\{ \bar{c}_j / \bar{a}_r^j : \bar{a}_r^j < 0 \right\}$$

Our criterion for selecting the pivot column,

$$\bar{c}_s / \bar{a}_r^s = \text{maximum} \left\{ \bar{c}_j / \bar{a}_r^j : \bar{a}_r^j < 0 \right\}$$

is sometimes expressed in an equivalent way as a "minimum ratio test" :

> *Pivot in the column which minimizes the ratio*
>
> $\dfrac{\bar{c}_j}{|\bar{a}_r^j|}$ *for all negative candidate pivot elements* $\bar{a}_r^j$

Let's continue with our example, applying the pivot selection rule which we've just derived:

−Z	1	2	3	4	5	B
1	0	4	1	0	1	−4
0	0	−2.5	0.5	1	−0.5	−1
0	1	−0.5	1.5	0	−0.5	2

Result of first pivot (row 3, col. 1)

Choosing the second row as the pivot row, and applying the minimum ratio test, we obtain

$$\text{minimum}\left(\frac{4}{2.5},\ \frac{1}{0.5}\right)=\frac{4}{2.5}$$

So we pivot on the element -2.5

This results in a tableau which is both primal and dual feasible, and therefore optimal:

¯Z	1	2	3	4	5	B
1	0	0	1.8	1.6	0.2	¯5.6
0	0	1	¯0.2	¯0.4	0.2	0.4
0	1	0	1.4	¯0.2	¯0.4	2.2

The optimal solution, therefore, is:

$Z = 5.6$, $X_1 = 2.2$, $X_2 = 0.4$, $X_3 = X_4 = X_5 = 0$

Summary of Dual Simplex Algorithm

- Start with a dual feasible basis, i.e., a solution which satisfies the optimality criterion (nonnegative reduced costs, if minimizing)
- Select a pivot row from among those having infeasibility, i.e., a negative value on the right-hand-side. If none, STOP.
  (A common "rule-of-thumb" is to choose the most negative RHS.)
- From among the columns having a negative value in the pivot row, select that which yields the minimum ratio of reduced cost to (absolute value of) candidate pivot element, i.e.,

$$\underset{\bar{a}_r^j < 0}{\text{minimum}} \left\{ \frac{\bar{c}_j}{\left|\bar{a}_r^j\right|} \right\}$$

reduced cost in current tableau

candidate pivot element in tableau

Compare the sequence of dual simplex pivots with the ordinary, primal simplex pivots applied to the tableau of the DUAL problem:

Maximize $3Y_1 + 4Y_2$
subject to

$$\begin{aligned} Y_1 + 2Y_2 &\leq 2 \\ 2Y_1 - Y_2 &\leq 3 \\ Y_1 + 3Y_2 &\leq 4 \\ Y_1 \geq 0,\ Y_2 &\geq 0 \end{aligned} \quad \Longrightarrow$$

−Z	1	2	3	4	5	B
1	3	4	0	0	0	0
0	1	2	1	0	0	2
0	2	−1	0	1	0	3
0	1	3	0	0	1	4

Primal Simplex Performed in Dual Tableau

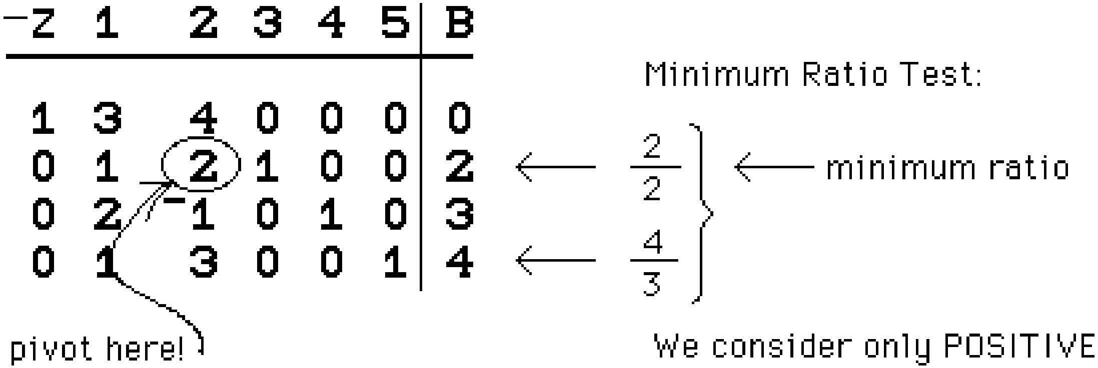
We see that the slack variables (columns 3,4,&5) can be used as an initial basis, so that (as before, when using the dual simplex method) artificial variables are not necessary.

$$\begin{array}{cccccc|c}
-Z & 1 & 2 & 3 & 4 & 5 & B \\
\hline
1 & \textcircled{3} & \textcircled{4} & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 2 \\
0 & 2 & -1 & 0 & 1 & 0 & 3 \\
0 & 1 & 3 & 0 & 0 & 1 & 4
\end{array}$$

Since we are MAXimizing, the solution is improved by entering either column 1 or 2 (with "relative profits" 3 & 4, respectively).

Let's choose to pivot in column 2 (which has the greatest relative profit).

In which row do we pivot?



-Z	1	2	3	4	5	B
1	3	4	0	0	0	0
0	1	2	1	0	0	2
0	2	-1	0	1	0	3
0	1	3	0	0	1	4

pivot here!

Minimum Ratio Test:

$\frac{2}{2}$ ← minimum ratio

$\frac{4}{3}$

We consider only POSITIVE elements as potential pivot elements!

The resulting tableau is:

$$\begin{array}{ccccccc|c}
{}^{-}Z & 1 & 2 & 3 & 4 & 5 & & B \\
\hline
1 & 1 & 0 & {}^{-}2 & 0 & 0 & & {}^{-}4 \\
0 & 0.5 & 1 & 0.5 & 0 & 0 & & 1 \\
0 & \boxed{2.5} & 0 & 0.5 & 1 & 0 & & 4 \\
0 & {}^{-}0.5 & 0 & {}^{-}1.5 & 0 & 1 & & 1
\end{array}$$

pivot here!

There are two candidate pivot rows:

Minimum ratio test:

$\longleftarrow 1/0.5$

$\longleftarrow 4/2.5$ ← *minimum ratio*

There is only one positive relative profit in this tableau, so we must pivot in column #1 , and the minimum ratio test selects the third row.

The tableau which results from this pivot satisfies the optimality conditions

$$\begin{array}{cccccc|c}
 & & & \pi_1 & \pi_2 & \pi_3 & \\
-Z & 1 & 2 & 3 & 4 & 5 & B \\ \hline
1 & 0 & 0 & -2.2 & -0.4 & 0 & -5.6 \\
0 & 0 & 1 & 0.4 & -0.2 & 0 & 0.2 \\
0 & 1 & 0 & 0.2 & 0.4 & 0 & 1.6 \\
0 & 0 & 0 & -1.4 & 0.2 & 1 & 1.8
\end{array}$$

Optimal solution:

$z = 5.6$

$Y_1 = 1.6$

$Y_2 = 0.2$

The relative profits of the slack variables give us the optimal simplex multipliers, since these relative profits are: $c^t - \pi A$ *or, in this case,* $[0\ \ 0\ \ 0] - \pi\,\mathbf{I} = -\pi$

$\Rightarrow\ \pi_1 = 2.2,\ \pi_2 = 0.4,\ \pi_3 = 0$ *This is the solution of the original problem!*

Dual Simplex pivots in primal tableau

-Z	1	2	3	4	5	B
1	2	3	4	0	0	0
0	¯1	¯2	¯1	1	0	¯3
0	¯2	1	¯3	0	1	¯4

-Z	1	2	3	4	5	B
1	0	4	1	0	1	¯4
0	0	¯2.5	0.5	1	¯0.5	¯1
0	1	¯0.5	1.5	0	¯0.5	2

-Z	1	2	3	4	5	B
1	0	0	1.8	1.6	0.2	¯5.6
0	0	1	¯0.2	¯0.4	0.2	0.4
0	1	0	1.4	¯0.2	¯0.4	2.2

Compare!

Primal Simplex pivots in dual tableau

-Z	1	2	3	4	5	B
1	3	4	0	0	0	0
0	1	2	1	0	0	2
0	2	¯1	0	1	0	3
0	1	3	0	0	1	4

-Z	1	2	3	4	5	B
1	1	0	¯2	0	0	¯4
0	0.5	1	0.5	0	0	1
0	2.5	0	0.5	1	0	4
0	¯0.5	0	¯1.5	0	1	1

-Z	1	2	3	4	5	B
1	0	0	¯2.2	¯0.4	0	¯5.6
0	0	1	0.4	¯0.2	0	0.2
0	1	0	0.2	0.4	0	1.6
0	0	0	¯1.4	0.2	1	1.8

While the two procedures are equivalent (dual simplex pivots in primal tableau, and primal simplex pivots in the dual tableau), in practice the effort may not be at all the same:

If n >> m, i.e., there are many more variables than constraints, the primal tableau will be much smaller than the dual tableau, each including the slack & surplus variables.

The Dual Simplex Method is especially useful when re-optimizing a tableau after changing the right-hand-side.

- For example, in a production planning problem which is solved each week, but with different demands &/or available resources (but same costs & profits).
- During parametric programming, in which the right-hand-side is varied to study the dependence of the optimal solution upon the right-hand-side

In these situations, the tableau after the right-hand-side is changed remains dual feasible!

Another example:

Minimize $2X_1 + 3X_2 + 5X_3 + 6X_4$

subject to

$$X_1 + 2X_2 + 3X_3 + X_4 \geq 2$$

$$-2X_1 + X_2 - X_3 + 3X_4 \leq -3$$

$$X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0,\ X_4 \geq 0$$

The initial tableau:

−Z	1	2	3	4	5	6	B
1	2	3	5	6	0	0	0
0	1	2	3	1	−1	0	2
0	−2	1	−1	3	0	1	−3

If we were to use the PRIMAL simplex method, we would have to introduce artificial variables and proceed with Phase One.

Instead, let's make columns 5 & 6 basic:

$-z$	1	2	3	4	5	6	B
1	2	3	5	6	0	0	0
0	1	2	3	1	$\boxed{-1}$	0	2
0	-2	1	-1	3	0	$\boxed{1}$	-3

The tableau which results is not feasible (since X_5 *and* X_6 *are both negative),*

−z	1	2	3	4	5	6	B
1	2	3	5	6	0	0	0
0	¯1	¯2	¯3	¯1	1	0	¯2
0	¯2	1	¯1	3	0	1	¯3

but the reduced costs are all nonnegative, so that (since we are MINimizing) the optimality conditions are satisfied.

This is a tableau then, with an initial dual feasible basis, so that we can apply the DUAL simplex method.

Let's choose the bottom row (with the greatest primal infeasibility) for the pivot row.

-Z	1	2	3	4	5	6	B
1	2	3	5	6	0	0	0
0	-1	-2	-3	-1	1	0	-2
0	-2	1	-1	3	0	1	-3

There are two candidate pivot columns. Use the dual minimum ratio test to make the selection.

candidates for pivot column

$$\begin{array}{ccccccc|c} & \downarrow & & \downarrow & & & & \\ -z & 1 & 2 & 3 & 4 & 5 & 6 & B \\ \hline 1 & 2 & 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 & -1 & 1 & 0 & -2 \\ 0 & \boxed{-2} & 1 & -1 & 3 & 0 & 1 & -3 \\ & \downarrow & & \downarrow & & & & \end{array} \longleftarrow \textit{pivot row}$$

$$\textit{minimum} \left\{ \frac{2}{2}, \quad \frac{5}{1} \right\} = \frac{2}{2}$$

So we pivot in column #1

We have gained some primal feasibility (the right-hand-side of the pivot row is now positive), and maintained dual feasibility (the reduced costs in the objective row remain nonnegative):

−z	1	2	3	4	5	6	B
1	0	4	4	9	0	1	¯3
0	0	¯2.5	¯2.5	¯2.5	1	¯0.5	¯0.5
0	1	¯0.5	0.5	¯1.5	0	¯0.5	1.5

Next we choose the second row (whose basic variable is negative) as the pivot row.

candidates for pivot column

$$\begin{array}{ccccccc|cl} & & \downarrow & \downarrow & \downarrow & & \downarrow & & \\ -z & 1 & 2 & 3 & 4 & 5 & 6 & B & \\ \hline 1 & 0 & 4 & 4 & 9 & 0 & 1 & {}^{-}3 & \\ 0 & 0 & {}^{-}2.5 & {}^{-}2.5 & {}^{-}2.5 & 1 & {}^{-}0.5 & {}^{-}0.5 & \leftarrow \text{pivot row} \\ 0 & 1 & {}^{-}0.5 & 0.5 & {}^{-}1.5 & 0 & {}^{-}0.5 & 1.5 & \end{array}$$

There are FOUR candidates for the pivot column

The minimum ratio test produces a tie in the choice of pivot column:

$$\text{minimum}\left\{\frac{4}{2.5}\quad\frac{4}{2.5}\quad\frac{9}{2.5}\quad\frac{1}{0.5}\right\} = \frac{4}{2.5}$$

Depending upon how the tie is broken, we get different solutions:

$-z$	1	2	3	4	5	6	B
1	0	4	4	9	0	1	-3
0	0	$\boxed{-2.5}$	$\boxed{-2.5}$	-2.5	1	-0.5	-0.5
0	1	-0.5	0.5	-1.5	0	-0.5	1.5

Select one of the two possible pivot elements before proceeding!

Pivoting in row 2, column 2, yields the following optimal tableau,

¯Z	1	2	3	4	5	6	B
1	0	0	0	5	1.6	0.2	¯3.8
0	0	1	1	1	¯0.4	0.2	0.2
0	1	0	1	¯1	¯0.2	¯0.4	1.6

with the optimal solution

$$Z = 3.8,\ X_1 = 1.6,\ X_2 = 0.2,\ X_3=X_4=X_5=X_6=0$$

Pivoting in row 2, column 3 yields the following optimal tableau:

¯z	1	2	3	4	5	6	B
1	0	0	0	5	1.6	0.2	¯3.8
0	0	1	1	1	¯0.4	0.2	0.2
0	1	¯1	0	¯2	0.2	¯0.6	1.4

with the optimal solution

$Z = 3.8$, $X_1 = 1.4$, $X_3 = 0.2$, $X_2 = X_4 = X_5 = X_6 = 0$

Two alternate optimal primal solutions:

$Z = 3.8,\ X_1 = 1.6,\ X_2 = 0.2,\ X_3 = X_4 = X_5 = X_6 = 0$

$Z = 3.8,\ X_1 = 1.4,\ X_3 = 0.2,\ X_2 = X_4 = X_5 = X_6 = 0$

The corresponding dual tableaus are *degenerate*, with at least one zero appearing on the RHS!

Path followed by Dual Simplex Method

Consider the following primal/dual pair of LPs:

Primal	*Dual*
Minimize $20x_1 + 10x_2$	Maximize $6y_1 + 8y_2$
subject to:	subject to:
$5x_1 + x_2 \geq 6$	$5y_1 + 2y_2 \leq 20$
$2x_1 + 2x_2 \geq 8$	$y_1 + 2y_2 \leq 10$
$x_1 \geq 0,\ x_2 \geq 0$	$y_1 \geq 0,\ y_2 \geq 0$

The primal system has $\binom{4}{2}$ = *6 basic solutions, of which 3 are feasible:*

Primal

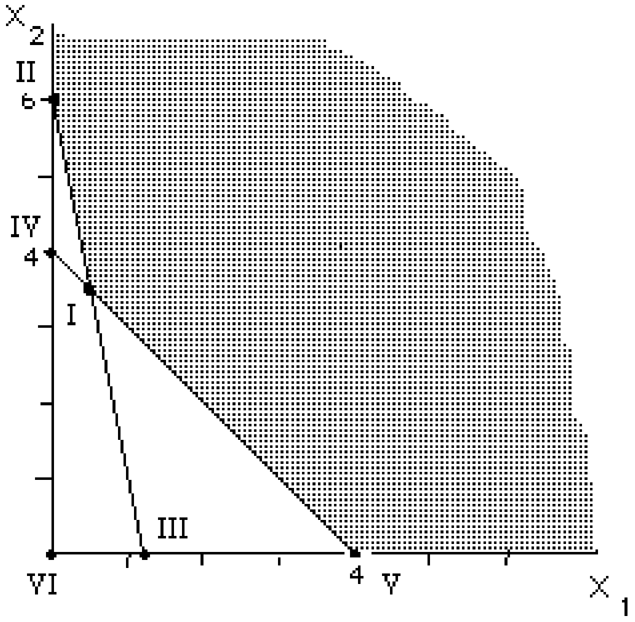
Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0,\ x_2 \geq 0$$



The dual system also has six basic solutions, 4 of them feasible:

Dual

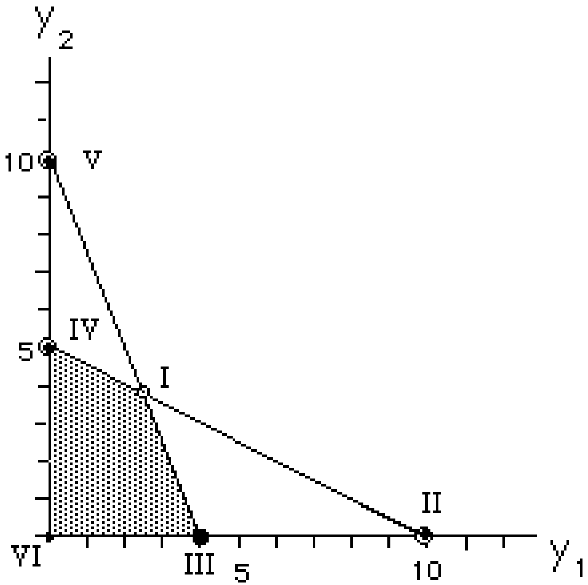
Maximize $6y_1 + 8y_2$

subject to:

$$5y_1 + 2y_2 \leq 20$$

$$y_1 + 2y_2 \leq 10$$

$$y_1 \geq 0, \quad y_2 \geq 0$$



Let's apply the Dual Simplex Algorithm to the primal problem:

Minimize $20x_1 + 10x_2$

subject to:

$$5x_1 + x_2 \geq 6$$

$$2x_1 + 2x_2 \geq 8$$

$$x_1 \geq 0,\ x_2 \geq 0$$

-Z	1	2	3	4	B
1	20	10	0	0	0
0	5	1	-1	0	6
0	2	2	0	-1	8

If we start with the surplus variables in the basis, we will have a dual feasible tableau:

Sequence of Dual Simplex Pivots:

¯Z	1	2	3	4	B
1	20	10	0	0	0
0	5	1	¯1	0	6
0	2	2	0	¯1	8

↓

¯Z	1	2	3	4	B
1	20	10	0	0	0
0	¯5	¯1	1	0	¯6
0	¯2	¯2	0	1	¯8

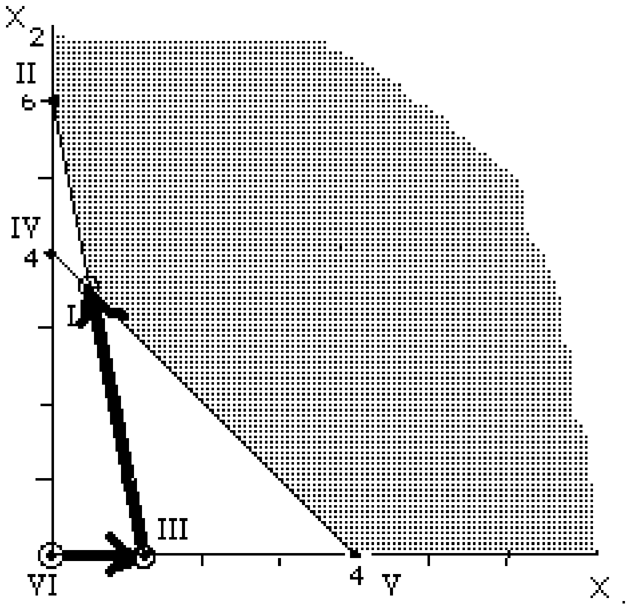
↗

¯Z	1	2	3	4	B
1	0	6	4	0	¯24
0	1	0.2	¯0.2	0	1.2
0	0	¯1.6	¯0.4	1	¯5.6

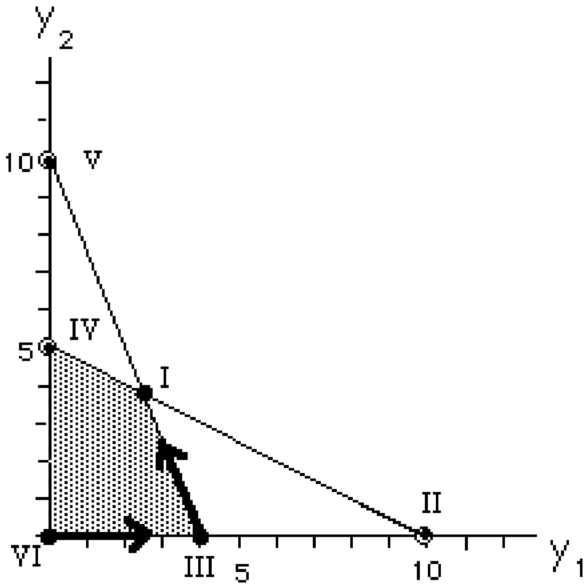
↓

¯Z	1	2	3	4	B
1	0	0	2.5	3.75	¯45
0	1	0	¯0.25	0.125	0.5
0	0	1	0.25	¯0.625	3.5

The path followed by the Dual Simplex Method is (in $X_1 X_2$-space):



In the dual space, the path followed by the dual simplex algorithm is:



	PRIMAL							DUAL				
extreme pt. #	X_1	X_2	X_3	X_4	feasible?	obj.	feasible?	Y_1	Y_2	Y_3	Y_4	
VI	4	0	14	0	✓	80		0	10	0	-10	
II	0	6	0	4	✓	60		10	0	-30	0	
I	.5	3.5	0	0	✓	45	✓	2.5	3.75	0	0	⟵ optimal
IV	0	4	-2	0		40	✓	0	5	10	0	
III	1.2	0	0	-5		24	✓	4	0	0	6	6
VI	0	0	-6	-8		0	✓	0	0	20	10	