56:272 Integer Programming & Network Flows
Final Examination -- December 14, 1998

Part A: Answer any **four** of the **five** problems. *(15 points each)*

1. Transportation problem ____
2. Integer LP Model Formulation ____
3. Balas' Additive Algorithm (Implicit Enumeration) ____
4. Generalized Assignment Problem ____
5. Benders' Decomposition of Capacitated Plant Location Problem ____

Part B: Answer any **seven** of the **eight** problems. *(10 points each)*

6. Branch-&-bound solution of knapsack problem ____
7. Asymetric Traveling salesman problem ____
8. Primal Simplex Algorithm for Networks ____
9. Gomory's Cutting Plane Algorithm ____
10. Chinese Postman Problem ____
11. Vertex Penalty Method for TSP ____
12. Shortest Path Problem ____
13. Location in a network ____

Total possible: 130 points

✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ **PART A** ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠

1. Transportation Problem: Consider the transportation problem with the tableau below:

	A	B	C	D	E	supply
F	6	3	4	1	2	4
G	4	2	5	6	3	7
H	5	1	7	2	4	9
demand	5	4	4	5	2	

a. If the ordinary simplex tableau were to be written for this problem, it would have ___ non-redundant constraint rows, not including the objective row, and ____ columns (not including -z and the right-hand-side).

b. This problem will have ____ basic variables (not including $\bar{}$z).

c. Find an initial basic feasible solution using the "Northwest Corner Method". *(Write the values of the variables in the tableau above.)*

d. What are the values of the dual variables for the solution in (c)? *(Note that* $\boldsymbol{V_A}$ *has been assigned the value zero)* U_F = ____ , U_G = ____ , U_H = ____ ,
V_A = __0_ , V_B = ____ , V_C = ____ , V_D = ____ , V_E = ____ .

e. What is the reduced cost of the variable X_{FD}? ___________

f. Will increasing X_{FD} improve the objective function? ___________

g. Regardless of whether the answer to (f) is "yes" or "no", what variable must leave the basis if X_{FD} enters?_________

h. What will be the value of X_{FD} if it is entered into the solution as in (g)? ___________

i. Which variables (if any), if it were entered into the solution, would result in a degenerate solution? _______

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2. Integer LP Model Formulation The board of directors of a large manufacturing firm is considering the set of investments shown below: Let R_i be the total revenue from investment i and C_i the cost (in

$millions) to make investment i. The board wishes to maximize total revenue and invest no more than a total of 50 million dollars.

Investment i	Revenue R_i	Cost C_i	Condition
1	1	5	None
2	2	8	Only if #1
3	3	12	Only if #2
4	4	18	Must if #1 and #2
5	5	24	Not if #3 or #4
6	6	27	Not if both #1 and #3
7	7	30	Only if #4 and not #1

Define $X_i = 1$ if investment i is selected, else 0.

a. Formulate this problem without the "side conditions" as an integer LP.

b. This is an example of a special class of integer programming problems called "___________" problems.

c. Add a constraint to enforce the condition "Investment #2 only if #1 is selected".

d.   Add a constraint to enforce the condition "Investment #3 only if #2 is selected".

e. Add a constraint to enforce the condition "Investment #4 must be selected if both #1 & #2 are selected".

f. Add a constraint to enforce the condition "Investment #5 cannot be selected if both #3 & #4 are selected".

g. Add a constraint to enforce the condition "Investment #6 cannot be selected if both #1 & #3 are selected".

h. Add a constraint to enforce the condition "Investment #7 only if #4 is selected but #1 is not selected".

i. Suppose that investments 3 and 4 are in products which compete against one another to some extent, so that 1 must be subtracted from the total revenue if <u>both</u> of these two investments are selected. Reformulate the problem (without the "side constraints" above.) *Define any new variable(s) which you use.*

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3. Balas' Additive Algorithm (Implicit Enumeration) Consider the problem

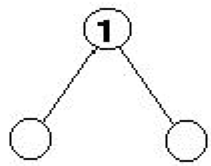
$$\begin{array}{ll} \text{Maximize} & X_1 + 8X_2 + 3X_3 - 5X_4 - 3X_5 \\ \text{subject to} & -2X_1 - X_2 + 2X_3 + 3X_4 - 3X_5 \quad 1 \\ & \quad X_2 - 2X_3 - 4X_4 - 2X_5 \quad -1 \\ & 2X_1 + 4X_3 + 3X_4 \quad 1 \\ & X_i = 0 \text{ or } 1,\ i=1,2,\ldots 5 \end{array}$$

a. Convert the problem to the standard form, i.e.,

$$\text{Minimize } \Sigma\, c_i x_i$$
$$\text{subject to } Ax \quad b$$
$$X_i = 0 \text{ or } 1$$

with $c_i \quad 0$ for all i

b. Suppose you are now at the node represented by J = {5, -2, 3} (where indices are of the variables in the standard form you found in (a).) In a search tree diagram, indicate the corresponding node.



Which are the free variables? _______________
Which are variables fixed at 0? _______________
Which are variables fixed at 1? _______________

Indicate any nodes which have already been fathomed with ⊗

c. Suppose that the current incumbent has cost equal to +6. Try to fathom the current node above by the appropriate fathoming tests of Balas' algorithm. Can it be fathomed?

d. Regardless of your answer in (c), suppose that the node is fathomed. Which node is to be considered next? J = { }

Indicate the node in the search tree diagram above with an ⊕.

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4. Generalized Assignment Problem: Consider the problem of assigning 6 jobs to 3 machines (each with limited capacity):

Resources used (a_{ij}):

Machine i	Job j: 1	2	3	4	5	6	Available b
1	18	22	24	21	24	16	20
2	23	21	18	15	24	6	27
3	15	21	14	15	13	14`	38

Costs (C_{ij}):

Machine i	Job j: 1	2	3	4	5	6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

a. Formulate this problem as a binary integer programming problem.

b. Suppose that the integer restrictions are relaxed and the problem were solved by the simplex LP algorithm. Will the optimal values of the variables of this LP relaxation necessarily be integer? ____

c. Suppose that the machine capacity constraints are relaxed, using the Lagrangian relaxation method with multipliers U_1, U_2, and U_3. The first 2 iterations of the subgradient optimization method to maximize the lower bound appears below, where the optimal value was estimated to be 120, and a stepsize parameter was assigned the value 0.75.

Lambda = 0.75
Upper bound Z* = 120

Iteration # 1

Multiplier vector U = 0 0 0
Objective function of relaxation:

machine \ job	1	2	3	4	5	6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

Dual value is [A]
Variables selected from GUB sets are:
1 3 1 1 2 1
Resources used are: 79 24 21, (Available: 20 27 38)
Subgradient of Dual Objective is [B | C | D]
Stepsize is 0.00861821

Multiplier vector U = 0.508475 0 0
Objective function of relaxation:

machine \ job	1	2	3	4	5	6
1	E	29.18	23.20	27.67	36.20	21.13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

Dual value is 77.8305
Variables selected from GUB sets are:
[F | G | H | I | 2 | J]
Resources used are: 0 53 50, (Available: 20 27 38)
Subgradient of Dual Objective is ¯20 26 12
Stepsize is [K]

d. Several values have been omitted from the output. ("Variables selected from GUB sets" refers to the machine selected for each of the jobs.) Compute their values:

A _____ B _____ C _____ D ______ E _____
F _____ G _____ H _____ I ______ J _____
K _____

e. What is the "integrality property" of a Lagrangian relaxation?

f. Does this particular Lagrangian relaxation have the integrality property? *Circle:* Yes No

g. What does your answer in (f) imply about the strength of the lower bound which can be obtained from this relaxation, compared to that of the LP relaxation in (b)? (Is it ≥ , ≤ , or = ?)

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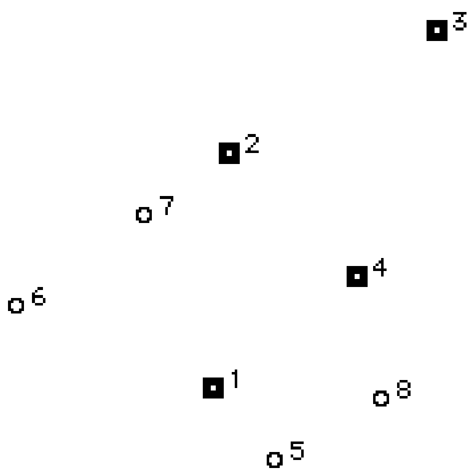
5. Benders' Decomposition of Capacitated Plant Location Problem

Consider the following problem in which demand in 8 cities is to be satisfied by plants to be built in one or more of cities 1,2,3, & 4:




```
Number of sources      = M = 4
Number of destinations = N = 8
Total demand: 40

K = capacity,
F = fixed cost
```

Costs, Supplies, Demands

i\j	1	2	3	4	5	6	7	8	K	F
1	0	42	75	33	17	38	33	30	19	498
2	42	0	43	32	56	47	19	52	15	23
3	75	43	0	46	82	90	62	67	17	89
4	33	32	46	0	36	61	40	22	17	129
Demand	6	1	2	5	5	8	3	10	68	

a. State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

A trial solution was evaluated, in which all four plants are to be open. The result was:

```
Subproblem Solution

Plants opened: # 1 2 3 4

Minimum transport cost = 666
Fixed cost of plants = 739
       Total = 1405
```

```
Generated support is αY+b, where
 α = 498 23 89 129
 & b = 666
That is, v(Y) ≥ αY+b

This is support # 1
```

The Master Problem was next optimized. (A constraint $\sum_{i=1}^{4} K_i Y_i \geq \sum_{j=1}^{8} D_j$ *was included in the master problem in order to guarantee that only solutions with sufficient capacity to meet the demand were produced.) The result was:*

```
Optimum of Master Problem

Optimal set of plants: 2 3 4
        with estimated cost [      ]
```

b. What is the value of the estimated cost of Y=(0,1,1,1) found by the Master problem ("blanked") above?

Next, the subproblem was solved, using the trial set of plants {2,3,4}, with the following results:

```
Subproblem Solution

Plants opened: # 2 3 4

Minimum transport cost = 1319
Fixed cost of plants = [      ]
               Total = [      ]
```

```
Generated support is αY+b, where
 α = 498 653 1364 639
 & b = ¯1096
That is, v(Y) ≥ αY+b

This is support # 2
```

c. What are the two values blanked above?

When the Master problem is solved once more, the result is:

```
Optimum of Master Problem

 Optimal set of plants: 1 2 4

         with estimated cost [      ]
```

d. Compute the estimated cost found by the master problem.

The subproblem is again solved, this time with trial set {1,2,4}, yielding:

```
Subproblem Solution

Plants opened: # 1 2 4

Minimum transport cost =  752
Fixed cost of plants   =  650
                 Total = 1402
```

```
Generated support is αY+b, where
 α = 1144 668 89 [      ]
 & b = ¯1270
That is, v(Y) ≥ αY+b

This is support # 3
```

e. Using the information below from the subproblem solution, compute the blanked value of α_4 above.

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	6	0	0	0	5	8	0	0	0
2	0	1	2	0	0	0	3	0	9
4	0	0	0	5	0	0	0	10	2

Dual Variables

Supply constraints

i=	1	2	3	4
U[i]=	34	43	0	43

Demand constraints

j=	1	2	3	4	5	6	7	8
V[j]=	¯34	¯43	0	¯43	¯17	4	¯24	¯21

(Demand pt #9 is dummy demand for excess capacity.)

Reduced costs: COST - U∘.+V

0	51	41	42	0	0	23	17
33	0	0	32	30	0	0	30
109	86	0	89	99	86	86	88
24	32	3	0	10	14	21	0

f. The solution of the transportation problem (the shipments) is degenerate. Based upon the information given, which variable(s) is/are basic but zero in the solution?

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6. Knapsack Problem: A "knapsack" is to be filled to maximize the value of the contents, subject to a weight restriction. A maximum of 1 unit of each item may be included.

Number of items: 4

Capacity of knapsack: 10

Item	Weight (kg)	Value ($)
1	3	5
2	5	9
3	2	3
4	4	7

Sorted by $/kg:

Item	Wt (kg)	Value($)	Value/Weight
2	5	9	1.8
4	4	7	1.75
1	3	5	1.66667
3	2	3	1.5

a. Formulate this problem as a 0-1 ILP problem:

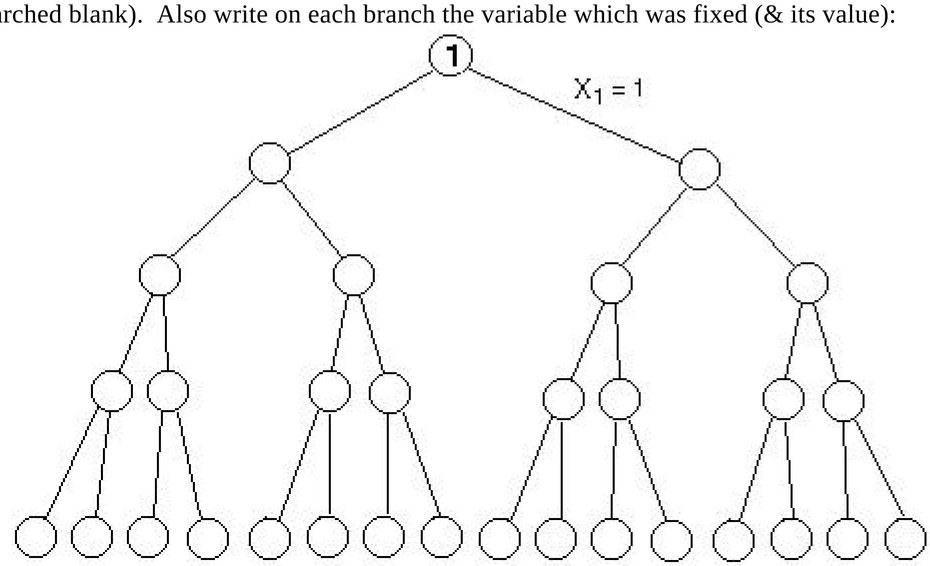
Output of the branch-&-bound algorithm for this problem appears below:

```
>>>Subproblem # 1
Forced in:
Forced out:
Free:         1  2  3  4
Fractional solution: selected items = 2 4
                    plus [        ] of item #[  ]
                    value = 17.6667
Rounding down yields value 16
*** NEW INCUMBENT! ***
   >>>Subproblem # 2
   Forced in:    1
   Forced out:
   Free:         2  3  4
   Fractional solution: selected items = 1 2
                       plus 0.5 of item # 4
                       value = 17.5
   Rounding down yields value 14
      >>>Subproblem # 3
      Forced in:    1  4
      Forced out:
      Free:         2  3
      Fractional solution: selected items = 1 4
                          plus 0.6 of item # 2
                          value = 17.4
      Rounding down yields value 12
         >>>Subproblem # 4
         Forced in:    1  2  4
         Forced out:
         Free:         3
         Infeasible!
         <<<Subproblem # 4 fathomed.
         >>>Subproblem # 5
         Forced in:    1  4
         Forced out:   2
         Free:         3
         Integer solution: selected items = 1 3 4
                             Value= 15
         <<<Subproblem # 5 fathomed.
```

```
    <<<Subproblem # 3 fathomed.
    >>>Subproblem # 6
    Forced in:    1
    Forced out:   4
    Free:         2  3
    Integer solution: selected items = 1 2 3
                           Value= 17
    *** NEW INCUMBENT! ***
    <<<Subproblem # 6 fathomed.
  <<<Subproblem # 2 fathomed.
  >>>Subproblem # 7
  Forced in:
  Forced out:   1
  Free:         2  3  4
  Fractional solution: selected items = [__________]
                       plus [____] of item # [____]
                       value = [_____]
  Rounding down yields value 16
  <<<Subproblem # 7 fathomed.
<<<Subproblem # 1 fathomed.
```

b. Complete the blanks in the output above.

c. Below is a "complete" search tree. Using the information in the above output, complete the numbering of the nodes which were explicitly searched (leaving other nodes which were implicitly searched blank). Also write on each branch the variable which was fixed (& its value):



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7. Asymmetric Traveling Salesman Problem. Five products are to be manufactured weekly on the same machine. The table below gives the cost of switching the machine from one product to another product. (Assume that this is also the cost of switching to the last product of the week to the first product to be scheduled the following week!)

	to:	A	B	C	D	E
from:	A	-	3	5	3	6
	B	2	-	3	2	8
	C	6	3	-	6	7
	D	3	1	4	-	8

E	6	4	2	3	-

a. The nearest neighbor heuristic, starting with product A, yields the product sequence
A →___→___→___→___→ A with cost ____.

After applying the "Hungarian Method" to the above matrix to solve the associated assignment problem (with large number inserted along the diagonal), we have:

from:	to:	A	B	C	D	E
	A	∞	3	3	0	0
	B	0	∞	2	0	3
	C	2	0	∞	2	0
	D	1	0	3	∞	3
	E	3	2	0	3	∞

b. What is the solution of this assignment problem?

___ →___
___ →___
___ →___
___ →___
___ →___

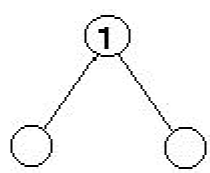
c. What is its cost? _________

d. Is it a valid product sequence? _______ If not, why not? _________________________

e. If not a valid sequence, what bound (circle: upper / lower) on the optimal cost does this result provide? ________

f. If not a valid sequence, what single constraint might be added to the assignment problem to eliminate the solution which you have obtained (but not eliminate any valid sequence)?

g. If the assignment problem does not yield a valid sequence, how might we branch to create subproblems in a branch and bound method? *(That is, specify the immediate descendents of the original problem below. Add additional descendents if more than two immediate descendents.)*



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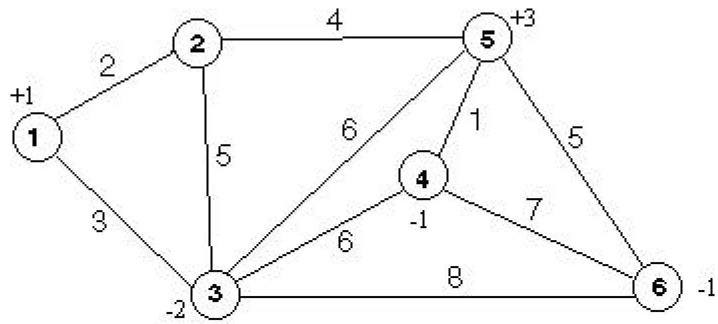
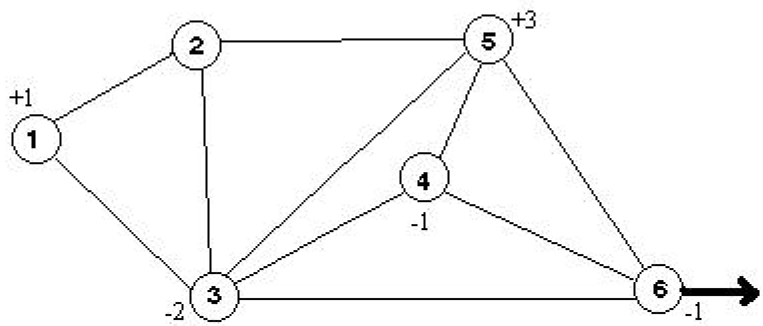
8. Primal Simplex Algorithm for Networks. Consider the undirected network below, where the number alongside each node represents supply or demand, i.e., node #1 has a supply of 1 units of a commodity, node #5 has 3 units, node #3 requires 2 units, while nodes 4 and 6 each require 1 unit. The numbers alongside the arcs represent unit shipping costs.

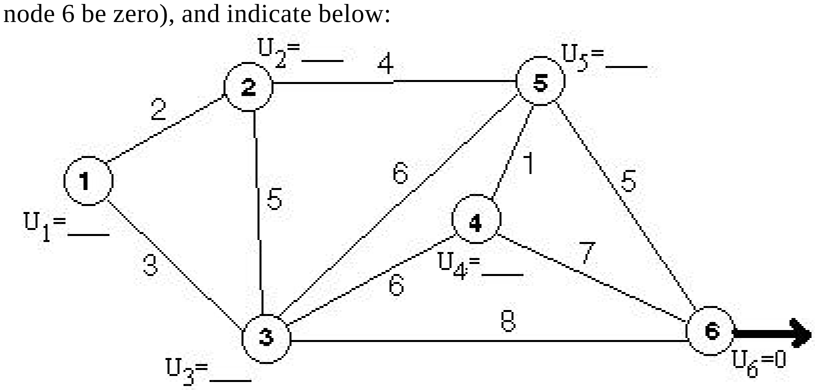


a. Find the minimum spanning tree of this network, and indicate it below.



b. Using the minimum spanning tree (plus artificial "root" arc at node 6) as an initial basis, compute the corresponding basic solution, i.e., flows. Indicate these flows above (indicating <u>direction</u> of the flow).

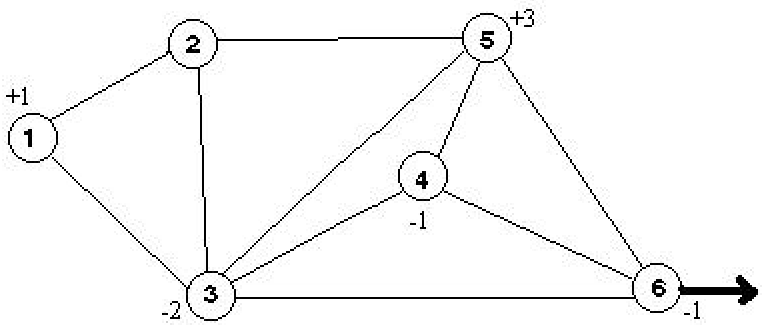
c. Using the same basis, compute the dual variables (simplex multipliers) for each node (letting the dual variable for node 6 be zero), and indicate below:



d. Choose one arc <u>not</u> in the rooted spanning tree, and "price" it, i.e., compute its reduced cost. *(Choose both an edge <u>and</u> a direction!)*

Would entering this arc into the basis result in an improvement? ____

e. Regardless of whether the arc you selected in (d) should enter the basis, explain how to enter the arc into the basis and how to choose the arc leaving the basis. Indicate the new basis on the network below:



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9. Gomory's Cutting Plane Algorithm. Consider the problem

Maximize $3X_1 + 5X_2$

subject to $-X_1 + X_2 \quad 5$

$9X_1 + 5X_2 \quad 45$

$X_1, X_2 \quad 0$, & integer

After adding slack & surplus variables X_3 & X_4, respectively, and solving the LP relaxation, we get the optimal tableau:

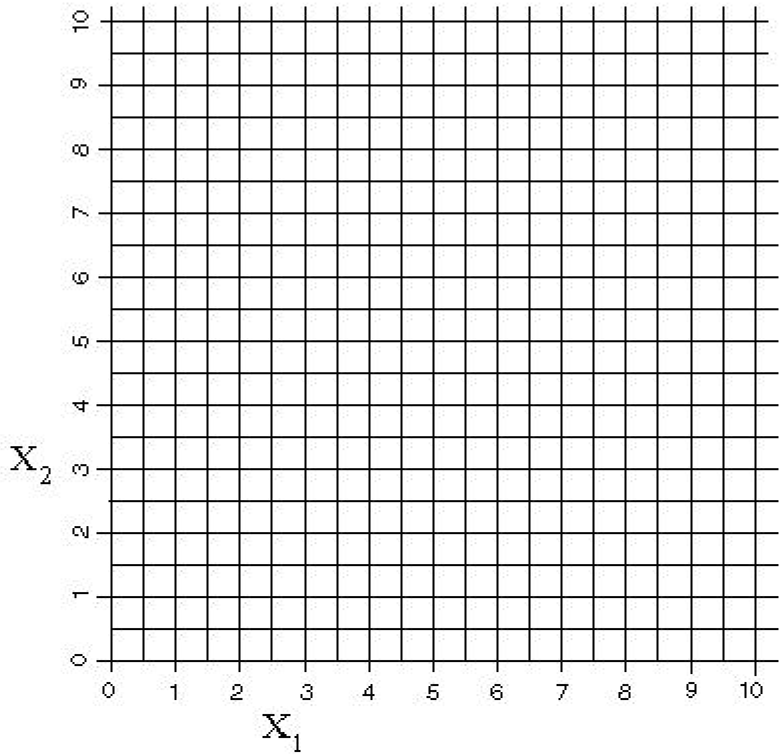
X_1	X_2	X_3	X_4	rhs	
0	0	0.571	2.143	36.429	(max)
0	1	0.071	0.643	6.429	
1	0	0.071	-0.357	1.429	

a. Using the bottom row of the tableau, state a constraint that may be added to the problem to exclude this extreme point of the feasible region of the LP relaxation without excluding any integer feasible solutions.

b. Express the constraint which you found in (a) in terms of the original variables X_1 and X_2:

_____ X_1 + ______ $X_2 \geq$ _____

c. Graph, in (X_1,X_2)-space, the original constraints, the optimum of the LP relaxation, and the new constraint which you chose in (b). Shade the feasible region after adding this constraint.



c. Add the new constraint in (b) to the tableau, and indicate by circling the next pivot:

	X_1	X_2	X_3	X_4	____	rhs
	0	0	0.571	2.143	36.429	(max)
	0	1	0.071	0.643	6.429	
	1	0	0.071	-0.357	1.429	
added row:						

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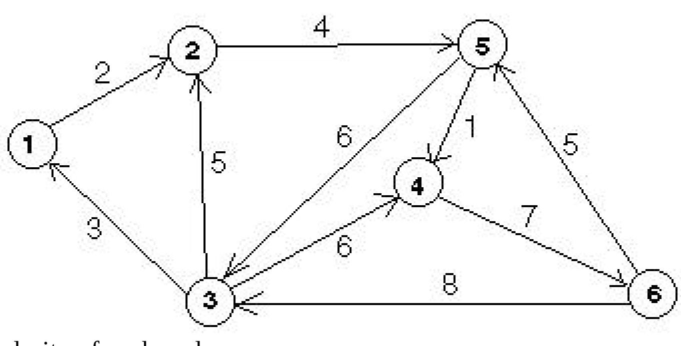
10. Chinese Postman Problem: Consider the directed graph below:



a. Compute the polarity of each node:

node:	1	2	3	4	5	6
polarity:	____	_____	_____	_____	_____	_____

b. Does this directed network possess an Euler tour?______ ... an Euler path? ______

c. In order for a delivery vehicle to travel each arc at least once with the minimum total distance traveled, which arcs should be traveled more than once?

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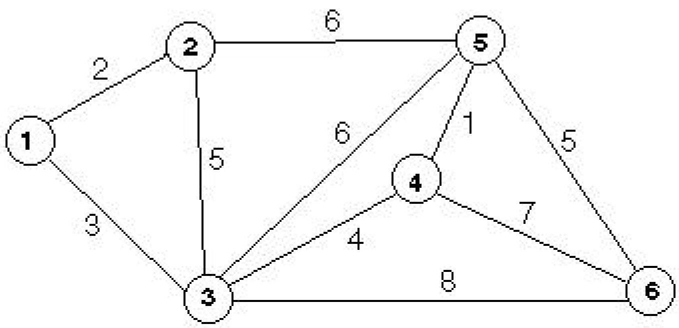
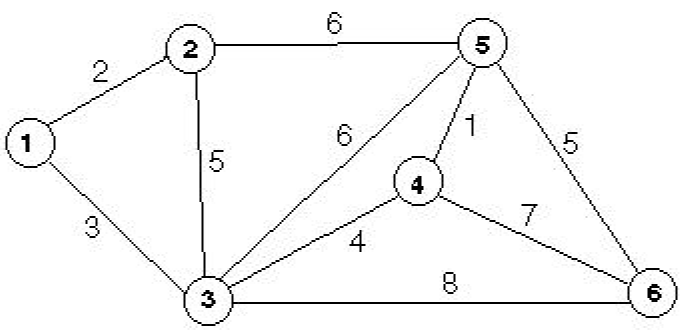
11. Vertex Penalty Method. Consider the network:



a. Indicate the minimum spanning tree of this network.

b. Indicate alongside each node its degree in the spanning tree.

c. Suppose that, using the vertex penalty method, we wish to find the shortest path (not tour) starting at node 1, which passes through each node of this network. Perform one iteration of this method, using a unit penalty of 1, and indicate the new spanning tree below:



Is this spanning tree a path?

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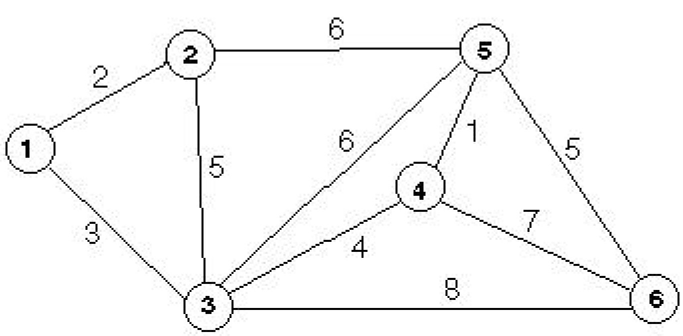
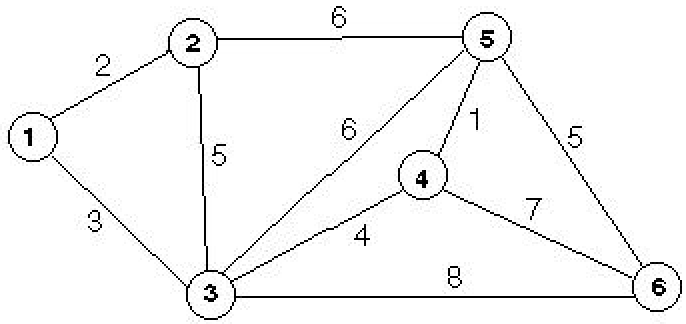
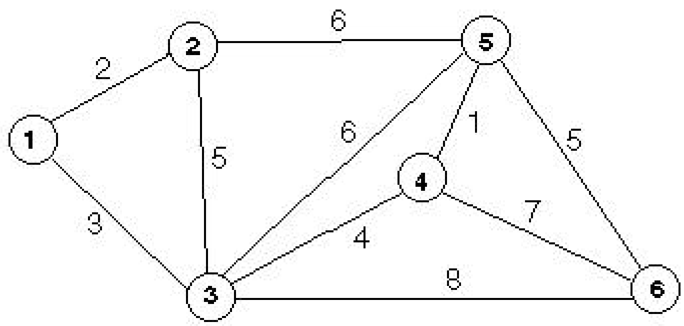
12. Shortest Path Problem. Consider again the network:



a. It is desired to find the shortest path from node **1** to node **6**, using the labeling algorithm of Dijkstra. Indicate the labels after the **first** iteration below:

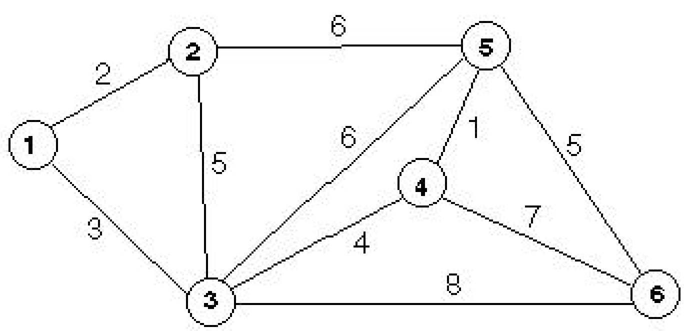


b. Indicate the labels after the second iteration below:



✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠ ✠

13. Location in a network: Consider again the network,



where the numbers on the edges are distances. Consider the demand at each node to be 1. The table of shortest path lengths found by Floyd's algorithm is:

4	2	3	7	8	11
2	4	5	7	6	11
3	5	6	4	5	8
7	7	4	2	1	6
8	6	5	1	2	5
11	11	8	6	5	10

a. At which node is the median (1-median) of the network? ___

b. What is the objective function value of the median problem at this node? ____

c. Consider the 2-median problem. What is the objective function value at the solution with nodes 3 and 4 selected? _____

d. Which node is the vertex center (node center) of the network? ____

e. What is the objective function of the center problem at this node? ____