DYNAMIC PROGRAMMING

- an example from *Introduction to Dynamic Programming*, by George Nemhauser (John Wiley & Sons, 1966), pages 71-76.

author

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A deterministic DP example with

- one state variable
- one decision variable
- three stages
- irregular (tabulated) returns & transitions

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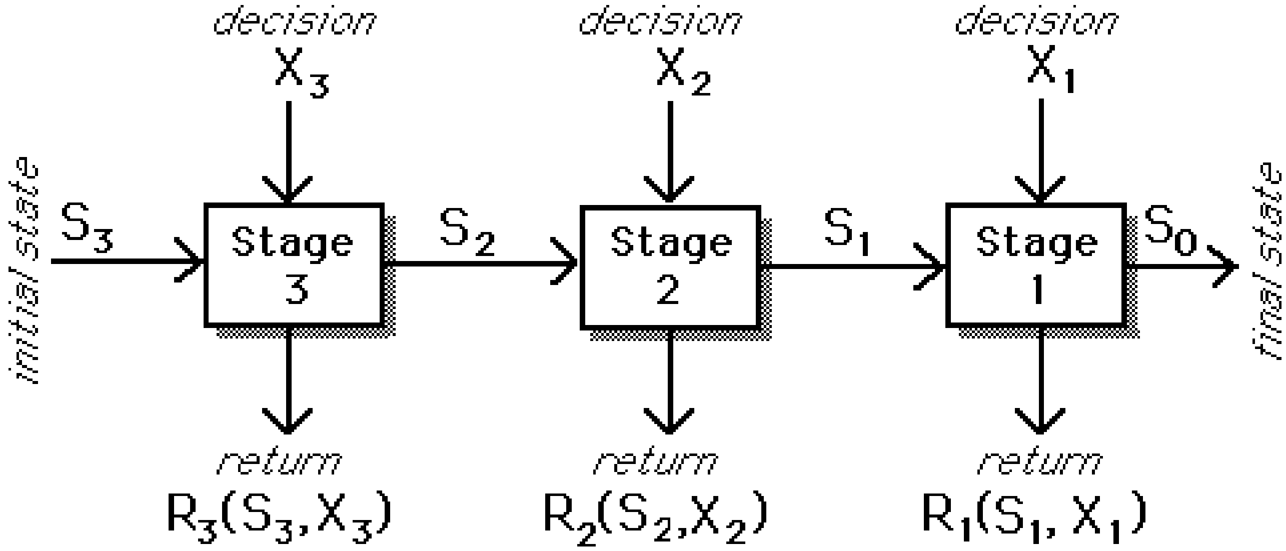

where

$$S_2=T_3(S_3,X_3)$$
$$S_1=T_2(S_2,X_2)$$
$$S_0=T_1(S_1, X_1)$$

The stages are numbered so that stage number is the number of ***remaining*** *stages!*

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Both state & decision at each stage are discrete, with possible values:

```
 State Vector
 ------------

 i    1 2 3 4 5
s[i] 1 2 3 4 5
```

```
Decision Vector
---------------

  i    1 2 3 4
 x[i] 1 2 3 4
```

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Returns

Stage 3

		decision X_3			
		1	2	3	4
state S_3	1	3	4	1	4
	2	2	4	3	3
	3	3	4	5	4
	4	4	2	3	2
	5	0	0	0	0

Stage 2

		decision X_2			
		1	2	3	4
state S_2	1	0	1	5	4
	2	5	4	2	0
	3	2	3	3	0
	4	3	5	4	2
	5	0	0	0	0

Stage 1

		decision X_1			
		1	2	3	4
state S_1	1	2	1	3	0
	2	4	3	2	0
	3	3	5	4	3
	4	0	4	3	5
	5	0	0	4	3

The return, $R_n(S_n,X_n)$, at stage n is a function of both the state S_n and the decision X_n.

E.g., $R_3(2,1) = 2$ is the return if the system is in state 2 at stage 3, and the decision is $X_3 = 1$.

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Transitions

Stage 3

state S_3	decision X_3			
	1	2	3	4
1	3	2	1	4
2	4	3	3	4
3	3	1	2	4
4	2	4	2	1
5	0	0	0	0

Stage 2

state S_2	decision X_2			
	1	2	3	4
1	0	2	5	1
2	3	4	3	0
3	4	5	4	0
4	3	4	2	3
5	0	0	0	0

Stage 1

state S_1	decision X_1			
	1	2	3	4
1	1	2	1	0
2	4	3	2	0
3	5	3	4	2
4	0	4	3	4
5	0	0	5	5

If the system is in state S_n at stage n, and the decision is X_n, a transition is made to the state $S_{n-1} = T_n(S_n,X_n)$, a function of both S_n & X_n.

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Optimal Value Function

$f_n(S_n) =$ maximum value of current & remaining stages, if state is S_n

Recursive definition:

$$f_n(S_n) = \begin{cases} \displaystyle\max_{X_n \in \{1,2,3,4\}} \left\{ R_n(S_n, X_n) + f_{n-1}\left(T_n(S_n, X_n)\right) \right\} & \text{for } n=3, 2, 1 \\ 0 & \text{for } n=0 \end{cases}$$

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APL code

Nemhauser DP example: tabulated returns & transitions

```
VALUE←F N;t
⍝
⍝      Optimal Value Function for Example DP Model
⍝
→LAST IF N=0
VALUE←MAX R[N;;]+(F N-1)[TRANSITION T[N;;]]
→0
LAST:VALUE←((⍴S)⍴0),-BIG
```

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Stage 1

s \ x	1	2	3	4
1	2.00	1.00	3.00	¯999.99
2	4.00	3.00	2.00	¯999.99
3	3.00	5.00	4.00	3.00
4	¯999.99	4.00	3.00	5.00
5	¯999.99	¯999.99	4.00	3.00

-999.99 indicates infeasible decision!

decision X_1

state S_1

	1	2	3	4
1	2	1	3	0
2	4	3	2	0
3	3	5	4	3
4	0	4	3	5
5	0	0	4	3

returns

decision X_1

state S_1

	1	2	3	4
1	1	2	1	0
2	4	3	2	0
3	5	3	4	2
4	0	4	3	4
5	0	0	5	5

transitions

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Stage 1

s \ x	1	2	3	4
1	2.00	1.00	3.00	¯999.99
2	4.00	3.00	2.00	¯999.99
3	3.00	5.00	4.00	3.00
4	¯999.99	4.00	3.00	5.00
5	¯999.99	¯999.99	4.00	3.00

-999.99 indicates infeasible decision!

State	Optimal Values	Optimal Decisions	Resulting State
1	3.00	3	1
2	4.00	1	4
3	5.00	2	3
4	5.00	4	4
5	4.00	3	5

$f_1(S_1)$ X_1^* $T_1(S_1,X_1^*)$

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Stage 2

s \ x	1	2	3	4
1	¯999.99	5.00	9.00	7.00
2	10.00	9.00	7.00	¯999.99
3	7.00	7.00	8.00	¯999.99
4	8.00	10.00	8.00	7.00

from previous computation

State	$f_1(S_1)$
1	3.00
2	4.00
3	5.00
4	5.00
5	4.00

decision X_2

state S_2	1	2	3	4
1	0	1	5	4
2	5	4	2	0
3	2	3	3	0
4	3	5	4	2
5	0	0	0	0

returns

decision X_2

state S_2	1	2	3	4
1	0	2	5	1
2	3	4	3	0
3	4	5	4	0
4	3	4	2	3
5	0	0	0	0

transitions

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Stage 2

s \ x	1	2	3	4
1	¯999.99	5.00	9.00	7.00
2	10.00	9.00	7.00	¯999.99
3	7.00	7.00	8.00	¯999.99
4	8.00	10.00	8.00	7.00

State	Optimal Values	Optimal Decisions	Resulting State
1	9.00	3	5
2	10.00	1	3
3	8.00	3	4
4	10.00	2	4

$\underbrace{\qquad\qquad}_{f_2(S_2)}$

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Stage 3

s \ x	1	2	3	4
1	11.00	14.00	10.00	14.00
2	12.00	12.00	11.00	13.00
3	11.00	13.00	15.00	14.00
4	14.00	12.00	13.00	11.00

from previous computation

State	$f_2(S_2)$
1	9.00
2	10.00
3	8.00
4	10.00

decision X_3

state S_3	1	2	3	4
1	3	4	1	4
2	2	4	3	3
3	3	4	5	4
4	4	2	3	2
5	0	0	0	0

returns

decision X_3

state S_3	1	2	3	4
1	3	2	1	4
2	4	3	3	4
3	3	1	2	4
4	2	4	2	1
5	0	0	0	0

transitions

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Stage 3

s \ x	1	2	3	4
1	11.00	14.00	10.00	14.00
2	12.00	12.00	11.00	13.00
3	11.00	13.00	15.00	14.00
4	14.00	12.00	13.00	11.00

State	Optimal Values	Optimal Decisions	Resulting State
1	14.00	2	2
		4	4
2	13.00	4	4
3	15.00	3	2
4	14.00	1	2

← *alternate optimal*

$\underbrace{\text{Optimal Values}}_{f_3(S_3)}$

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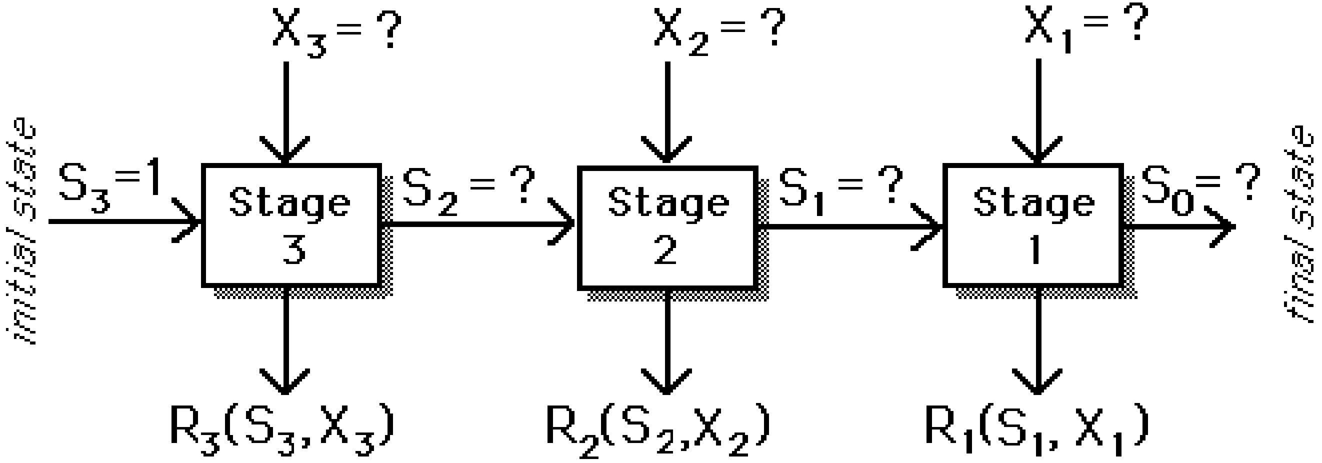
Suppose that the system begins in the state $S_3 = 1$.

What is the maximum return?

What is the optimal sequence of states (trajectory) and decisions?



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Optimal Solution No. 1

STAGE	STATE	DECISION
3	1	2
2	2	1
1	3	2
0	3	

Optimal Solution No. 2

STAGE	STATE	DECISION
3	1	4
2	4	2
1	4	4
0	4	

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