The finite element method combines some concepts from spectral methods with others from finite-volume methods.

**General Formulation:**
Divide computational region into grid cells.

The following must be performed for each grid cell:

\[
\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \alpha \nabla^2 \phi \quad \text{for } x \in V \tag{1}
\]

\[
\begin{align*}
\text{BC:} & \quad \phi = \phi_0(x, t) \quad \text{on } x \in S_1 \\
& \quad \mathbf{c}_1 \phi + \frac{\partial \phi}{\partial n} = c_2 \quad \text{on } x \in S_2 \tag{2}
\end{align*}
\]

Multiply (1) by a **test function** \( W(x, t) \) and integrate over \( V \) to get

\[
\int_V \left( \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi - \alpha \nabla^2 \phi \right) W \, dv = 0. \tag{3}
\]

(4)
Use divergence theorem to write

\[ \iint_S \nabla \cdot (w \phi) \, dv = \int_S \nabla w \cdot \phi \, da \]

\[ = \iint_S \left( \nabla w \cdot \phi + w \nabla \phi \right) \, dv \]

to get

\[ \iint_S w \nabla \phi \, dv = \int_S w \frac{\partial \phi}{\partial n} \, da - \iint_S \nabla w \cdot \phi \, dv. \]

Plug this into (3) to get

\[ \iint_S \left[ \frac{\partial w}{\partial t} + (u \cdot \nabla) w + \alpha \nabla \cdot \nabla w \right] \, dv + \int_{S_2} c_1 \phi \nabla \cdot w \, da \]

\[ = \int_{S_2} c_2 w \, da \]

- weak form of advection-diffusion equation

when we require test functions \( w(x,t) \) to satisfy

\[ w(x,t) = 0 \quad \text{on } x \in S_1. \]
Consider a set of basis functions \( y_n(x) \).

Approximate \( \phi(x,t) \) by a function \( \hat{\phi}(x,t) \), defined by

\[
\hat{\phi}(x,t) = \sum_{j=1}^{N} a_j(t) y_j(x) + \phi_0(x,t)
\]  

(5)

Substitute (5) into (4) with \( w = w_i(x) \).

Galerkin method: Let \( w_i = y_i(x) \) (test function = basis function)

\[
\begin{align*}
\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{S} \mathbf{u} &= \mathbf{f} \\
\text{for } i = 1, 2, \ldots, N
\end{align*}
\]

In direct notation,

\[
M \frac{d^2 a}{dt^2} + Sa = f
\]

- solve using IVP solver
• Divide \( V \) up into elements (e.g., triangles, tetrahedrons, etc).

• Choose \( \psi_i(\mathbf{x}) \) such that

\[
\psi_i(\mathbf{x}_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

and

\( \psi_i(\mathbf{x}) \) is a polynomial for each element

• Perform integration in (6) using a numerical integration method (e.g., using Newton-Cotes rules).

• If integration is such that mass matrix is diagonal we refer to it as a lumped mass matrix. (Occurs when polynomial used to interpolation is of same order as \( \psi_i(\mathbf{x}) \)).

• Lumped mass matrices are useful for explicit IVP solvers.

• Exhibits spurious oscillations for convection-dominated problems, similar to using centered finite differences.
Methods for Convection-Dominated Problems

A. Streamline-Upwind Petrov-Galerkin Method (SUPG)

Idea: Choose test function as

\[ w = \phi_i + p_i \]

where \( \phi_i(x) \) = basis function
\( p_i(x) \) = upwinding correction.

Integral form of advection-diffusion eqn:

\[ \int_V \left[ \frac{\partial \phi}{\partial t} w + (u \cdot \nabla \phi)w - \nabla \cdot (\mu \nabla \phi) w \right] \, dv = 0 \]

Substitute \( w = \phi_i + p_i \) to get

\[ \int_V \left[ \frac{\partial \phi_i}{\partial t} + u \cdot \nabla \phi_i - \nabla \cdot (\mu \nabla \phi_i) \right] \phi_i \, dv \]

= \[ - \int_V \left[ \frac{\partial p_i}{\partial t} + u \cdot \nabla p_i - \nabla \cdot (\mu \nabla p_i) \right] p_i \, dv \], \( i = 1, \ldots, N \)

- artificial diffusion term (used to suppress the instability)

- remainder of solution same as for standard Galerkin FEM
choice for \( p_i(x) \):

\[
p_i(x) = \frac{h}{2|y|} \mathbf{u} \cdot \nabla y_i(x)
\]

(other choices are also sometimes used)

This choice yields an artificial viscosity of the form

\[
\frac{h|\mathbf{u}|}{2} \mathbf{r}^2 \phi
\]

**Comments**

- Many other variations of SUPG method, which vary by the choice of \( p_i(x) \)

- Other variants of FEM also can deal with convection
  - Taylor-Galerkin method
  - Characteristic Galerkin method