Markov Chains

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Discrete Time Markov Chain

X_n = state of system at time n

Consider a system with a **finite** set of **states**: $\{s_1, s_2, \ldots s_N\}$

The system is observed at a certain sequence of points in time, or **stages**.

The system may make a **transition** from one state to another between observations, according to known probability distributions.

Transition Probabilities

$$P_{ij}^{n-1,n} = \mathbf{P}\{X_n = \mathbf{j} \mid X_{n-1} = \mathbf{i}\}$$

If the Markov chain is ***stationary***, then the transition probabilities are the same at every stage, i.e.,

$$\mathbf{p}_{ij}^{n-1,n} = \mathbf{p}_{ij} = \mathbf{P}\{X_n = \mathbf{j} \mid X_{n-1} = \mathbf{i}\}$$

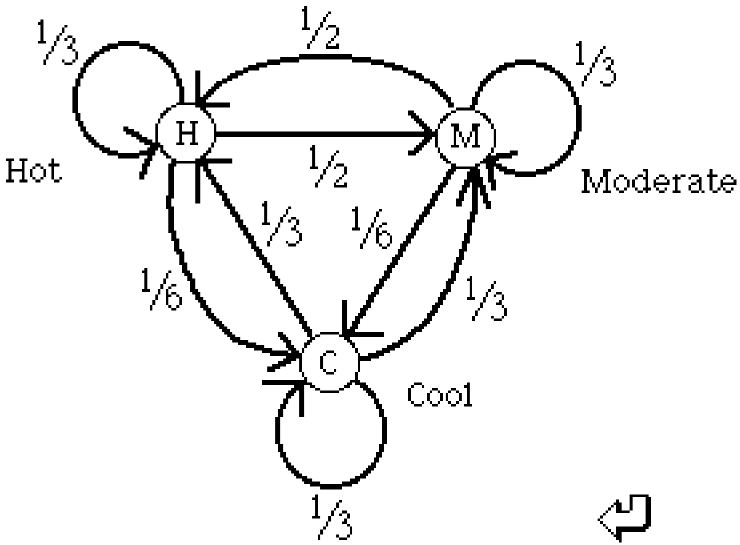
> *Note that the state at stage n+1 may depend ONLY on the state in the immediately preceding stage, n, and NOT on any earlier history of the system.*

Notation

- p_{ij} transition probability
- $p_{ij}^{(n)}$ n-stage transition probability
- π_j steady-state probability
- $f_{ij}^{(n)}$ first-passage probability
- N_{ij} first-passage time
- m_{ij} mean first-passage time

Example: Summer Weather

-- a model for use by a utility company in planning day-to-day repairs & maintenance



$$P = \begin{array}{c|ccc} & H & M & C \\ \hline H & \tfrac{1}{3} & \tfrac{1}{2} & \tfrac{1}{6} \\ M & \tfrac{1}{2} & \tfrac{1}{3} & \tfrac{1}{6} \\ C & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \end{array}$$

stage= 1 day

Suppose that on Wednesday the weather is HOT.

What is the probability that on Saturday the weather is either MODERATE or COOL?

Row 1 of the matrix P gives the probability distribution of the weather on Thursday:

$$P = \begin{array}{c} \\ H \\ M \\ C \end{array} \begin{array}{c} \begin{array}{ccc} H & M & C \end{array} \\ \begin{bmatrix} 1/3 & 1/2 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

What is the distribution of the weather on Friday?

Compute the probability that it is Cool on Friday by conditioning on Thursday's weather:

$$P\{\text{Cool on Friday} \mid \text{Hot on Wednesday}\}$$

$$= P\{\text{Cool Friday} \mid \text{Hot Thursday}\} \times P\{\text{Hot Thursday} \mid \text{Hot Wed.}\}$$

$$+ P\{\text{Cool Friday} \mid \text{Moderate Thurs}\} \times P\{\text{Moderate Thurs} \mid \text{Hot Wed.}\}$$

$$+ P\{\text{Cool Friday} \mid \text{Cool Thursday}\} \times P\{\text{Cool Thursday} \mid \text{Hot Wed.}\}$$

$$= p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33}$$

$$P\{\text{Cool on Friday} \mid \text{Hot on Wednesday}\}$$

$$= p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \quad = \quad \underbrace{\begin{bmatrix} p_{11} & p_{12} & p_{13} \end{bmatrix}}_{\textit{row 1 of P}} \overbrace{\begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix}}^{\textit{column 3 of P}}$$

$$= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{6}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{6}\right) + \left(\tfrac{1}{6}\right)\left(\tfrac{1}{3}\right)$$

element in row 1, column 3 of P^2

$$= 0.1944444$$

The probability of going from state 1 (HOT) to state 3 (Cool) in TWO stages (days)

In general, the probability that the system makes a transition from state i to state j in 2 stages is

$$P_{ij}^{n,n+2} = \sum_{k} p_{ik} p_{kj}$$

which is the element in row i & column j of P^2

$$P^2 = \begin{bmatrix} 0.4167 & 0.3889 & 0.1944 \\ 0.3889 & 0.4167 & 0.1944 \\ 0.3889 & 0.3889 & 0.2222 \end{bmatrix}$$

Three-day Forecast

$$P^3 = \begin{array}{c} \\ H \\ M \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \quad H \quad & \quad M \quad & \quad C \quad \end{array} \\ \begin{bmatrix} 0.39815 & 0.40278 & 0.19907 \\ 0.40278 & 0.39815 & 0.19907 \\ 0.39815 & 0.39815 & 0.2037 \end{bmatrix} \end{array}$$

So, if Wednesday is HOT, the probability that Saturday (three days hence) is Moderate is 0.40278,
and the probability that it is Cool is 0.19907

The utility company can be 60.185% certain, then, that Saturday will NOT be hot.

The Chapman-Kolmogorov Equation

Let $p_{ij}^{(n)} =$ P{system is in state j at stage n, given that system is in state j initially}

Then

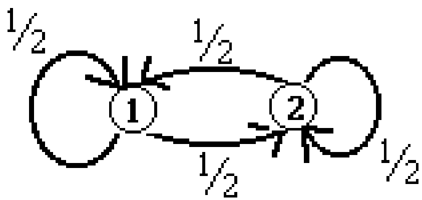
$$\boxed{p_{ij}^{(n)} = \sum_k p_{ik}^{(r)}\, p_{kj}^{(n-r)}}$$

for any i&j, and r such that $0 \le r \le n$

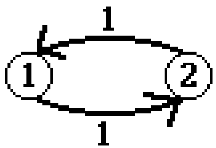
That is, $p_{ij}^{(n)}$ *is the inner product of row #i of* P^r *and column #j of* P^{n-r}

Long-Run Behavior of Markov Chains

A Markov chain is **regular** if
there is some k such that its transition probability matrix P, raised to the power k, has strictly positive elements only.



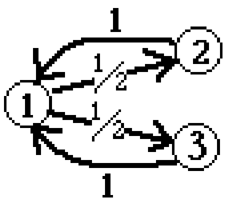
$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Not regular:

$$P^n = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$



$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Not regular:

If a Markov chain is REGULAR, then

$$\lim_{n \to +\infty} p_{ij}^{(n)} = \pi_j$$

(independent of the initial state i)

That is,

$$\lim_{n \to +\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \\ \pi_1 & \pi_2 & \cdots & \pi_N \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_N \end{bmatrix}$$

The limiting probability π_j is called a *steady-state probability*

Chapman-Kolmogorov Equation:

$$\boxed{p_{ij}^{(n)} = \sum_k p_{ik}^{(r)}\, p_{kj}^{(n-r)}}$$

for any i&j, and r such that $0 \le r \le n$

$$\Longrightarrow \quad p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)}\, p_{kj}$$

$$\Longrightarrow \lim_{n\to\infty} p_{ij}^{(n)} = \sum_k \lim_{n\to\infty} p_{ik}^{(n-1)}\, p_{kj}$$

$$\Longrightarrow \quad \boxed{\pi_j = \sum_k \pi_k\, p_{kj}}$$

$$\boxed{\lim_{n\to\infty} p_{ij}^{(n)} \equiv \pi_j}$$

Therefore, the limiting probabilities must satisfy the conditions:

1) $\sum_j \pi_j = 1$

2) $\pi_j = \sum_i \pi_i p_{ij}$ ← in matrix form, $\boldsymbol{\pi} = \boldsymbol{\pi} P$

3) $\pi_j > 0$

Summer Weather Example

$$\pi = \pi P$$

$$P = \begin{array}{cc} & \begin{array}{ccc} H & M & C \end{array} \\ \begin{array}{c} H \\ M \\ C \end{array} & \begin{bmatrix} 1/3 & 1/2 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

π_1 is the product of π and column 1 of P:

$$\pi_1 = p_{11}\pi_1 + p_{21}\pi_2 + p_{31}\pi_3$$

$$= 1/3\,\pi_1 + 1/2\,\pi_2 + 1/3\,\pi_3$$

$$\Rightarrow\ 2/3\,\pi_1 - 1/2\,\pi_2 - 1/3\,\pi_3 = 0$$

$$\pi_2 = p_{12}\pi_1 + p_{22}\pi_2 + p_{32}\pi_3$$
$$= \tfrac{1}{2}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{3}\pi_3$$

π_2 is the product of π and column #2 of P

$$\Rightarrow -\tfrac{1}{2}\pi_1 + \tfrac{2}{3}\pi_2 - \tfrac{1}{3}\pi_3 = 0$$

$$\pi_3 = p_{13}\pi_1 + p_{23}\pi_2 + p_{33}\pi_3$$
$$= \tfrac{1}{6}\pi_1 + \tfrac{1}{6}\pi_2 + \tfrac{1}{3}\pi_3$$

π_3 is the product of π and column #3 of P

$$\Rightarrow -\tfrac{1}{6}\pi_1 - \tfrac{1}{6}\pi_2 + \tfrac{2}{3}\pi_3 = 0$$

$$\begin{cases} {}^{2}\!/\!_{3}\pi_1 - {}^{1}\!/\!_{2}\pi_2 - {}^{1}\!/\!_{3}\pi_3 = 0 \\ -{}^{1}\!/\!_{2}\pi_1 + {}^{2}\!/\!_{3}\pi_2 - {}^{1}\!/\!_{3}\pi_3 = 0 \\ -{}^{1}\!/\!_{6}\pi_1 - {}^{1}\!/\!_{6}\pi_2 + {}^{2}\!/\!_{3}\pi_3 = 0 \end{cases}$$

This system of equations is linearly dependent (the sum of the left sides is zero!)

We need also the equation $\pi_1 + \pi_2 + \pi_3 = 1$

$$\begin{cases} \frac{2}{3}\pi_1 - \frac{1}{2}\pi_2 - \frac{1}{3}\pi_3 = 0 \\ -\frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 - \frac{1}{3}\pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Discarding any one of the first three equations gives us a system with full rank!

The solution: $\pi_1 = \frac{2}{5}$ $\pi_2 = \frac{2}{5}$ $\pi_3 = \frac{1}{5}$

"Long Range Forecast"

That is, "in the long run", summer days will be HOT or MODERATE with probability 40% each, and COOL with probability 20%.

Example

Consider an engine repair shop which specializes in the repair of two types of automobile engines: gasoline & diesel

The overhaul of a diesel engine requires two days, while the overhaul of a gasoline engine requires a single day.

Each morning, the probability of receiving a diesel engine for overhaul is $p_D = 1/3$.

The probability of receiving a gasoline engine for overhaul is
$p_G = 1/2$.

The profit per day for overhauling a diesel engine is \$20, and for a gasoline engine is \$23.

Work which cannot be done on the day received is lost to competitor repair shops.

What is the best policy for accepting jobs?

- If only 1 day's work is complete on a diesel engine, any jobs which arrive must be refused.
- Otherwise, if only one engine type is received, that job should be accepted.
- If not in the midst of overhauling a diesel engine, and BOTH engine types arrive, we can

  a) give preference to the DIESEL engine

or

  b) give preference to the GASOLINE engine

Which is the better choice?

Solution

Markov Chain Model

Let's assume that the system (repair shop) is observed at midday each day.

What are the possible states of the system?

(1) repair shop is idle

(2) first day of work on diesel engine is in progress

(3) second day of work on diesel engine is in progress

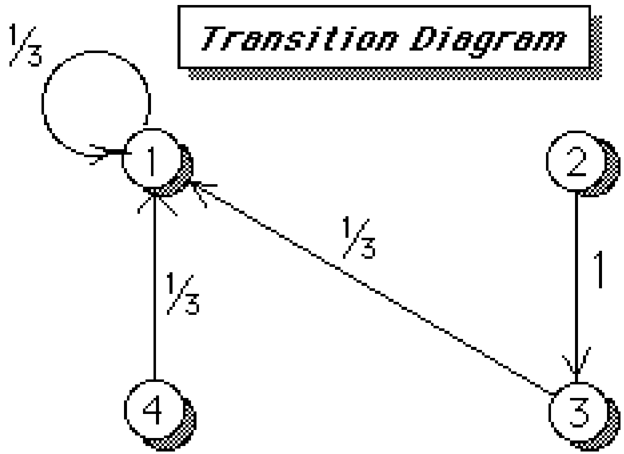
(4) work on gasoline engine is in progress

Transition Diagram

States

(1) shop is idle
(2) day #1 on diesel
(3) day #2 on diesel
(4) work on gasoline engine

What transitions are possible?
What is the probability of each transition?

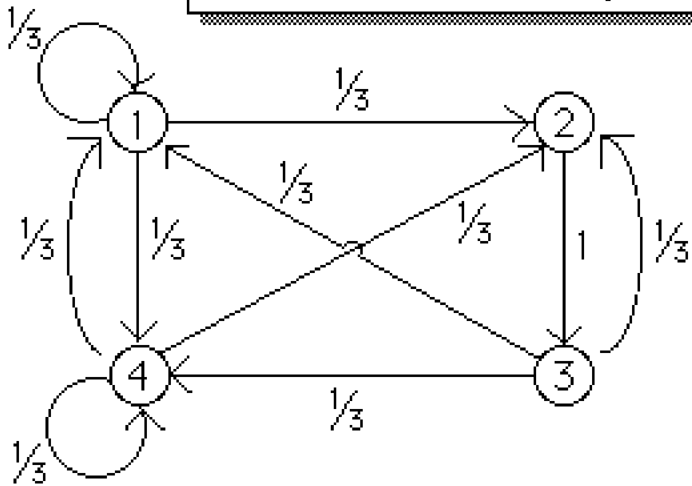


States

(1) shop is idle
(2) day #1 on diesel
(3) day #2 on diesel
(4) work on gasoline engine

Some transitions occur with probabilities which are the same, whether preference is given to diesel or gasoline engines.

Alternative A: Give preference to diesel engines



States

(1) shop is idle
(2) day #1 on diesel
(3) day #2 on diesel
(4) work on gasoline engine

Probability of NO arrivals: $(1-p_D)(1-p_G) = (1-\frac{1}{3})(1-\frac{1}{2}) = \frac{1}{3}$

Probability that gasoline engine arrives, but no diesel:
$(1-p_D)p_G = (1-\frac{1}{3})\frac{1}{2} = \frac{1}{3}$

Alternative A: Give preference to diesel engines

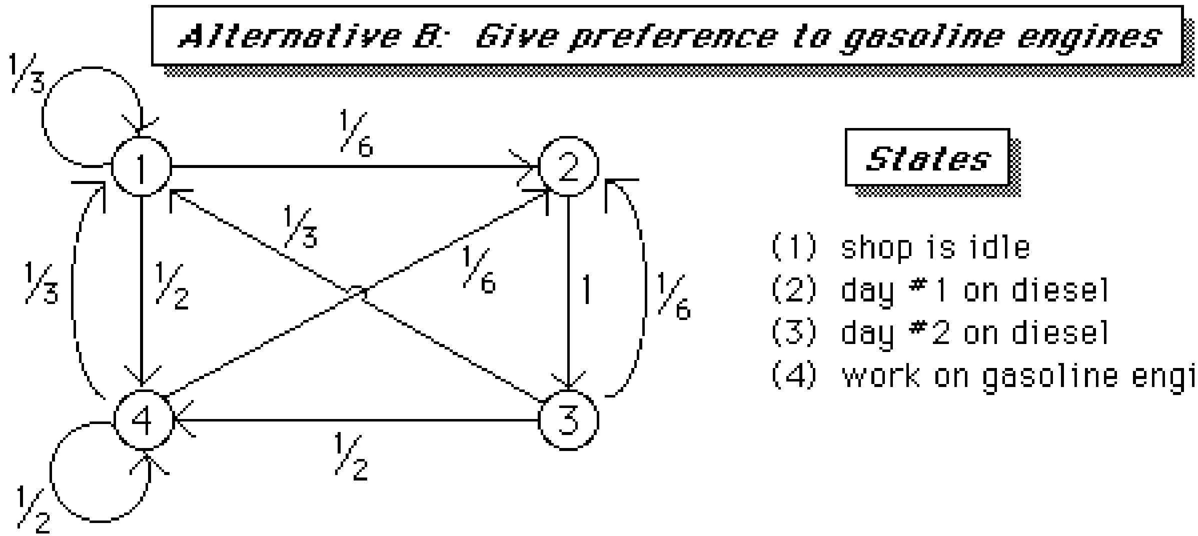
$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Steady-State Distribution

i	π_i
1	0.25
2	0.25
3	0.25
4	0.25

Expected profit/day:

$$\$0\,\pi_1 + \$20\,\pi_2 + \$20\,\pi_3 + \$23\,\pi_4 = \$15.75$$



States

(1) shop is idle
(2) day #1 on diesel
(3) day #2 on diesel
(4) work on gasoline engine

Probability that diesel engine arrives, but no gasoline engine:

$$p_D(1-p_G) = \tfrac{1}{3}(1-\tfrac{1}{2}) = \tfrac{1}{6}$$

Alternative B: Give preference to gasoline engines

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} \end{bmatrix}$$

Steady-State Distribution

i	π_i
1	0.285714
2	0.142857
3	0.142857
4	0.428571

Expected profit/day:

$$\$0\,\pi_1 + \$20\,\pi_2 + \$20\,\pi_3 + \$23\,\pi_4 = \$15.571413$$

Policy	(a) Prefer diesel engine	(b) Prefer gasoline engine
Expected profit/day	$15.75	$15.571413

The better policy is to accept the diesel engine when the shop is ready for the next engine, and both types arrive.

Example A **Self-Service Elevator** in a four-story building operates solely according to the buttons pushed **inside** the elevator. That is, a person on the outside cannot "call" the elevator to the floor he where he is. (Consequently, the only way to get the elevator is for someone else to get off at your floor.)

Of the passengers entering the building at the first floor and wishing to use the elevator, half go to the second floor and the other half divides equally between the third and fourth floors.

Passengers above the first floor want to go to the first floor in 80% of the cases. Otherwise, they are equally likely to want to go to the other two floors.

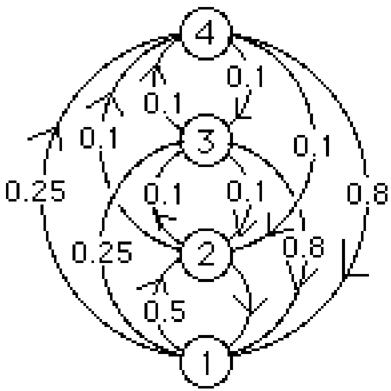
Questions

- If you enter the building on the first floor, what is the probability that you will find the elevator there?
- If the elevator is not at the first floor but at the second floor, how many trips is it expected to make before returning to the first floor?

Markov Chain Model

State i = location of the elevator (i=1,2,3,4) at end of trip

Stage n = trip number



Transition Probabilities

$$\begin{array}{c|cccc} \text{from} \backslash \text{to} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0.5 & 0.25 & 0.25 \\ 2 & 0.8 & 0 & 0.1 & 0.1 \\ 3 & 0.8 & 0.1 & 0 & 0.1 \\ 4 & 0.8 & 0.1 & 0.1 & 0 \end{array}$$

First Passage Time

Define the random variable

N_{ij} = the number of the stage at which the system, starting in state **i**, ***first*** reaches state **j**.

First-Passage Probabilities

$f_{ij}^{(n)}$ = probability that the system, starting in state **i**, will **first** reach state **j** in **exactly n** steps

$= P\{N_{ij} = n\}$

Recall that $p_{ij}^{(n)}$ *is the probability that, starting in state i, the system is in state j after n (but perhaps NOT for the first visit!)*

⇨ **computation**

Computation of $f_{ij}^{(n)}$

We express $p_{ij}^{(n)}$ by conditioning on the step, k, at which the system first reaches state j:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} P\left\{ \begin{array}{c} \text{system in} \\ \text{state j at} \\ \text{step n} \end{array} \middle| \begin{array}{l} \text{system first} \\ \text{reaches state} \\ \text{j at step k} \end{array} \right\} \times P\left\{ \begin{array}{l} \text{system first} \\ \text{reaches state j} \\ \text{at step k} \end{array} \right\}$$

$$= \sum_{k=1}^{n} p_{jj}^{(n-k)} \times f_{ij}^{(k)}$$

Solve for $f_{ij}^{(n)}$:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} p_{jj}^{(n-k)} \times f_{ij}^{(k)}$$

$$= \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)} + p_{jj}^{(0)} \; f_{ij}^{(n)}$$

$$\Rightarrow \quad p_{jj}^{(0)} \; f_{ij}^{(n)} = \boxed{f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)}}$$

$$\boxed{f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)}}$$

Recursive Computation:

Compute the powers of P, i.e., $p_{ij}^{(n)}$

With $f_{ij}^{(1)} = p_{ij}$, compute $f_{ij}^{(2)}$

Then, knowing $f_{ij}^{(1)}$ and $f_{ij}^{(2)}$, compute $f_{ij}^{(3)}$, etc.

Elevator Example

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)}$$

$$P$$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0.5 & 0.25 & 0.25 \\ 2 & 0.8 & 0 & 0.1 & 0.1 \\ 3 & 0.8 & 0.1 & 0 & 0.1 \\ 4 & 0.8 & 0.1 & 0.1 & 0 \end{array}$$

$$P^2$$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.8 & 0.05 & 0.075 & 0.075 \\ 2 & 0.16 & 0.42 & 0.21 & 0.21 \\ 3 & 0.16 & 0.41 & 0.22 & 0.21 \\ 4 & 0.16 & 0.41 & 0.21 & 0.22 \end{array}$$

$$f_{21}^{(1)} = p_{21}^{(1)} = 0.8$$

$$f_{21}^{(2)} = p_{21}^{(2)} - \sum_{k=1}^{1} p_{11}^{(2-k)} f_{21}^{(k)}$$

$$= p_{21}^{(2)} - p_{11}^{(1)} f_{21}^{(1)}$$

$$= 0.16 - 0.0 \times 0.8$$

$$= 0.16$$

$$f_{21}^{(1)} = 0.8$$

$$f_{21}^{(2)} = 0.16$$

$$\boxed{f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)}}$$

$$P^3$$

$$\begin{array}{c|c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.16 & 0.415 & 0.2125 & 0.2125 \\ 0.672 & 0.122 & 0.103 & 0.103 \\ 0.672 & 0.123 & 0.102 & 0.103 \\ 0.672 & 0.123 & 0.103 & 0.102 \end{bmatrix}
\end{array}$$

$$f_{21}^{(3)} = p_{21}^{(3)} - \sum_{k=1}^{2} p_{11}^{(3-k)} f_{21}^{(k)}$$

$$= p_{21}^{(3)} - p_{11}^{(2)} f_{21}^{(1)} - p_{11}^{(1)} f_{21}^{(2)}$$

$$\Rightarrow \quad f_{21}^{(3)} = 0.672 - 0.8 \times 0.8 - 0.0 \times 0.16$$

$$= 0.032$$

$$f_{21}^{(1)} = 0.8$$

$$f_{21}^{(2)} = 0.16$$

$$f_{21}^{(3)} = 0.032$$

$$\boxed{f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{jj}^{(n-k)} \times f_{ij}^{(k)}}$$

$$f_{21}^{(4)} = p_{21}^{(4)} - \sum_{k=1}^{3} p_{11}^{(4-k)} f_{21}^{(k)}$$

$$P^4$$

$$\begin{bmatrix} 0.672 & 0.1225 & 0.10275 & 0.10275 \\ 0.2624 & 0.3566 & 0.1905 & 0.1905 \\ 0.2624 & 0.3565 & 0.1906 & 0.1905 \\ 0.2624 & 0.3565 & 0.1905 & 0.1906 \end{bmatrix}$$

$$\Rightarrow \quad f_{21}^{(4)} = p_{21}^{(4)} - p_{11}^{(3)} f_{21}^{(1)} - p_{11}^{(2)} f_{21}^{(2)} - p_{11}^{(1)} f_{21}^{(3)}$$

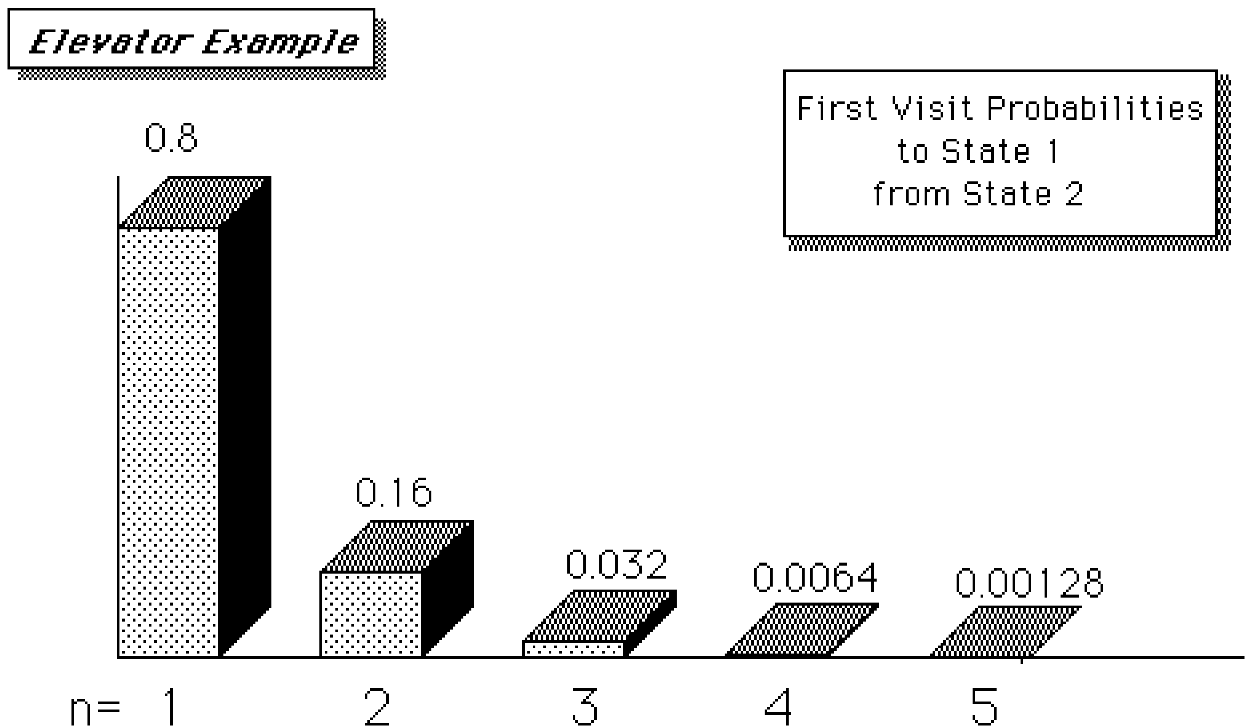
$$= 0.2624 - 0 - 0.8 \times 0.16 - 0.16 \times 0.8$$

$$= 0.0064$$

Elevator Example

First Visit Probabilities to State 1 from State 2

n	P
1	0.8
2	0.16
3	0.032
4	0.0064
5	0.00128


Elevator Example
First Visit Probabilities
to State 1
from State 2
0.8
0.16
0.032
0.0064
0.00128
n= 1
2
3
4
5

Mean First Passage Time

m_{ij} = expected number of stage at which the system, starting in state **i**, first reaches state **j**

$$\mathbf{m}_{ij} = \mathrm{E}N_{ij} = \sum_{n=1}^{\infty} \mathbf{n}\,\mathbf{f}_{ij}^{(n)}$$

where

N_{ij} = the number of the stage at which the system, starting in state **i**, *first* reaches state **j**.

$P\{N_{ij} = n\} = f_{ij}^{(n)}$

computation

Mean First Passage Time

$$m_{ij} = E\{N_{ij}\} = \sum_{n=1}^{\infty} \mathbf{n}\, \mathbf{f}_{ij}^{(n)}$$

Computation of m_{21}

n	$f_{21}^{(n)}$	$n f_{21}^{(n)}$
1	0.8	0.8
2	0.16	0.32
3	0.032	0.096
4	0.0064	0.0256
5	0.00128	0.0064
6	0.000256	0.001536
7	0.0000512	0.0003584
8	0.000002048	0.00008192
		$\approx$ 1.25

Elevator Example

Mean First Passage Time

$$m_{ij} = E\{N_{ij}\} = \sum_{n=1}^{\infty} n\, f_{ij}^{(n)}$$

An alternative to computing this infinite sum:

$$E\{N_{ij}\} = \sum_{k} E\{N_{ij} \mid X_1 = k\} \times P\{X_1 = k\}$$

$$= \underbrace{E\{N_{ij} \mid X_1 = j\}}_{1} \underbrace{P\{X_1 = j\}}_{p_{ij}} + \sum_{k \neq j} \underbrace{E\{N_{ij} \mid X_1 = k\}}_{1 + E\{N_{kj}\}} \times \underbrace{P\{X_1 = k\}}_{p_{ik}}$$

$$= 1 \times p_{ij} + \sum_{k \neq j} [\, 1 + E\{N_{kj}\}\,]\, p_{ik}$$

$$E\{N_{ij}\} = 1 \times p_{ij} + \sum_{k \neq j} [\,1 + E\{N_{kj}\}\,]\, p_{ik}$$

$$\underbrace{E\{N_{ij}\}}_{m_{ij}} = \underbrace{p_{ij} + \sum_{k \neq j} p_{ik}}_{1} + \sum_{k \neq j} \underbrace{E\{N_{kj}\}}_{m_{kj}}\, p_{ik}$$

$$\Rightarrow \boxed{m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}}$$

If we form these equations for a fixed j and all possible values of i, we get a system of linear equations in

m_{11}, m_{21}, m_{31}, & m_{41}

Elevator Example

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

What is m_{21}, i.e., the expected number of trips required to reach floor #1 if the elevator is currently on floor #2?

To compute m_{21} from the above equation requires that we also compute m_{11}, m_{31}, and m_{41}

i.e., m_{k1}, $k \neq 2$

$$\boxed{m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}}$$

The above equation, with j fixed at the value 1 and i=1, 2, 3, & 4, yields:

$$\begin{cases} i=1: & m_{11} = 1 + p_{12} m_{21} + p_{13} m_{31} + p_{14} m_{41} \\ i=2: & m_{21} = 1 + p_{22} m_{21} + p_{23} m_{31} + p_{24} m_{41} \\ i=3: & m_{31} = 1 + p_{32} m_{21} + p_{33} m_{31} + p_{34} m_{41} \\ i=4: & m_{41} = 1 + p_{42} m_{21} + p_{43} m_{31} + p_{44} m_{41} \end{cases}$$

$$P = \begin{bmatrix} 0 & 0.5 & 0.25 & 0.25 \\ 0.8 & 0 & 0.1 & 0.1 \\ 0.8 & 0.1 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 & 0 \end{bmatrix} \qquad \begin{cases} m_{11} = 1 + p_{12} m_{21} + p_{13} m_{31} + p_{14} m_{41} \\ m_{21} = 1 + p_{22} m_{21} + p_{23} m_{31} + p_{24} m_{41} \\ m_{31} = 1 + p_{32} m_{21} + p_{33} m_{31} + p_{34} m_{41} \\ m_{41} = 1 + p_{42} m_{21} + p_{43} m_{31} + p_{44} m_{41} \end{cases}$$

$$\Rightarrow \begin{cases} m_{11} = 1 + 0.5\, m_{21} + 0.25\, m_{31} + 0.25\, m_{41} \\ m_{21} = 1 + 0\, m_{21} + 0.1\, m_{31} + 0.1\, m_{41} \\ m_{31} = 1 + 0.1\, m_{21} + 0\, m_{31} + 0.1\, m_{41} \\ m_{41} = 1 + 0.1\, m_{21} + 0.1\, m_{31} + 0\, m_{41} \end{cases} \Rightarrow \begin{cases} m_{11} = 2.25 \\ m_{21} = 1.25 \\ m_{31} = 1.25 \\ m_{41} = 1.25 \end{cases}$$

By solving four sets of four equations, we obtain all the mean first passage times:

Mean First Passage Times

Elevator

from \ to	1	2	3	4
1	2.25	2.8	5.5	5.5
2	1.25	3.96	6	6
3	1.25	3.6	6.6	6
4	1.25	3.6	6	6.6

The expected number of stages between visits to a state #i ("mean recurrence time") is the reciprocal of the steady-state probability of state #i:

$$\boxed{m_{ii} = \frac{1}{\pi_i}} \quad \forall i$$

Simulation results:

0	1	2	3	4	5	6	7	8	9	0
2	1	2	3	1	2	1	3	1	3	4
2	1	2	1	2	4	1	2	1	2	1
2	1	2	1	3	1	2	1	2	1	4
2	1	4	1	3	4	1	3	1	2	1

The array RUN has now been globally defined in the workspace.
Each row of the array represents a repetition of the simulation.
Note: Column 1 represents stage 0, i.e. the initial state.