PS 3 solution

Problem 1

a) The solution is: \( \phi(x,t) = e^{-(x-ut)^2} \), here \( u = 10 \)

Plot \( \phi(x,t) \) versus \( x \) at times 0, 0.5, and 1.0:

\[ \begin{array}{ccc}
\text{t = 0} & \text{t = 0.5} & \text{t = 1.0} \\
\end{array} \]

b) The general solution is: \( \phi(x,t) = \frac{1}{\left[ 4\alpha(t+c) \right]^{1/2}} e^{-\frac{x^2}{4\alpha(t+c)}} \), here \( c = 1/4\alpha, \alpha = 2 \)

Plot \( \phi(x,t) \) versus \( x \) at times 0, 0.1, and 0.2:

\[ \begin{array}{ccc}
\text{t = 0} & \text{t = 0.1} & \text{t = 0.2} \\
\end{array} \]
Problem 2

a) \( a = C^2 \quad b = 0 \quad c = -1 \quad b^2 - 4ac > 0 \), hyperbolic PDE

\[ \phi(x,0) = \phi_1(x) \]

Initial value problem, the initial conditions given by \((t > 0)\):

\[ \frac{\partial \phi}{\partial t}(x,0) = \phi_2(x) \]

b) \( a = 1 \quad b = 0 \quad c = 1 \quad b^2 - 4ac < 0 \), elliptic PDE

Boundary value problem, the boundary conditions given by: \( \phi(x,y) \) at \( S \) or \( \frac{\partial \phi}{\partial n}(x,y) \) at \( S \), \( S \) is the entire boundary of solving control volume.

c) \( a = \alpha \quad b = 0 \quad c = 0 \quad b^2 - 4ac = 0 \), parabolic PDE

Initial value problem, the initial conditions given by \((t > 0)\):

\( \phi(x,0) = \phi_0(x) \)

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Problem 3

When transverse loading is constant, the governing equation can be written by:

\[ EI \frac{\partial^4 \phi}{\partial x^4} + P \frac{\partial^2 \phi}{\partial x^2} = q_0 \]

The exact solution has form:

\[ \phi(x) = c_1 + c_2 x + \frac{q_0}{2P} x^2 + c_3 \sin \left( \sqrt{\frac{P}{EI}} x \right) + c_4 \cos \left( \sqrt{\frac{P}{EI}} x \right) \]

Apply four BCs, the constants can be determined as:

\[ c_1 = -\frac{q_0 EI}{P^2}, c_2 = -\frac{q_0}{2P}, c_3 = \frac{q_0 EI}{P^2} \left[ \frac{1 - \cos \sqrt{P/\sqrt{EI}}}{\sin \sqrt{P/\sqrt{EI}}} \right], c_4 = \frac{q_0 EI}{P^2} \]

Since \( EI = 1, P = 1, q_0 = 5 \), the solution can be simplified as:

\[ \phi(x) = 2.7315 \sin x + 5 \cos x + 2.5x^2 - 2.5x - 5 \]

The maximum deflection is:

\[ \phi(0.5) = 0.07246 \]

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Problem 4

Using central differencing in the space, rewrite PDE as:

\[ \phi_{i+2} - 4\phi_{i+1} + 6\phi_i - 4\phi_{i-1} + \phi_{i-2} + h^2 \left( \phi_{i+1} - 2\phi_i + \phi_{i-1} \right) = 5h^4 \]

1) \( N=5, h=0.25 \)

BCs: \( \phi_{i} = \phi_{5} = 0 \) and \( \phi_{0} = -\phi_{2}, \phi_{6} = -\phi_{4} \)
Finally the equations simplified to:

\[
\begin{bmatrix}
5 - 2h^2 & -4 + h^2 & 1 \\
-4 + h^2 & 6 - 2h^2 & -4 + h^2 \\
1 & -4 + h^2 & 5 - 2h^2 \\
\end{bmatrix}
\begin{bmatrix}
\phi_2 \\
\phi_3 \\
\phi_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
5h^4 \\
5h^4 \\
5h^4 \\
\end{bmatrix}
\]

Solve by Matlab:
Maximum deflection is: 0.07656.

2) N=10, h=0.1
BCs: \( \phi_1 = \phi_{11} = 0 \) and \( \phi_0 = -\phi_2, \phi_{12} = -\phi_{10} \)

\[
\begin{bmatrix}
5 - 2h^2 & -4 + h^2 & 1 & & & & & & 0 \\
-4 + h^2 & 6 - 2h^2 & -4 + h^2 & 1 & & & & & \vdots \\
1 & -4 + h^2 & 6 - 2h^2 & -4 + h^2 & 1 & & & & \vdots \\
& & & & & & & & \vdots \\
0 & & & & & & & & 1 & -4 + h^2 & 6 - 2h^2 & -4 + h^2 \end{bmatrix}
\begin{bmatrix}
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_{10} \\
\end{bmatrix}
= 
\begin{bmatrix}
5h^4 \\
5h^4 \\
5h^4 \\
5h^4 \\
\end{bmatrix}
\]

Maximum deflection is: 0.07312.

2) N=20, h=0.05
BCs: \( \phi_1 = \phi_{21} = 0 \) and \( \phi_0 = -\phi_2, \phi_{22} = -\phi_{20} \)

Maximum deflection is: 0.07263.

<table>
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<th>Maximum deflection</th>
<th>Difference with the exact</th>
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<td>Exact</td>
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<td>/</td>
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<td>0.07312</td>
<td>0.91%</td>
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<td>N=20</td>
<td>0.07263</td>
<td>0.23%</td>
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</table>

Exact solution                          Exact solution and Num. solution (N=5)
Problem 5

a) Integrate PDE over $V_i$

$$\frac{d}{dt} \int_{V_i} \phi dV = \alpha \int_{V_i} \nabla^2 \phi dV = \alpha \int_{S_i} \mathbf{n} \cdot \nabla \phi dS$$

Further derive:

$$\frac{d}{dt} (\phi_{i,j} A) = \alpha \left[ \frac{k}{h} (\phi_{i+1,j} - \phi_{i,j}) - \frac{k}{h} (\phi_{i,j} - \phi_{i-1,j}) + \frac{h}{k} (\phi_{i,j+1} - \phi_{i,j}) - \frac{h}{k} (\phi_{i,j} - \phi_{i,j-1}) \right]$$

Where: $A = hk$, $k = y_{j+1} - y_j$, $h = x_{i+1} - x_i$, $\frac{d}{dt} = \frac{d_2}{d_1} = \frac{d_2}{d_1} = h/2$ in x-direction

b) Along the boundaries $\nabla \phi = 0$, so if this problem is solved using FVM, summarize all equations in each grid cell $\frac{d}{dt} \sum_{i,j} (\phi_{i,j} A) = 0$. That means no loss of chlorine with time, the total amount of chlorine remains constant.

Problem 6

Using backwards differencing in time and central differencing in space, the heat equation can be discretized as:

$$\frac{\phi_m^n - \phi_m^{n-1}}{\Delta t} = \alpha \frac{\phi_{m+1}^n - 2\phi_m^n + \phi_{m-1}^n}{\Delta x^2}$$

Simplified as:

$$(1 + 2S)\phi_m^n = S(\phi_{m+1}^n + \phi_{m-1}^n)$$

where $S = \frac{\alpha \Delta t}{\Delta x^2}$
Perturb \( \phi^n_m \) by the form:

\[
\delta^n_m = A_0 \xi^n e^{i\phi_n},
\]

where \( x_m = m\Delta x \)

Substitute the perturb term into the differential equation and simplified:

\[
\xi = \frac{1}{1 + 2S - 2S \cos(\beta \Delta t)} = \frac{1}{1 + 4S \sin^2(\beta \Delta t)}
\]

So if \( S > 0 \), the FDM will be stable, which means the stability condition is: \( \frac{\alpha \Delta t}{\Delta x^2} > 0 \)