Markov Chain Model of (s,S) Inventory System

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

Daily demand for an item is random, with the probability distribution:

d	0	1	2	3	4
P{D=d}	0.1	0.2	0.3	0.3	0.1

At the end of each day, the stock on hand is observed. If it exceeds **s** = 2 (the reorder point), no action is taken; otherwise, the inventory is replenished by an amount which brings the level up to **S** = 6 units at the beginning of the next day.

Questions

- What is the average stock-on-hand for this inventory system?
- What is the frequency of replenishments?
- What is the average number of days between stockouts?

Questions

If the initial stock-on-hand is 6,

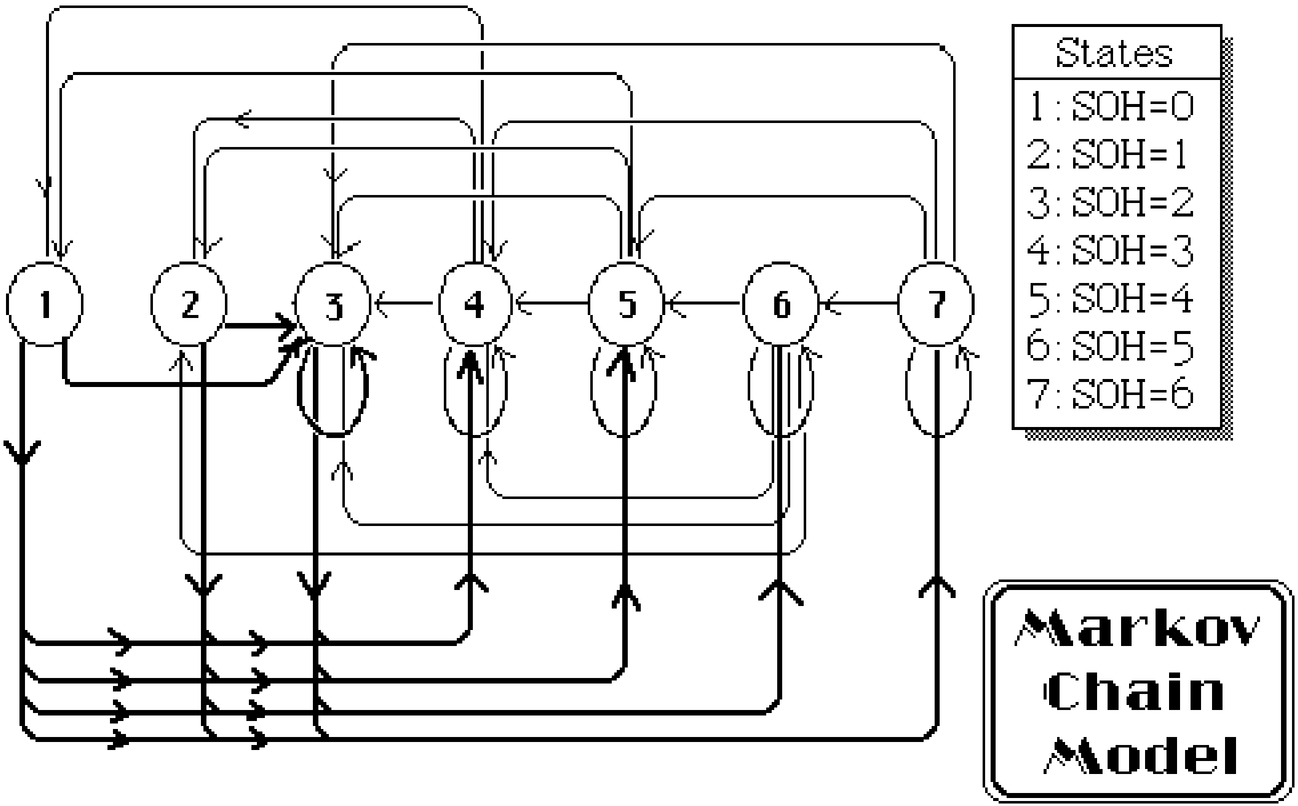
- what is the expected number of days until a stockout occurs?
- what is the probability that the first stockout occurs 5 days hence?
- what is the probability that a replenishment occurs 3 days hence?
- what is the expected number of stockouts during the next 30 days?
- what is the expected number of replenishments during the next 30 days?

- ☞ Markov chain model
- ☞ Simulation of the Markov chain
- ☞ Powers of the transition probability matrix
- ☞ Steadystate distribution
- ☞ Expected number of visits
- ☞ First-passage probabilities
- ☞ Mean first-passage time

Markov Chain Model

Define the state of the system according to the stock-on-hand (SOH) at the end of the day (before replenishment occurs)

X_n =	1	2	3	4	5	6	7
SOH=	0	1	2	3	4	5	6


1
2
3
4
5
6
7
States
1 : SOH=0
2 : SOH=1
3 : SOH=2
4 : SOH=3
5 : SOH=4
6 : SOH=5
7 : SOH=6
Markov
Chain
Model

Transition Probabilities

$$P_{ij} = P\{X_n = j \mid X_{n-1} = i\}$$

If i>3 (SOH>2), no replenishment occurs:

$$P_{ij} = \begin{cases} P\{D = (i-j)\} & \text{for } j>1 \text{ (SOH>0)} \\ P\{D \geq (i-j)\} & \text{for } j=1 \text{ (SOH=0)} \end{cases}$$

For example,

$$P_{42} = P\{D=2\} = 0.3$$

$$P_{41} = P\{D \geq 3\} = P\{D=3\}+P\{D=4\} = 0.3 + 0.1 = 0.4$$

Transition Probabilities

$$P_{ij} = P\{X_n = j \mid X_{n-1} = i\}$$

If i≤3 (SOH≤2), the SOH at the beginning of the next day is 6:

$$P_{ij} = P\{D = (6-[j-1])\}$$

For example,

$$P_{25} = P\{D = 2\} = 0.3$$

(s,S) system: s=2, S=6

Transition Probability Matrix

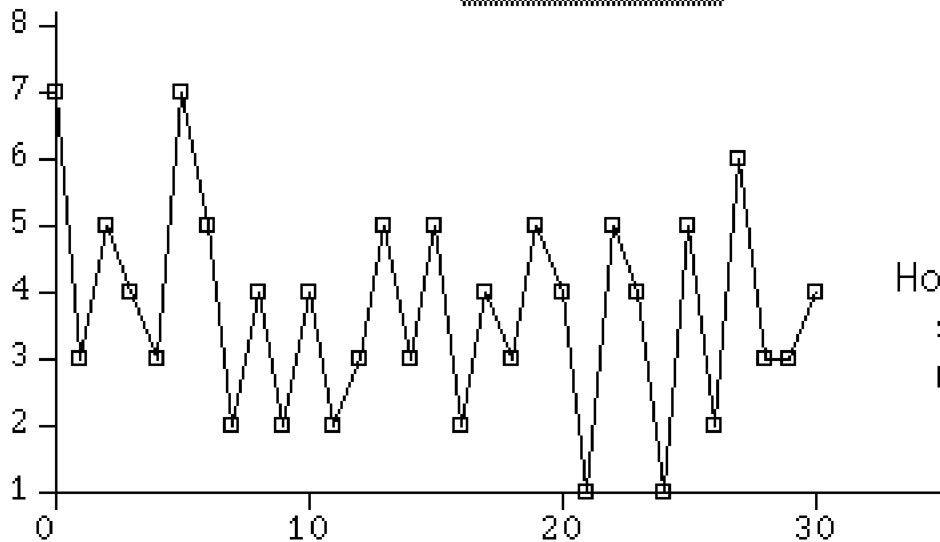
from \ to	1	2	3	4	5	6	7
1	0	0	0.1	0.3	0.3	0.2	0.1
2	0	0	0.1	0.3	0.3	0.2	0.1
3	0	0	0.1	0.3	0.3	0.2	0.1
4	0.4	0.3	0.2	0.1	0	0	0
5	0.1	0.3	0.3	0.2	0.1	0	0
6	0	0.1	0.3	0.3	0.2	0.1	0
7	0	0	0.1	0.3	0.3	0.2	0.1

States

i	name
1	SOH=0
2	SOH=1
3	SOH=2
4	SOH=3
5	SOH=4
6	SOH=5
7	SOH=6

Simulation of
30 days' operation

Repetition #1



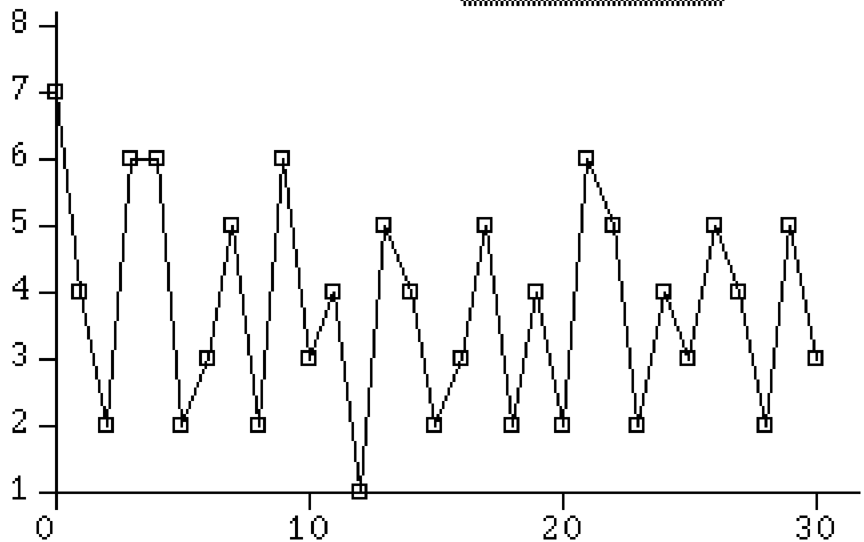
States

i	name
1	SOH=0
2	SOH=1
3	SOH=2
4	SOH=3
5	SOH=4
6	SOH=5
7	SOH=6

How many
stockouts?
replenishments?

SIMULATION

Repetition #2



States

i	name
1	SOH=0
2	SOH=1
3	SOH=2
4	SOH=3
5	SOH=4
6	SOH=5
7	SOH=6

How many
stockouts?
replenishments?

SIMULATION

Simulation results

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
7	5	2	5	3	6	6	4	1	3	6	2	4	2	6	3	5	3	6	2	5	5	2	5	4	1	6	4	1	3	3
7	5	3	5	3	3	5	1	7	5	2	4	1	4	2	4	3	4	1	5	5	4	4	1	4	1	5	2	5	1	4
7	5	4	2	3	6	2	5	4	4	1	4	2	6	6	5	1	5	4	1	7	5	1	4	4	1	6	2	5	2	5
7	4	2	6	5	5	2	3	6	5	1	4	2	5	2	4	4	1	6	4	1	5	2	4	3	5	3	4	1	4	1
7	4	1	6	4	1	3	5	5	2	5	4	3	5	1	5	3	5	1	3	5	4	2	7	4	3	7	7	3	6	5
7	6	3	6	3	5	5	5	3	4	2	7	4	1	4	1	4	1	5	2	4	2	7	6	5	1	6	4	1	6	3
7	7	4	1	7	6	6	2	5	2	7	5	3	4	3	4	4	1	3	5	3	6	5	1	3	3	5	4	1	4	1
7	6	4	1	5	2	6	4	2	6	5	3	4	1	5	2	4	3	5	2	6	5	1	6	6	2	5	2	5	1	4
7	6	4	2	5	1	3	4	2	5	2	5	4	1	4	2	5	1	4	1	4	1	5	1	6	4	1	5	3	3	7
7	4	3	4	2	4	4	3	3	4	3	5	2	3	5	2	3	4	1	4	1	4	1	6	3	5	2	3	5	1	7

10 simulations of 30 stages,
beginning in state #7
(Stock-on-hand=6)

2^{nd} Power

from \ to	1	2	3	4	5	6	7
1	0.15	0.2	0.23	0.21	0.13	0.06	0.02
2	0.15	0.2	0.23	0.21	0.13	0.06	0.02
3	0.15	0.2	0.23	0.21	0.13	0.06	0.02
4	0.04	0.03	0.11	0.28	0.27	0.18	0.09
5	0.09	0.09	0.14	0.25	0.22	0.14	0.07
6	0.14	0.16	0.19	0.22	0.16	0.09	0.04
7	0.15	0.2	0.23	0.21	0.13	0.06	0.02

3rd Power

from \ to	1	2	3	4	5	6	7
1	0.097	0.108	0.159	0.245	0.205	0.126	0.06
2	0.097	0.108	0.159	0.245	0.205	0.126	0.06
3	0.097	0.108	0.159	0.245	0.205	0.126	0.06
4	0.139	0.183	0.218	0.217	0.144	0.072	0.027
5	0.122	0.155	0.197	0.228	0.167	0.092	0.039
6	0.104	0.123	0.172	0.24	0.193	0.115	0.053
7	0.097	0.108	0.159	0.245	0.205	0.126	0.06

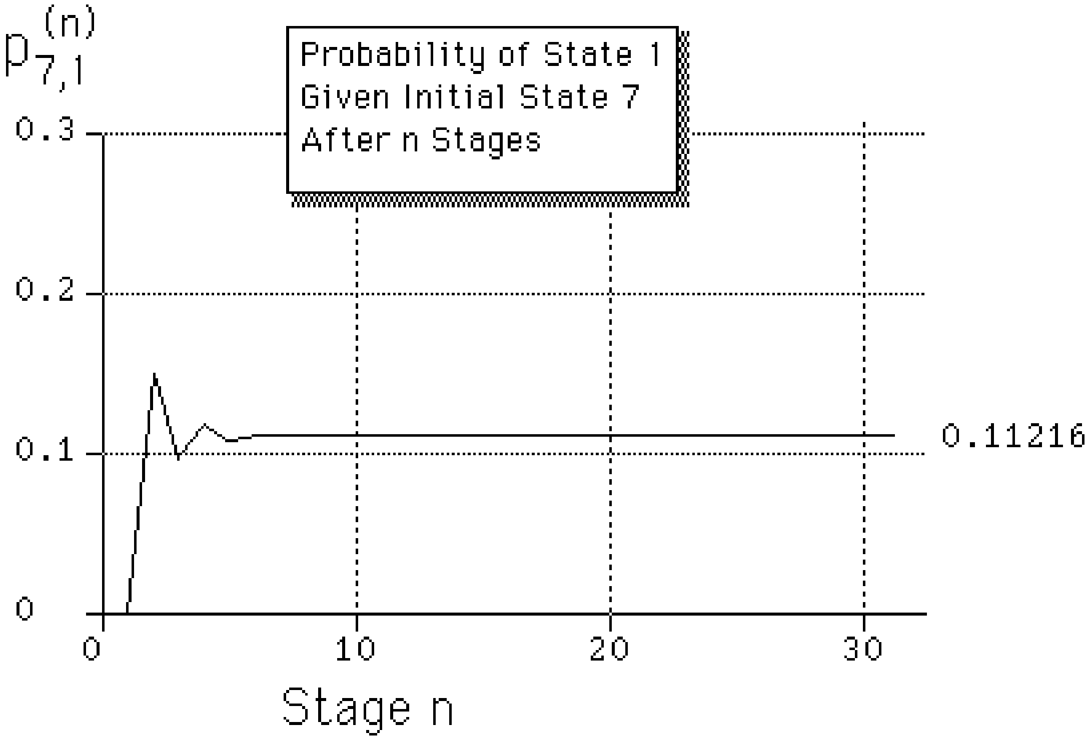
4^{th} Power

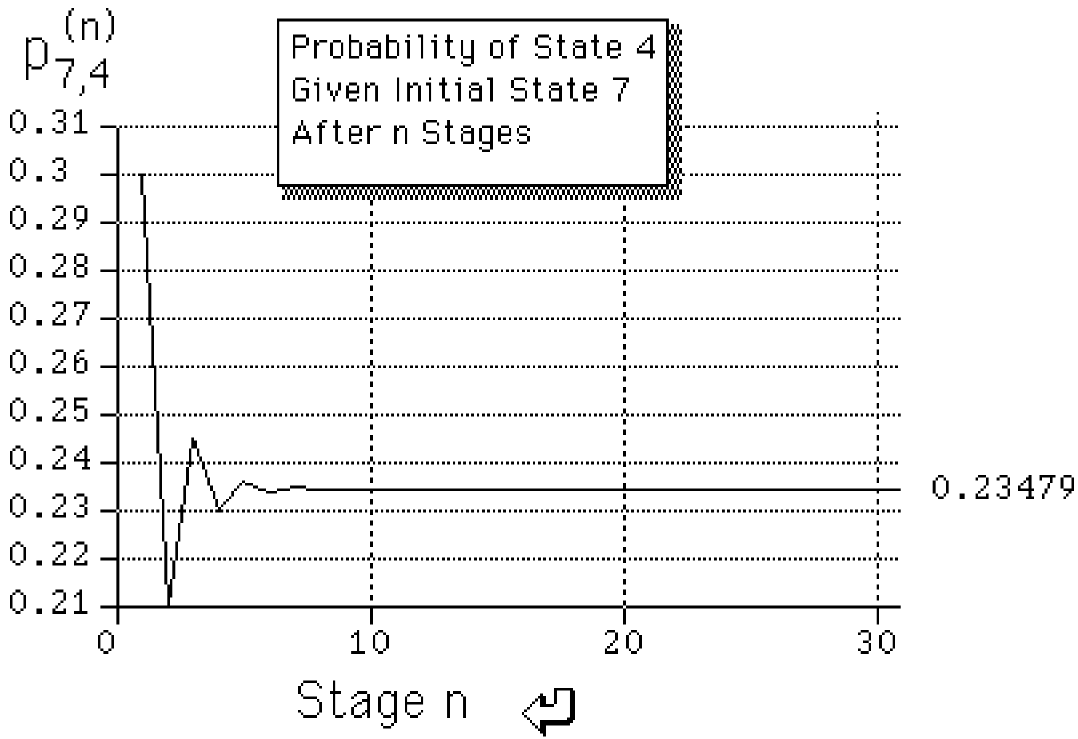
from \ to	1	2	3	4	5	6	7
1	0.1185	0.1476	0.1907	0.2305	0.1729	0.0974	0.0424
2	0.1185	0.1476	0.1907	0.2305	0.1729	0.0974	0.0424
3	0.1185	0.1476	0.1907	0.2305	0.1729	0.0974	0.0424
4	0.1012	0.1155	0.1649	0.2422	0.1989	0.1206	0.0567
5	0.1079	0.1277	0.1746	0.2377	0.189	0.1118	0.0513
6	0.1153	0.1414	0.1856	0.2327	0.1779	0.1019	0.0452
7	0.1185	0.1476	0.1907	0.2305	0.1729	0.0974	0.0424

30th Power

P^{30}

from \ to	1	2	3	4	5	6	7
1	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
2	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
3	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
4	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
5	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
6	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678
7	0.11216	0.13578	0.18116	0.23479	0.18247	0.10595	0.047678


$p_{7,1}^{(n)}$
Probability of State 1
Given Initial State 7
After n Stages
0.3
0.2
0.1
0
0.11216
0
10
20
30
Stage n



$p_{7,4}^{(n)}$
Probability of State 4
Given Initial State 7
After n Stages
0.31
0.3
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
0.23479
0
10
20
30
Stage n

Steady State Distribution π

i	name	P{i}
1	SOH=0	0.11216
2	SOH=1	0.13578
3	SOH=2	0.18116
4	SOH=3	0.23479
5	SOH=4	0.18247
6	SOH=5	0.10595
7	SOH=6	0.047678

Average Stock-on-Hand $\sum_{i=1}^{7} (i-1)\pi_i$

i	State	Pi	C	Pi×C
1	SOH=0	0.11216	0	0
2	SOH=1	0.13578	1	0.13578
3	SOH=2	0.18116	2	0.36233
4	SOH=3	0.23479	3	0.70438
5	SOH=4	0.18247	4	0.72989
6	SOH=5	0.10595	5	0.52976
7	SOH=6	0.04767	6	0.28607

The average cost/period in steady state is 2.7482

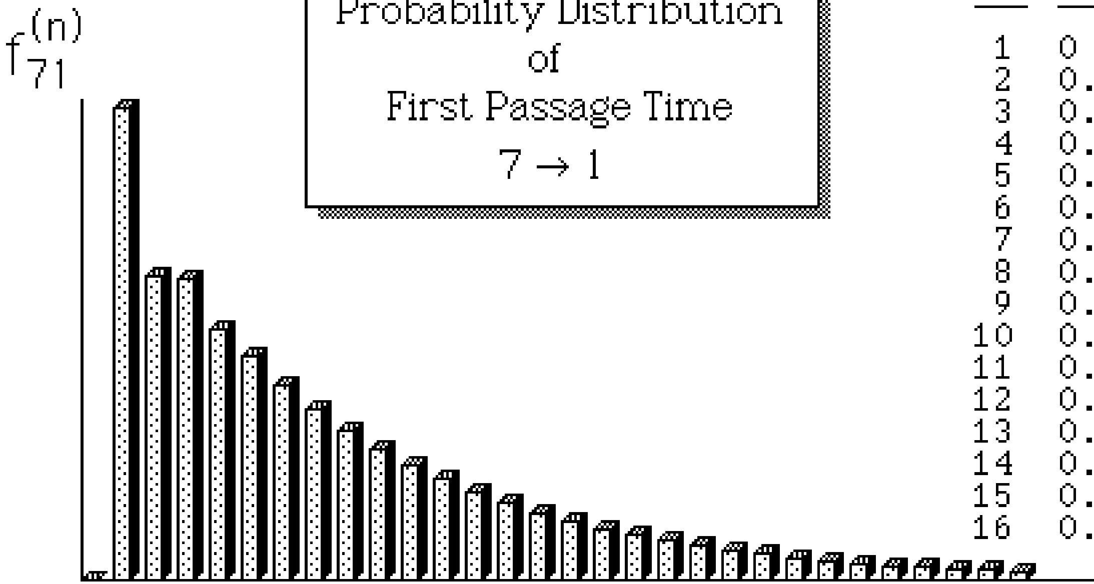
(Here, "cost" = SOH)

Expected no. of visits during first 30 stages

$$\sum_{n=1}^{30} p_{ij}^{(n)}$$

from \ to	1	2	3	4	5	6	7
1	3.2799	3.9822	5.3871	7.0914	5.555	3.2407	1.4636
2	3.2799	3.9822	5.3871	7.0914	5.555	3.2407	1.4636
3	3.2799	3.9822	5.3871	7.0914	5.555	3.2407	1.4636
4	3.5997	4.1647	5.408	6.9416	5.3523	3.123	1.4106
5	3.3375	4.2052	5.5238	7.0195	5.4183	3.0968	1.3989
6	3.2747	4.0529	5.5565	7.098	5.4765	3.1629	1.3786
7	3.2799	3.9822	5.3871	7.0914	5.555	3.2407	1.4636

Probability Distribution of First Passage Time $7 \rightarrow 1$



n	$f_{71}^{(n)}$
1	0
2	0.15
3	0.097
4	0.096
5	0.08039
6	0.071713
7	0.062397
8	0.054822
9	0.047983
10	0.04206
11	0.036847
12	0.032287
13	0.028289
14	0.024787
15	0.021718
16	0.019029

(s,S) system: s=2, S=6

m_{ij}

Mean First Passage Times

from \ to	1	2	3	4	5	6	7
1	8.9155	7.3651	5.5199	3.6212	4.7312	8.7037	20.974
2	8.9155	7.3651	5.5199	3.6212	4.7312	8.7037	20.974
3	8.9155	7.3651	5.5199	3.6212	4.7312	8.7037	20.974
4	6.0641	6.0212	5.4043	4.2591	5.8423	9.8148	22.085
5	8.4023	5.7225	4.7653	3.9276	5.4803	10.062	22.332
6	8.9621	6.8449	4.5848	3.5933	5.1613	9.4383	22.757
7	8.9155	7.3651	5.5199	3.6212	4.7312	8.7037	20.974

States

i	name
1	SOH=0
2	SOH=1
3	SOH=2
4	SOH=3
5	SOH=4
6	SOH=5
7	SOH=6