Spectral Method

Sturm-Liouville Theory:

Problem: Consider an equation of the form

\[(r(x) y')' + \lambda [p(x) + \lambda] y = 0 \quad (1)\]

where \( p(x) > 0 \) on \((a, b)\) on an interval \((a, b)\) subject to the BC:

1. **Homogeneous**
   
   \[y' + k_1 y = 0 \quad \text{if } x = a\]
   \[y' + k_2 y = 0 \quad \text{if } x = b\]

2. **Periodic**
   
   \[y(x) = y(x + b - a)\]

3. **Singular**
   
   \[r(x) = 0 \quad \text{at } x = a, b\]

Differential Eigenproblem:

- Differential equivalent of \( Ay = \lambda y \)
  
  \((A = \text{symmetric matrix})\)

- No solution of (1) other than \( y = 0 \), except for certain special values of \( \lambda \).

- \( \lambda_n \) = eigenvalue
  
  \[y_n(x) = \text{eigenfunction}\]
  
  \[p(x) = \text{weighting function}\]
Properties of SL Problem:

1. Eigenvalues $\lambda_n$ are real.

2. Eigenfunctions $y_n(x)$ are orthogonal
   \[ \int_a^b y_n(x) y_m(x) \, dx = 0 \quad \text{if} \quad n \neq m. \]

3. Eigenfunctions $y_n(x)$ are complete.
   - Any function $f(x)$ subject to same BC as $y_n(x)$ can be expanded in $(a,b)$ as
   \[ f(x) = \sum_{n=1}^{\infty} a_n y_n(x). \]

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Idea of Spectral Method

\[ \frac{\partial^2 \phi}{\partial t^2} + u \cdot \nabla \phi = \alpha \phi. \]

1. Expand $\phi(x,t)$ in terms of orthogonal test function

   \[ \phi(x,t) = \sum_{n=1}^{\infty} a_n(t) y_n(x), \]

   where $y_n(x)$ satisfy same BC as $\phi(x,t)$ on $x = a, b$. 

(2)
2. Plug into PDE (assume \( y = \text{const} \) for the moment)

\[
\sum_{n=1}^{\infty} a_n \frac{\partial y_n(x)}{\partial t} + \sum_{n=1}^{\infty} a_n(t) \frac{\partial y_n(x)}{\partial x} = \alpha \sum_{n=1}^{\infty} a_n(t) \frac{\partial^2 y_n(x)}{\partial x^2}
\]

3. Write \( y' \) and \( y'' \) in terms of \( y_n(x) \).

4. Multiply PDE by \( p(x) y_m(x) \)

5. Integrate over \((a,b)\) and eliminate terms using orthogonality.

6. Solve resulting ODE system for \( a_n(t) \).

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Note: Inversion of Eigenfunction Expansion

\[
p(x) = \sum_{n=1}^{\infty} a_n y_n(x)
\]

Mult. by \( p(x) y_m(x) \) & integrate over \((a,b)\):

\[
\int_{a}^{b} p(x) f(x) \, dx = \sum_{n=1}^{\infty} a_n \int_{a}^{b} p(x) y_m(x) \, dy_m(x) \, dx
\]

\[
= a_m \int_{a}^{b} y_m^2(x) \, dx
\]

\[
\therefore \quad a_m = \frac{\int_{a}^{b} p(x) f(x) \, dx}{\int_{a}^{b} y_m^2(x) \, dx}
\]
Example: Periodic domain, constant velocity

\[ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}, \quad x \in (0, 2\pi) \]  

BC: \( \phi(x, t) = \phi(x + \frac{2\pi}{L}, t) \)

IC: \( \phi(x, 0) = f(x) \)

\[ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \frac{n}{N} \frac{\partial^2 \phi}{\partial x^2}, \quad N = \text{even} \]  

\[ \phi(x, t) = \sum_{n=-N/2}^{N/2} a_n(t) e^{inx}, \quad \text{where } a_n(t) = \text{complex-valued function such that } \sum_{n=-N/2}^{N/2} a_n(t) = \overline{a_n(t)} \]  

(to ensure that \( \phi \) is real-valued).

(Note: \( a + \overline{a} = \text{real} \) for any complex number \( a \))

**Step 1:** Expand \( \phi(x, t) \) in a Fourier series

**Step 2:** Plug into PDE

\[ \sum_{n=-N/2}^{N/2} \left( \frac{d}{dt} a_n + inu a_n + n^2 a_n \right) e^{inx} = 0 \]

**Step 3:** Multiply PDE by \( e^{inx} \) and integrate over \((0, 2\pi)\)

\[ \sum_{n=-N/2}^{N/2} \left( \frac{d}{dt} a_n + inu a_n + n^2 a_n \right) \int_{0}^{2\pi} e^{inx} e^{inx} dx = 0 \]
Step 4
Apply Orthogonality property

\[ \int_0^{2\pi} e^{inx} e^{-inx} \, dx = \begin{cases} 0, & n \neq m \\ 2\pi, & n = m \end{cases} \]

where

\[ \int_0^{2\pi} e^{inx} \, dx = \int_0^{2\pi} \cos(nx) + i\sin(nx) \, dx = \pi + \pi = 2\pi. \]

to get

\[ \frac{da_m}{dt} + i\mu a_m + \alpha m^2 a_m = 0 \quad (3) \]

- spectral form of advection-diffusion eqn.

Step 5: Solve IVP (3)

\[ a_m(t) = a_m(0) e^{-i(\mu + \alpha m^2)t} \quad (4) \]

Step 6: Plug(4) back into Fourier series (1) to obtain

\[ \Phi(x, t) \]

Step 7: 

\[ \Phi(x, 0) = f(x) = \sum_{m=-M}^{M} a_m(0) e^{inx} \]

To invert, multiply by \( e^{-inx} \) and integrate over \([0, 2\pi]\)

\[ \int_0^{2\pi} f(x) e^{-inx} \, dx = 2\pi a_m(0) \rightarrow \text{solve for } a_m(0). \]
Advantages

- Very fast
- Very high accuracy (exponential convergence)
  Index of convergence: $a_n \sim O(1/n^k)$ as $n \to \infty$
  Algebraic convergence
  Exponential convergence: constants $q$ and $r > 0$ exist such that
  $a_n \sim O(\exp(qn^r))$ as $n \to \infty$

[Note: $\lim_{n \to \infty} n^k \exp(-qn^r) = 0$ for all $k$ and $r > 0$ (exp. conv. is always most accurate)]

Disadvantages

- Requires simple geometries and boundary conditions
- Subject to Gibbs oscillations in presence of sharp gradients
Example: \(\frac{\partial \phi}{\partial t} + u(x) \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}\), \(x \in (0, 2\pi)\)

BC: \(\phi(x, t) = \phi(x + 2\pi, t)\)

IC: \(\phi(x, 0) = f(x)\)

Try spectral approach:

Step 1: \(\phi(x, t) = \sum_{n=1}^{N/2} a_n(t) e^{inx}\)

Step 2: Plug into PDE

\[
\sum_{n=1}^{N/2} \left( \frac{da_n}{dt} + in\alpha u(x) + \alpha n^2 a_n \right) e^{inx} = 0
\]

Problem: can't invert because \(u(x)e^{inx}\) is not orthogonal
Pseudo-Spectral Approach

Step 1: March one forward in time (e.g., Forward Euler)

\[ \phi(x, t_{n+1}) - \phi(x, t_n) = h \left[ \frac{\partial^2 \phi}{\partial x^2} - u \frac{\partial \phi}{\partial x} \right]_{t_n} \]  

Step 2: Invert \( \phi(x, t_n) = \sum_{n = -N/2}^{N/2} a_n(t_n) e^{i n x} \) to obtain \( a_n(t_n) \)

[Multiply by \( e^{-i n x} \) and integrate over \( (0, 2\pi) \)]

Step 3: Evaluate integrals of derivatives

\[ \frac{\partial \phi}{\partial x} \bigg|_{t_n} = \sum_{n = -N/2}^{N/2} \text{i} n a_n(t_n) e^{i n x} \]

\[ \frac{\partial^2 \phi}{\partial x^2} \bigg|_{t_n} = \sum_{n = -N/2}^{N/2} -n^2 a_n(t_n) e^{i n x} \]

Step 4: Step (4) one step forward and repeat