Incorporating new information in the decision tree

- Bayes' Rule
- PROTRAC, Inc. Problem
- Farmer Jones' Problem
Given

\( S_1, S_2, \ldots, S_n \) possible states of nature

\( P[S_i] \) prior probabilities

\( O_1, O_2, \ldots, O_m \) possible outcomes of an experiment

\( P[O_j | S_i] \) likelihood of an outcome

Calculate

\( P[S_i | O_j] \) posterior probabilities
By the definition of conditional probability,

\[
P(S_i|O_j) = \frac{P(S_i \cap O_j)}{P(O_j)}
\]

\[\Rightarrow P(S_i \cap O_j) = P(S_i|O_j)P(O_j) = P(O_j|S_i)P(S_i)\]
Bayes' Rule

\[ P(S_i \cap O_j) = P(S_i | O_j) \cdot P(O_j) = P(O_j | S_i) \cdot P(S_i) \]

\[ \Rightarrow P(S_i | O_j) = \frac{P(O_j | S_i) \cdot P(S_i)}{P(O_j)} \]
Incorporating New Information

Suppose that in the PROTRAC example, a market research study can be made before deciding which strategy (A, B, or C) to select. The results of this study can then be used to more accurately estimate the probabilities of a "Strong" or "Weak" market.
Test results are either
- Encouraging
- Discouraging

Reliability of the market study: "The past results with our test have tended to be in the 'right direction'. Specifically, in 60% of the instances when the market has been strong, the preceding market study was 'Encouraging', while in 70% of the instances when the market has been weak, the preceding market study was 'Discouraging'."
The statement about "reliability" of the market study provides:

**Conditional Probabilities:**

"In 60% of the instances when the market has been strong, the preceding market study was 'encouraging' "

\[ P(E|S) = 60\% \]

\[ P(D|S) = 40\% \]

"In 70% of the instances when the market has been weak, the preceding market study was 'discouraging' "

\[ P(E|W) = 30\% \]

\[ P(D|W) = 70\% \]
Bayes' Rule can now be used to find the values for \( P(S|E), P(S|D) \), etc. For example,

\[
P(S|E) = \frac{P(E|S)P(S)}{P(E)} = \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|W)P(W)}
\]

\[
= \frac{(0.6)(0.45)}{(0.6)(0.45) + (0.3)(0.55)} = \frac{0.27}{0.27 + 0.165} = \frac{0.27}{0.435} = 0.621
\]
The posterior probability of a Strong market is given by:

\[ P(S|E) = \frac{P(E|S)P(S)}{P(E)} = \frac{(0.6)(0.45)}{(0.6)(0.45) + (0.3)(0.55)} = \frac{0.27}{0.27 + 0.165} = \frac{0.27}{0.435} = 0.621 \]

- \( P(E|S) = 60\% \)
- \( P(D|S) = 40\% \)
- \( P(E|W) = 30\% \)
- \( P(D|W) = 70\% \)
The decision tree is now drawn with the decision nodes *following* the (random) outcome of the market study:

This node represents the outcome of the market study.
"Folding back the tree"
The maximum expected payoff which can be attained is 13.46
Expected Value of Sample Information

EVWSI: "Expected Value With Sample Information"
EVWOI: "Expected Value Without Information"
EVSI: "Expected Value of Sample Information"

\[ EVSI = EVWSI - EVWOI \]
**EXAMPLE**

\[ \text{EVSI} = \text{EVWSI} - \text{EVWOI} \]

In the "PROTRAC" decision problem,

\[
\text{EVWOI} = 12.85 \quad \text{Expected payoff with no market study}
\]

\[
\text{EVWSI} = 13.46 \quad \text{Expected payoff using market study}
\]

\[
\text{EVSI} = 13.46 - 12.85 = 0.61
\]
EXPECTED VALUE OF PERFECT INFORMATION

EVWPI: "Expected Value With Perfect Information"

EVWOI: "Expected Value Without Information"

EVPI = EVWPI - EVWOI
EXAMPLE

PROTRAC decision problem

To calculate EVWPI ("Expected Value With Perfect Information"), we draw the decision tree in which the decision-maker has full knowledge of which state has occurred before the decision must be made.
The Payoff Table

<table>
<thead>
<tr>
<th>Decision</th>
<th>State of &quot;Nature&quot;</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S: strong</td>
<td>W: weak</td>
</tr>
<tr>
<td>A</td>
<td>0.45</td>
<td>0.55</td>
</tr>
<tr>
<td>B</td>
<td>30</td>
<td>-8</td>
</tr>
<tr>
<td>C</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
We draw the tree with the state of the economy known before the decision is made.
The decision is made, knowing the state of the economy.
Folding back:
0.45 \times 30 + 0.55 \times 15

15.65

EVWPI
\[ \text{EVPI} = \text{EVWPI} - \text{EVWOI} \]

\[ = 15.65 - 12.85 \]

\[ = 2.8 \]

Expected Value of Perfect Information
EXAMPLE

Farmer Jones must determine whether to plant corn or soybeans on a certain piece of land.

His "payoff" depends upon the weather conditions during the summer growing season:
- If he plants **corn** and the weather is **warm**, he earns $8000
- If he plants **corn** and the weather is **cold**, he earns $5000
- If he plants **soybeans** and the weather is **warm**, he earns $7000
- If he plants **soybeans** and the weather is **cold**, he earns $6500.
In the past, 40% of all years have been cold, and 60% have been warm.

Before planting, farmer Jones can pay $600 for an expert weather forecast.

If the year will actually be cold, there is a 90% chance that the forecaster will be correct, i.e., predict a cold year.

If the year will actually be warm, there is a 80% chance that the forecaster will be correct, i.e., will predict a warm year.
CONSTRUCTING DECISION TREE

FOLDING BACK TREE

OPTIMAL DECISIONS

EXPECTED VALUE OF FORECAST

EXPECTED VALUE OF PERFECT INFORMATION
Jones must first decide whether to **Hire Forecaster (HF)**, or **Not Hire Forecaster (NHF)**.
If he decides against hiring the forecaster, then he must next decide whether to plant:

- Corn, or
- Soybeans
After he plants his crop, the weather happens to be
- **Warm**, or
- **Cold**

Prior Probabilities

\[
P(\text{Warm}) = 60% \\
P(\text{Cold}) = 40%
\]
His payoff depends upon the crop which he planted, and the weather conditions.
What is
$P\{WF\} \& P\{CF\}$?

If he hires the forecaster, he is given
Warm Forecast, or
Cold Forecast
Condition the event "Warm Forecast" on the events "Warm weather" and "Cold weather":

\[ P\{\text{Warm Forecast}\} \]
\[ = P\{\text{Warm Forecast} | \text{Warm}\} P\{\text{Warm weather}\} \]
\[ \quad \text{(correct in warm season.)} \]
\[ + P\{\text{Warm Forecast} | \text{Cold}\} P\{\text{Cold weather}\} \]
\[ \quad \text{(error in cold season.)} \]

\[ P\{\text{WF}\} = P\{\text{WF} | \text{W}\} P\{\text{W}\} + P\{\text{WF} | \text{C}\} P\{\text{C}\} \]
\[ = 0.8 \times 0.6 + 0.1 \times 0.4 \]
\[ = 0.52 \]
\[ P\{ \text{Cold Forecast} \} \]
\[ = P\{ \text{Cold Forecast} \mid \text{Warm} \} P\{ \text{Warm weather} \} \]
\[ \text{(error in warm season.)} \]
\[ + P\{ \text{Cold Forecast} \mid \text{Cold} \} P\{ \text{Cold weather} \} \]
\[ \text{(correct in cold season.)} \]

\[ P\{ \text{CF} \} = P\{ \text{CF} \mid W \} P\{ W \} + P\{ \text{CF} \mid C \} P\{ C \} \]
\[ = 0.2 \times 0.6 + 0.9 \times 0.4 \]
\[ = 0.48 = 1 - P\{ \text{WF} \} \]
After receiving the forecast, Jones must decide upon the crop
Then, depending upon the weather, he collects a payoff.
Revised probabilities after receiving forecast

\[ P\{\text{Warm weather} \mid \text{Warm Forecast} \} \]

\[ P\{W \mid WF\} = \frac{P\{WF \mid W\} P\{W\}}{P\{WF\}} = \frac{0.8 \times 0.6}{0.52} = 0.9231 \]

\[ \text{Bayes' Rule} \]

\[ P\{\text{Cold weather} \mid \text{Warm Forecast} \} \]

\[ P\{C \mid WF\} = 1 - P\{W \mid WF\} = 0.0769 \]
\[ P\{W \mid WF\} = 0.9231 \]
\[ P\{C \mid WF\} = 0.0769 \]
Revised probabilities after receiving forecast

\[ P\{\text{Cold} \mid \text{Cold Forecast} \} = \frac{P\{\text{CF} \mid C\} P\{C\}}{P\{\text{CF}\}} = \frac{0.9 \times 0.4}{0.48} = 0.75 \]

**Bayes' Rule**

\[ P\{\text{Warm} \mid \text{Cold Forecast} \} = 1 - P\{\text{C} \mid \text{CF}\} = 1 - 0.75 = 0.25 \]
\[
P(C | CF) = 0.75
\]
\[
P(W | CF) = 0.25
\]
Now we begin “folding back” the nodes of the tree...
Folding back nodes 10 & 11:

\[ 8(0.25) + 5(0.75) = 5.75 \]
\[ 7(0.25) + 6.5(0.75) = 6.625 \]
Folding back nodes 8 & 9:

\[ 8(0.9231) + 5(0.0769) = 7.769 \]
\[ 7(0.9231) + 6.5(0.0769) = 6.962 \]
Next we fold back nodes 6 & 7 by choosing the maximum payoff.
Next we fold back node 5:

\[
(0.52)(7.769) + (0.48)(6.635) = 7.22
\]
Next, fold back nodes 3 & 4:

\[
8(0.6) + 5(0.4) = 6.8 \\
7(0.6) + 6.5(0.4) = 6.8
\]
Folding back node 2: Jones is indifferent to planting either crop.

1. NHF
   - $0.6K
   - 7.22K

2. NHF
   - $6.8K
   - S

3. 3
   - $6.8K
   - w
   - Cold
   - $5K
   - $8K
   - 0.6
   - 0.4

4. 4
   - $6.8K
   - w
   - Cold
   - $7K
   - 0.6
   - 0.4
   - $6.5K

5. 5
   - 7.22K
   - WF
   - 0.52
   - CF
   - $6.625K
   - 0.48
   - 0.52

6. 6
   - $7.769K
   - c
   - $7.769

7. 7
   - $6.625K
   - c
   - $6.625

8. 8
   - $7.769

9. 9
   - 6.962

10. 10
    - 5.75

11. 11
    - 6.625
Folding back node 1:
Choose NHF since 6.8 > 6.62

Decision Analysis 3  8/21/00
Solution:
Do not hire forecaster, plant either crop.

Decision tree diagram with payoffs and probabilities:
- **Node 1**: HF (High Flooding) with a chance of $6.8K
- **Node 2**: NHF (No High Flooding), $6.8K
- **Node 3**: 0.6 chance of $8K, 0.4 chance of $5K
- **Node 4**: 0.6 chance of $7K, 0.4 chance of $6.5K
- **Node 5**: WF (Wet Flooding) with a chance of $6.8K
- **Node 6**: C (Cold), $C
- **Node 7**: CF (Cold Flooding), $C
- **Node 8**: 0.6 chance of $8K, 0.4 chance of $5K
- **Node 9**: 0.6 chance of $7K, 0.4 chance of $6.5K
- **Node 10**: C (Cold), $C
- **Node 11**: CF (Cold Flooding), $C
**EVSI**

What is the expected value of the forecast?

If the forecast were "free", Jones' expected payoff, using the forecast, would be $7,22K, or $420 more than his expected payoff without the forecast.

\[
\text{EVSI} = 420
\]
**EVPI**

*What is the expected value of perfect information?*

*Imagine that Jones obtained a forecast which was 100% accurate*
Folding back the tree:

\[ 8(0.6) + 6.5(0.4) = 7.4 \]
$$EVPI = EVWPI - EVWOI$$

$$= 7400 - 6800 = 600$$

Expected Value With Perfect Information

Expected Value Without Information
The NBS TV network earns an average of $400K from a hit show, and loses an average of $100K on a flop. Of all shows reviewed by the network, 25% turn out to be hits and 75% flops. For $40K, a market research firm will have an audience view a prospective show and give its view about whether the show will be a hit or flop.
If a show is actually going to be a hit, there is a 90% chance that the market research firm will predict a hit; if the show is actually going to be a flop, there is an 80% chance that the firm will predict a flop. What is the optimal strategy? What is EVSI? What is EVPI?