

Deterministic Equipment Replacement Problem

Suppose that a new car costs \$20,000 and that the annual operating cost and resale value for a car is determined only by its age:

Age i (years)	Operating cost C_i next year (\$)	Resale value V_i (\$)
0	400	
1	600	16,000
2	900	13,000
3	1300	9,000
4	1800	6,000
5	2400	4,000
6	3100	2,000
7	3900	1,000
8	4800	1,000
9	5800	1,000
10	7000	000

What is the optimal replacement strategy for the next 10 years, given that you begin with a new car, and at the end of the ten years, you receive the trade-in value for your current car?

DP Model #1

Define stages corresponding to the years, with

stage n = # years remaining in the planning period

state s = age of car at current stage

decision $x = 0$ if "keep", 1 if "replace"

$f_n(s)$ = minimum cost of last n years if the age of the car is currently s

Transition: If the age at stage n is s , then the age at stage $n-1$ is

$$\begin{cases} s+1 & \text{if } x = 0 \text{ ("keep")} \\ 1 & \text{if } x = 1 \text{ ("replace")} \end{cases}$$

Recursion:

$$f_n(s) = \min \{ C_s + f_{n-1}(s+1), p - V_s + C_0 + f_{n-1}(1) \}$$

where the first & second elements in the minimand correspond to the decisions "keep" & "replace", respectively.

APL Function Definition

```
▽ z←F N;operating_cost;OC;PC;TIV
[1]  A
[2]  A Value Function for deterministic auto replacement
[3]  A where s = age, x = [0, 1] = [keep, replace]
[4]  A
[5]  :if N=0    A Terminal conditions
[6]      z←(-(ρs)ρV),BIG
[7]      A Penalty appended to prevent infeasible states
[8]  :else
[9]      A recursion
[10]     OC←C[s|s°.x~x]    A Operating cost
[11]     PC←((ρs)ρPP)°.xx  A Purchase cost
[12]     TIV←V°.xx        A Trade-in value
[13]     z←MIN (OC+PC-TIV)+(F N-1)[TRANSITION 1+s°.x~x]
[14]  :endif
▽
```

---Stage 1---

s \ x:	0	1	Minimum
1	12400	11600	12400
2	8100	8600	8600
3	4700	4600	4700
4	2200	1600	2200
5	400	400	400
6	2100	2400	2100
7	2900	3400	2900
8	3800	3400	3400
9	5800	3400	3400

---Stage 2---

s \ x:	0	1	Minimum
1	8000	8000	8000
2	3800	5000	5000
3	900	1000	1000
4	2200	2000	2000
5	4500	4000	4000
6	6000	6000	6000
7	7300	7000	7000
8	8200	7000	7000

---Stage 3---

s \ x:	0	1	Minimum
1	4400	3600	4400
2	100	600	600
3	3300	3400	3300
4	5800	6400	5800
5	8400	8400	8400
6	10100	10400	10100
7	10900	11400	10900

---Stage 4---

s \ x:	0	1	Minimum
1	0	0	0
2	4200	3000	3000
3	7100	7000	7000
4	10200	10000	10000
5	12500	12000	12000
6	14000	14000	14000

---Stage 5---

s \ x:	0	1	Minimum
1	3600	4400	3600
2	7900	7400	7400
3	11300	11400	11300
4	13800	14400	13800
5	16400	16400	16400

---Stage 6---

s \ x:	0	1	Minimum
1	8000	8000	8000
2	12200	11000	11000
3	15100	15000	15000
4	18200	18000	18000

---Stage 7---

s \ x:	0	1	Minimum
1	11600	12400	11600
2	15900	15400	15400
3	19300	19400	19300

---Stage 8---

s \ x:	0	1	Minimum
1	16000	16000	16000
2	20200	19000	19000

---Stage 9---

s \ x:	0	1	Minimum
1	19600	20400	19600

---Stage 10---

s \ x:	0	1	Minimum
0	20000	22000	20000

*** Optimal value is 20000.00 ***

<u>stage</u>	<u>state</u>	<u>decision</u>
10	0	Keep
9	1	Keep
8	2	Replace
7	1	Keep
6	2	Replace
5	1	Keep
4	2	Replace
3	1	Keep
2	2	Replace
1	1	Keep
0	2	

DP Model #2

Define the optimal value function

$g(n)$ = minimum cost of last n years, if at the beginning of that time one has a new car.

We wish to determine $g(10)$.

Decision variable: y = # of years to keep car before replacement

Recursion:

$$g(n) = \min_{1 \leq y \leq n} \left\{ \sum_{i=0}^{y-1} C_i + (p - V_y) + g(n - y) \right\}, \quad n = 1, 2, \dots$$
$$g(0) = 0$$

APL Code for Recursive Computation

```
[0] Solve N;OC;RC;TV;z;m
[1]  A
[2]  A   Deterministic equipment replacement
[3]  A   Assumes machine is NOT replaced at final stage!
[4]  g+y+,0
[5]
[6]  :for n :in 1N
[7]    m+Max_AgeIn
[8]    OC++\m+C           A Operating cost
[9]    RC+(((m-1)pPP),0)-m+V   A Replacement cost
[10]   z+OC+RC+m+Φg       A Total
[11]   g+g,L/z           A Find minimum
[12]   y+y,z 1L/z       A Find minimizer
[13]  :endfor
```

Details of Solution

n = 1

y	OC	RC	g[°]	Total
1	400	-16000	0	-15600

Minimum is -15600 at y = 1

n = 2

y	OC	RC	g[°]	Total
1	400	4000	-15600	-11200
2	1000	-13000	0	-12000

Minimum is -12000 at y = 2

n = 3

y	OC	RC	g[°]	Total
1	400	4000	-12000	-7600
2	1000	7000	-15600	-7600
3	1900	-9000	0	-7100

Minimum is -7600 at y = 1

n = 4

y	OC	RC	g[°]	Total
1	400	4000	-7600	-3200
2	1000	7000	-12000	-4000
3	1900	11000	-15600	-2700
4	3200	-6000	0	-2800

Minimum is -4000 at y = 2

n = 5

y	OC	RC	g[°]	Total
1	400	4000	-4000	400
2	1000	7000	-7600	400
3	1900	11000	-12000	900
4	3200	14000	-15600	1600
5	5000	-4000	0	1000

Minimum is 400 at y = 1

n = 6

y	OC	RC	g[°]	Total
1	400	4000	400	4800
2	1000	7000	-4000	4000
3	1900	11000	-7600	5300
4	3200	14000	-12000	5200
5	5000	16000	-15600	5400
6	7400	-2000	0	5400

Minimum is 4000 at y = 2

n = 7

y	OC	RC	g[°]	Total
1	400	4000	4000	8400
2	1000	7000	400	8400
3	1900	11000	-4000	8900
4	3200	14000	-7600	9600
5	5000	16000	-12000	9000
6	7400	18000	-15600	9800
7	10500	-1000	0	9500

Minimum is 8400 at y = 1

n = 8

y	OC	RC	g[°]	Total
1	400	4000	8400	12800
2	1000	7000	4000	12000
3	1900	11000	400	13300
4	3200	14000	-4000	13200
5	5000	16000	-7600	13400
6	7400	18000	-12000	13400
7	10500	19000	-15600	13900
8	14400	-1000	0	13400

Minimum is 12000 at y = 2

n = 9

y	OC	RC	g[°]	Total
1	400	4000	12000	16400
2	1000	7000	8400	16400
3	1900	11000	4000	16900
4	3200	14000	400	17600
5	5000	16000	-4000	17000
6	7400	18000	-7600	17800
7	10500	19000	-12000	17500
8	14400	19000	-15600	17800
9	19200	-1000	0	18200

Minimum is 16400 at y = 1

n = 10

y	OC	RC	g[°]	Total
1	400	4000	16400	20800
2	1000	7000	12000	20000
3	1900	11000	8400	21300
4	3200	14000	4000	21200
5	5000	16000	400	21400
6	7400	18000	-4000	21400
7	10500	19000	-7600	21900
8	14400	19000	-12000	21400
9	19200	19000	-15600	22600
10	25000	0	0	25000

Minimum is 20000 at y = 2

Summary of Results

n	g(n)	y(n)
0	0	0
1	-15600	1
2	-12000	2
3	-7600	1
4	-4000	2
5	400	1
6	4000	2
7	8400	1
8	12000	2
9	16400	1
10	20000	2

Verify that this is the same replacement plan as that of the first model!

The second DP model, based upon a renewal process, requires significantly less computation than the first.

How can the stochastic nature of the real process be incorporated into each DP model?