OPTIMAL LOT SIZE

by Dynamic Programming

Dennis Bricker,
Dept. of Industrial Engineering,
University of Iowa
dennis-bricker@uiowa.edu
A company requires \( n \) units of a customized electronic component, which is ordered from a supplier.

When a lot is received, it is immediately inspected, and the company pays an amount \( c \) for each unit passing inspection.

The rejection rate is \( q = 1 - p \).

Any units in surplus of the number required yields a salvage value \( v \) per unit.

If insufficient acceptable units are received, another lot must be ordered. There is a fixed cost \( K \) for reordering.
The smallest lotsize for which the expected yield of acceptable units is equal to at least $n$ is, of course, $\left\lceil \frac{n}{p} \right\rceil$, but the optimal lot size will, in general, be larger in order to avoid the reordering cost $K$.

**Example data**

- $n = 20$ units
- $q =$ rejection rate = 15%
- $c =$ cost per acceptable unit = $20$
- $v =$ salvage value for surplus units = $5$
- $K =$ reordering cost = $500$
We will assume that the outcome of each inspection is independent and identically distributed, so that the acceptable yield of a lot of size $N$ would have binomial distribution with parameters $(N, p)$. Hence we would expect that a lot size of $\left\lceil \frac{20}{0.85} \right\rceil = \left\lceil 23.5294 \right\rceil = 24$ would yield the required 20 units. However, there would be approximately

$$\sum_{j=0}^{19} p_x(j) = 28.66\%$$

probability that a deficit would remain so that reordering would be required, where

$$p_x(j) = \binom{x}{j} p^j (1 - p)^{x-j}$$

is the probability that $j$ units of a lot of size $x$ will pass inspection.
### Binomial Distribution Table

\[ P \{j \text{ units accepted} \mid x \text{ units ordered}\} \]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(j)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>15000</td>
<td>85000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>25500</td>
<td>72250</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>05738</td>
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<td>61412</td>
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<td></td>
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<tr>
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<td></td>
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<td>04145</td>
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<td></td>
</tr>
</tbody>
</table>

For example, if 6 units are ordered, the probability that exactly 4 units are accepted is 0.17618.
For the original \( n \) required units and each possible deficit, what are the lot sizes which will minimize the total expected cost (minus salvage value received for surplus units)?

**Dynamic Programming Model**

Define an optimal value function

\[
f(n) = \text{minimum expected cost of acquiring } n \text{ acceptable units.}
\]

\[
x^*(n) = \text{optimal lot size when } n \text{ acceptable units are required.}
\]

*We wish to determine the values of \( f(20) \) and \( x^*(20) \).*
Recursive Definition of the Optimal Value Function

\[
f(n) = \min_{x \geq n} \left\{ c \sum_{j=0}^{x} j p_{x}(j) - v \sum_{j=n+1}^{x} (j-n) p_{x}(j) + \sum_{j=0}^{n-1} \left[ K + f(n-j) \right] p_{x}(j) \right\}
\]

where

- \( c \sum_{j=0}^{x} j p_{x}(j) \) is the expected cost of acceptable units in a lot of size \( x \)
- \( v \sum_{j=n+1}^{x} (j-n) p_{x}(j) \) is the expected salvage value of surplus units
- \( \sum_{j=0}^{n-1} \left[ K + f(n-j) \right] p_{x}(j) \) is the expected cost of reordering

Note that \( f(n) \) appears on both left and right of the "="!
Denote the optimal \( x \) by \( \hat{x} \).

\[
f(n) - p_{\hat{x}}(0)f(n) = c \sum_{j=0}^{\hat{x}} jp_{\hat{x}}(j) - v \sum_{j=n+1}^{\hat{x}} (j-n)p_{\hat{x}}(j) + \sum_{j=0}^{n-1} \left[ K + f(n-j) \right] p_{\hat{x}}(j) + Kp_{\hat{x}}(0)
\]

Solving for \( f(n) \) yields the recursion

\[
f(n) = \min_{x \geq n} \left\{ c \sum_{j=0}^{x} j p_x(j) - v \sum_{j=n+1}^{x} (j-n) p_x(j) + \sum_{j=0}^{n-1} \left[ K + f(n-j) \right] p_x(j) + Kp_x(0) \right\} / \left( 1 - p_x(0) \right)
\]
Computation of $f(1)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>purchase</th>
<th>salvage</th>
<th>reorder</th>
<th>Total</th>
</tr>
</thead>
<tbody>
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<td>-3.61250</td>
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<td>51</td>
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<tr>
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<td>0.00569</td>
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<td>119</td>
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<td>136</td>
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<td>107.0002</td>
</tr>
</tbody>
</table>

$f(1) = 42.5959$ with lotsize = 2
Example Calculation: Suppose the lotsize is $x=3$, so that the probability distribution of the number of acceptable pieces is

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
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<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0038</td>
<td>05738</td>
<td>32513</td>
<td>61412</td>
</tr>
</tbody>
</table>

$$p_x(j) = \binom{x}{j} p^j (1-p)^{x-j}$$
Then the expected purchase price is \( c \sum_{j=1}^{x} j p_x(j) = \)

\[
20\left[ 1(0.05738) + 2(0.32513) + 3(0.61412) \right] \\
= 20[0.057375 + 0.65025 + 1.84237] = 20[2.55] = \$51
\]

The expected **salvage value** is \( v \sum_{j=n+1}^{x} (j-n) p_x(j) = \)

\[
5 \left[ 1 \times 0.32513 + 2 \times 0.61412 \right] = 5[0.325125 + 1.22825] \\
= 5[1.55338] = \$7.77
\]

The expected **reorder cost** is \( \sum_{j=0}^{n-1} \left[ K + f(n-j) \right] p_x(j) + K p_x(0) = \)

\[
500 \times 0.00338 = \$1.6875
\]

Summing and dividing by \( 1 - p_x(0) = 0.996625 \) yields

\[
\frac{51 - 7.77 + 1.6875}{0.996625} = \$45.07
\]
**Computation of f(2):**

<table>
<thead>
<tr>
<th>x</th>
<th>purchase</th>
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<th>reorder</th>
<th>Total</th>
</tr>
</thead>
<tbody>
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</table>

\[ f(2) = 66.9837 \text{ with lotsize } = 4 \]
Computation of $f(3)$:

<table>
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<tr>
<th>$x$</th>
<th>purchase</th>
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<th>reorder</th>
<th>Total</th>
</tr>
</thead>
<tbody>
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$f(3) = 92.0151$ with lotsize = 5
Note that the minimand is unimodal, although not convex:
Computation of $f(4)$:

<table>
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</tr>
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</tr>
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<td>153</td>
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<td>$135.066$</td>
</tr>
<tr>
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</table>

$f(4) = 115.289$ with lotsize $= 7$

*etc.*
<table>
<thead>
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<th># Required</th>
<th>Lotsize</th>
<th>Expected yield</th>
<th>Expected cost</th>
</tr>
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<td>0.0000</td>
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<td>66.9837</td>
</tr>
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<td>5</td>
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<tr>
<td>20</td>
<td>27</td>
<td>22.95</td>
<td>464.5606</td>
</tr>
</tbody>
</table>
Summary

If 20 usable parts are required, a lot of size 27 should be ordered. The expected yield is 22.95 (nearly 23, i.e., 3 more than required), and the expected cost is $464.56.

If, for example, the yield is 23, the cost would be $20 \times 23 = $460, and the extra 3 parts could be salvaged for $5 \times 3 = $15, a net cost of $445 (about $19.56 less than the expected cost).

If the yield were only 18, however, the cost of this lot would be $20 \times 18 = $360, and two additional parts are needed, so that another lot of size 4 should be ordered. (This would cost an additional $500 for re-ordering, plus the cost of the acceptable parts, etc.)