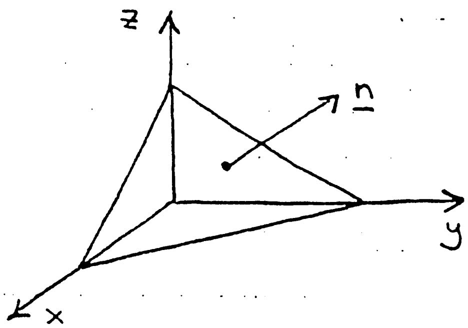
5-1. Isotropic Property of Pressure

Show that pressure is an isotropic stress.

Stress tetrahedron:



S_o = area of slanted surface (with unit normal $\underline{n}$)

P_i = pressure on surface normal to $\underline{e}_i$

P = pressure on slanted surface

What is desired is to show that $P_x = P_y = P_z = P$.
Start with stress equilibrium, eq. (5.3-16):

$$\lim_{S\to 0} \frac{1}{S} \int_S \underline{s}(\underline{n})\, dS = 0$$

$$S_i = \text{area of surface normal to } \underline{e}_i = (\underline{n}\cdot\underline{e}_i)\, S_o \qquad \text{(eq. (5.3-19))}$$

$$\lim_{S\to 0} \frac{S_o}{S}\left[s(\underline{n}) + \sum_i (\underline{n}\cdot\underline{e}_i)\, s(-\underline{e}_i)\right] = 0 \qquad \text{(eq. (5.3-20))}$$

With pressure the only stress in a static fluid,

$$\underline{s}(\underline{n}) = -P\,\underline{n} \quad , \quad \underline{s}(\underline{e}_i) = -P_i\,\underline{e}_i$$

From eq. (5.3-18),

$$\underline{s}(-\underline{e}_i) = -\underline{s}(\underline{e}_i)$$

$$
\begin{aligned}
0 &= \underline{s}(\underline{n}) + \sum_i (\underline{n}\cdot\underline{e}_i)\,\underline{s}(-\underline{e}_i) \\
&= \underline{s}(\underline{n}) - \sum_i (\underline{n}\cdot\underline{e}_i)\,\underline{s}(\underline{e}_i) \\
&= \underline{s}(\underline{n}) - \sum_i n_i\,\underline{s}(\underline{e}_i) \\
&= -P\,\underline{n} - \sum_i n_i\,(-P_i\,\underline{e}_i)
\end{aligned}
$$

$$P\,\underline{n} = \sum_i P_i\, n_i\, \underline{e}_i$$

$$P \sum_i n_i\, \underline{e}_i = \sum_i P_i\, n_i\, \underline{e}_i$$

This will hold for arbitrary $\underline{n}$ only if $P_i = P$. Q.E.D.

5-2. Transport of Vorticity

(a) Starting with the Navier-Stokes eq., show that

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\underline{\nabla}\underline{v} + \nu\nabla^2\underline{\omega} \quad \text{where} \quad \underline{\omega} = \underline{\nabla}\times\underline{v}.$$

Expanding $D\underline{v}/Dt$ and dividing by ρ, the N-S eq. [eq. 5.8-2] is

$$\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\underline{v} = -\frac{1}{\rho}\underline{\nabla}\mathcal{P} + \nu\nabla^2\underline{v}$$

Take the curl $(\underline{\nabla}\times)$ of each term:

$$\underline{\nabla}\times\frac{\partial\underline{v}}{\partial t} = \frac{\partial}{\partial t}(\underline{\nabla}\times\underline{v}) = \frac{\partial\underline{\omega}}{\partial t}$$

$$\underline{\nabla}\times(\underline{v}\cdot\underline{\nabla}\underline{v}) = \;? \qquad \text{[see below]}$$

$$\underline{\nabla}\times\left(\frac{1}{\rho}\underline{\nabla}\mathcal{P}\right) = \frac{1}{\rho}\underline{\nabla}\times\underline{\nabla}\mathcal{P} = \underline{0} \qquad \text{[(7) of Table A-1]}$$

$$\underline{\nabla}\times(\nu\nabla^2\underline{v}) = \nu\underline{\nabla}\times(\nabla^2\underline{v}) = \;? \qquad \text{[see below]}$$

Evaluation of $\underline{\nabla}\times(\underline{v}\cdot\underline{\nabla}\underline{v})$:

$$\underline{v}\cdot\underline{\nabla}\underline{v} = \underline{\nabla}\left(\frac{v^2}{2}\right) - \underline{v}\times(\underline{\nabla}\times\underline{v}) \qquad \text{[(11) of Table A-1, with } \underline{a}=\underline{b}=\underline{v}\text{]}$$

$$= \underline{\nabla}\left(\frac{v^2}{2}\right) - \underline{v}\times\underline{\omega}$$

$$\underline{\nabla}\times(\underline{v}\cdot\underline{\nabla}\underline{v}) = \underbrace{\underline{\nabla}\times\underline{\nabla}\left(\frac{v^2}{2}\right)}_{0} - \underline{\nabla}\times(\underline{v}\times\underline{\omega})$$

$$\underline{\nabla}\times(\underline{v}\times\underline{\omega}) = \underline{\omega}\cdot\underline{\nabla}\underline{v} - \underline{v}\cdot\underline{\nabla}\underline{\omega} + \underline{v}(\underline{\nabla}\cdot\underline{\omega}) - \underline{\omega}\underbrace{(\underline{\nabla}\cdot\underline{v})}_{0}$$

[(6) of Table A-1]

$$\underline{\nabla} \cdot \underline{\omega} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{v}) = 0 \qquad \text{[(9) of Table A-1]}$$

$$\therefore \quad \underline{\nabla} \times (\underline{v} \cdot \underline{\nabla}\,\underline{v}) = \underline{v} \cdot \underline{\nabla}\,\underline{\omega} - \underline{\omega} \cdot \underline{\nabla}\,\underline{v}$$

Evaluation of $\underline{\nabla} \times (\nabla^2 \underline{v})$:

$$\nabla^2 \underline{v} = \underline{\nabla}\,(\underbrace{\underline{\nabla} \cdot \underline{v}}_{0}) - \underline{\nabla} \times (\underbrace{\underline{\nabla} \times \underline{v}}_{\underline{\omega}}) \qquad \text{[(10) of Table A-1]}$$

$$\underline{\nabla} \times (\nabla^2 \underline{v}) = - \underline{\nabla} \times (\underline{\nabla} \times \underline{\omega})$$

$$= - \underline{\nabla}\,(\underbrace{\underline{\nabla} \cdot \underline{\omega}}_{0}) + \nabla^2 \underline{\omega} \qquad \text{[(10) of Table A-1]}$$

$$\therefore \quad \underline{\nabla} \times (\nabla^2 \underline{v}) = \nabla^2 \underline{\omega}$$

Collecting all of the terms,

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v} \cdot \underline{\nabla}\,\underline{\omega} - \underline{\omega} \cdot \underline{\nabla}\,\underline{v} = \nu \nabla^2 \underline{\omega}$$

or

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\nabla}\,\underline{v} + \nu \nabla^2 \underline{\omega} \qquad \text{Q.E.D.}$$

(b) Show that for a planar flow, the result in (a) becomes

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \qquad \text{where} \qquad \omega = |\underline{\omega}|.$$

For either kind of planar flow, (x,y) or (r,θ), this is obtained from the z-component of the eq. in (a):

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\underline{\nabla}\underline{v} + \nu\nabla^2\underline{\omega} \qquad [\text{general}]$$

$$\underline{\omega} = \omega_z \underline{e}_z = \omega\,\underline{e}_z \qquad [\text{planar flow}]$$

$$\left.\begin{aligned} \underline{e}_z\cdot\frac{D\underline{\omega}}{Dt} &= \frac{D\omega}{Dt} \\ \underline{e}_z\cdot\nabla^2\underline{\omega} &= \nabla^2\omega \end{aligned}\right\} \qquad [\underline{e}_z = \text{const.}]$$

$$\underline{e}_z\cdot(\underline{\omega}\cdot\underline{\nabla}\underline{v}) = \underline{e}_z\cdot\Big(\omega\underbrace{\frac{\partial\underline{v}}{\partial z}}_{\underline{0}}\Big) = 0 \qquad [\underline{v} = \underline{v}(x,y,t)\text{ only}]$$

This completes the proof.

The simplified equation in (b) does not hold for axisymmetric flows because $\underline{e}_i \neq$ const. in $\underline{\omega} = \omega\underline{e}_i$. That is, $\underline{e}_i = \underline{e}_\theta$ for the (r,z) case and $\underline{e}_i = \underline{e}_\phi$ for the spherical (r,θ) case, and $\underline{e}_\theta$ and $\underline{e}_\phi$ both depend on position. As a result, $\underline{\omega}\cdot\underline{\nabla}\underline{v} \neq \underline{0}$ and $\underline{e}_i\cdot\nabla^2\underline{\omega} \neq \nabla^2\omega_i$. The correct vorticity eqs. for the axisymmetric cases are

$$\frac{D\omega_\theta}{Dt} = \frac{\omega_\theta v_r}{r} + \nu\left(\nabla^2\omega_\theta - \frac{\omega_\theta}{r^2}\right) \qquad [\text{cyl. } (r,z)]$$

$$\frac{D\omega_\phi}{Dt} = \frac{\omega_\phi}{r\sin\theta}\left(v_r\sin\theta + v_\theta\cos\theta\right) + \nu\left(\nabla^2\omega_\phi - \frac{\omega_\phi}{r^2\sin^2\theta}\right) \qquad [\text{spherical } (r,\theta)]$$

(c) Derive the result in (b) from the stream function eqs.

From Table 5-11, for the planar (x, y) case:

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \left[\frac{\partial\psi}{\partial x}\frac{\partial(\nabla^2\psi)}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial(\nabla^2\psi)}{\partial x}\right] = \nu\nabla^4\psi$$

Also, from the definition of ψ and from eq. (5.9-13a),

$$\frac{\partial\psi}{\partial x} = -v_y, \quad \frac{\partial\psi}{\partial y} = v_x, \quad \nabla^2\psi = -w_z = -w$$

Substitute these into the stream function eq.:

$$\frac{\partial}{\partial t}(-w) - \left[(-v_y)\left(-\frac{\partial w}{\partial y}\right) - v_x\left(-\frac{\partial w}{\partial x}\right)\right] = \nu\nabla^2(-w)$$

$$\frac{\partial w}{\partial t} + \underbrace{v_x\frac{\partial w}{\partial x} + v_y\frac{\partial w}{\partial y}}_{\underline{v}\cdot\underline{\nabla}w} = \nu\nabla^2 w$$

$$\therefore \quad \frac{\partial w}{\partial t} + \underline{v}\cdot\underline{\nabla}w = \frac{D\underline{w}}{Dt} = \nu\nabla^2 w.$$

The derivation for the planar (r, θ) case is very similar:

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{r}\left[\frac{\partial\psi}{\partial r}\frac{\partial(\nabla^2\psi)}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial(\nabla^2\psi)}{\partial r}\right] = \nu\nabla^4\psi$$

$$\frac{\partial\psi}{\partial r} = -v_\theta, \quad \frac{\partial\psi}{\partial\theta} = r v_r, \quad \nabla^2\psi = -w_z = -w$$

$$\frac{\partial}{\partial t}(-w) - \frac{1}{r}\left[-v_\theta\left(-\frac{\partial w}{\partial\theta}\right) - r v_r\left(-\frac{\partial w}{\partial r}\right)\right] = \nu\nabla^2(-w)$$

$$\underbrace{\frac{\partial w}{\partial t} + \underbrace{v_r \frac{\partial w}{\partial r} + \frac{v_\theta}{r}\frac{\partial w}{\partial \theta}}_{\underline{v}\cdot\underline{\nabla}\underline{w}}}_{\frac{Dw}{Dt}} = \nu \nabla^2 w$$

$$\frac{Dw}{Dt} = \nu \nabla^2 w \qquad \text{(as before).}$$

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5-3. Velocity and Pressure Calculations

Given: $v_x(x,y) = C x^m$ in 2-D flow with $\rho = $ const.

$v_y(x,0) = 0$, $P(0,0) = P_0$.

(a) Determine $v_y(x,y)$. Use the continuity eq.:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_y(x,y) - \underbrace{v_y(x,0)}_{0} = -\int_0^y \frac{\partial v_x}{\partial x}\, dy$$

$$\frac{\partial v_x}{\partial x} = C m x^{m-1}$$

$$v_y = - C m x^{m-1} y$$

(b) Determine values of m for which $\underline{w} = \underline{0}$.

$$\underline{w} = \underline{\nabla} \times \underline{v} = w_z \underline{e}_z \quad \text{for} \quad \underline{v}(x,y).$$

$$w_z = \frac{\partial v_y}{\partial x} - \underbrace{\frac{\partial v_x}{\partial y}}_{0} = - C m(m-1) x^{m-2} y$$

$\therefore$ Flow is irrotational ($w_z = 0$) for arbitrary x and y only if $m=0$ or $m=1$.

5-3 (cont.)

(c) Determine values of m that satisfy N-S eq. and find $\mathcal{P}(x,y)$.

Navier-Stokes :

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla}\mathcal{P} + \mu \nabla^2 \underline{v}$$

$$\rho \frac{D\underline{v}}{Dt} = \rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{0} + \underline{v}\cdot\underline{\nabla}\underline{v}\right) = \rho\, \underline{v}\cdot\underline{\nabla}\underline{v} \quad \text{for steady flow}$$

X-component :

$$\frac{\partial \mathcal{P}}{\partial x} = \mu \nabla^2 v_x - \rho\, \underline{v}\cdot\underline{\nabla} v_x$$

$$\nabla^2 v_x = \frac{\partial^2 v_x}{\partial x^2} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{0} = C m (m-1)\, x^{m-2}$$

$$\underline{v}\cdot\underline{\nabla} v_x = v_x \frac{\partial v_x}{\partial x} + v_y \underbrace{\cancel{\frac{\partial v_x}{\partial y}}}_{0} = (C x^m)(C m x^{m-1}) = C^2 m\, x^{2m-1}$$

$$\frac{\partial \mathcal{P}}{\partial x} = \mu C m (m-1)\, x^{m-2} - \rho C^2 m\, x^{2m-1}$$

$$\mathcal{P}(x,y) = \mu C m\, x^{m-1} - \frac{\rho C^2}{2} x^{2m} + f(y)$$

y-component :

$$\frac{\partial \mathcal{P}}{\partial y} = \mu \nabla^2 v_y - \rho\, \underline{v}\cdot\underline{\nabla} v_y$$

$$\nabla^2 v_y = \frac{\partial^2 v_y}{\partial x^2} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{0} = -C m (m-1)(m-2)\, x^{m-3} y$$

5-3 (Cont.)

$$\underline{v} \cdot \underline{\nabla} v_y = v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}$$

$$= (Cx^m)\left(-Cm(m-1)x^{m-2}y\right) + \left(-Cmx^{m-1}y\right)\left(-Cmx^{m-1}\right)$$

$$= -C^2 m(m-1)x^{2m-2}y + C^2m^2x^{2m-2}y$$

$$= C^2 m x^{2m-2} y\left[m-(m-1)\right] = C^2 m x^{2m-2} y$$

$$\frac{\partial P}{\partial y} = -\mu C m(m-1)(m-2)x^{m-3}y - \rho C^2 m x^{2m-2} y$$

$$P(x,y) = -\frac{\mu C m(m-1)(m-2)}{2} x^{m-3}y^2 - \frac{\rho C^2 m}{2} x^{2m-2}y^2 + g(x)$$

Comparing the two expressions for $P(x,y)$, we conclude that they can be equal only if

$$-\frac{\mu C m(m-1)(m-2)}{2} x^{m-3}y^2 = f_1(y) \qquad (i)$$

$$-\frac{\rho C^2 m}{2} x^{2m-2}y^2 = f_2(y) \qquad (ii)$$

where $f(y) = f_1(y) + f_2(y)$. For (i) to hold we need $m = 0, 1, 2$ or 3; (ii) holds only if $m=0$ or $m=1$.

Conclusion: The N-S eqn. is satisfied only if $m=0$ or $m=1$.

(These were also the values of m found to give irrotational flow.)

5-3 (cont.)

Calculate $\mathcal{P}$ for $m=0$:

$$\left.\begin{aligned}\mathcal{P}(x,y) &= -\frac{\rho C^2}{2} + f(y)\\ \mathcal{P}(x,y) &= g(x)\end{aligned}\right\} \Rightarrow f(y) = \text{const.},\ g(x) = \text{const.}$$

$$\therefore \quad \mathcal{P}(x,y) = \text{const.} = \mathcal{P}_0$$

Calculate $\mathcal{P}$ for $m=1$:

$$\left.\begin{aligned}\mathcal{P}(x,y) &= \mu C - \frac{\rho C^2}{2}x^2 + f(y)\\ \mathcal{P}(x,y) &= -\frac{\rho C^2}{2}y^2 + g(x)\end{aligned}\right\} \Rightarrow \begin{aligned} f(y) &= -\frac{\rho C^2}{2}y^2 + \text{const.}\\ g(x) &= \mu C - \frac{\rho C^2}{2}x^2 + \text{const.}\end{aligned}$$

$$\therefore \quad \mathcal{P}(x,y) = -\frac{\rho C^2}{2}(x^2+y^2) + \text{const.}$$

$$= \mathcal{P}_0 - \frac{\rho C^2}{2}(x^2+y^2)$$

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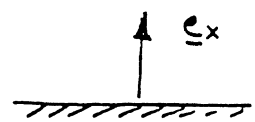
5-4. Normal Viscous Stress at a Surface

For a Newtonian fluid of const. ρ at an impermeable solid surface, show that the normal component of the viscous stress is zero.

Approach: Consider planar, cylindrical, and spherical surfaces. Fix coordinates on surface.

Planar:



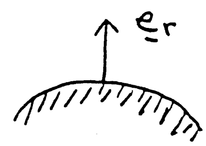
$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} \qquad \text{(Table 5-4)}$$

$$\underline{\nabla} \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(continuity)}$$

$$\underbrace{\left.\frac{\partial v_x}{\partial x}\right|_s}_{\text{at surface}} = -\left(\underbrace{\left.\frac{\partial v_y}{\partial y}\right|_s}_{\substack{0 \\ \text{no slip}}} + \underbrace{\left.\frac{\partial v_z}{\partial z}\right|_s}_{\substack{0 \\ \text{no slip}}} \right)$$

$$\therefore \quad \tau_{xx}\big|_s = 0$$

Cylindrical:



$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \qquad \text{(Table 5-5)}$$

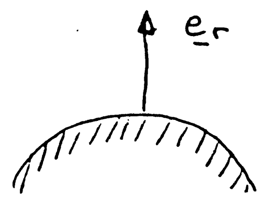
$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) = \frac{\partial v_r}{\partial r} + \frac{v_r}{r}$$

$$\left.\frac{\partial v_r}{\partial r}\right|_s = -\underbrace{\left.\left(\cancel{\frac{v_r}{r}}\right)\right|_s}_{\substack{0\\ \text{no penetration}}} - \underbrace{\left.\left(\frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}\right)\right|_s}_{\substack{0\\ \text{no slip}}} - \underbrace{\left.\cancel{\frac{\partial v_z}{\partial z}}\right|_s}_{\substack{0\\ \text{no slip}}} = 0$$

$$\therefore \quad \tau_{rr}\big|_s = 0$$

Spherical :



$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \qquad (\text{Table 5-6})$$

$$\underline{\nabla}\cdot\underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$= 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{\partial v_r}{\partial r} + \frac{2 v_r}{r}$$

$$\left.\frac{\partial v_r}{\partial r}\right|_s = -\underbrace{\left.\left(\cancel{\frac{2 v_r}{r}}\right)\right|_s}_{\substack{0\\ \text{no penetration}}} - \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\cancel{v_\theta}\sin\theta\right)}_{\substack{0\\ \text{no slip}}} - \underbrace{\frac{1}{r\sin\theta}\cancel{\frac{\partial v_\phi}{\partial\phi}}}_{\substack{0\\ \text{no slip}}}$$

$$\therefore \quad \tau_{rr}\big|_s = 0$$