Lecture 16. Finite-Difference Method

Idea: Replace derivatives in PDE with a difference formula.

Advantages:
- Simple (easy to code & analyze)
- Can be very accurate by using high-order differences
- Can lead to very fast matrix solutions (tridiagonal matrices, etc)

Disadvantages:
- Requires boundary-fitted coordinate system in multiple dimensions
One-Dimensional Boundary-Value Problem

Steady-State Thermal diffusion with a heat source and thermal loss
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{f(x) \phi}{h^2} = 0 \]

- Thermal loss through insulation is proportional to temperature

Use centered difference approximation
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} \]

\[ h = x_{i+1} - x_i \]

to write governing eqn as

\[ \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} - f_i \phi_i = 0, \quad i = 2, n-1. \quad (1) \]

Boundary conditions

\[ \phi(x_i) = C_i \quad \text{Dirichlet (Const. Temperature)} \]

\[ \frac{\partial \phi}{\partial x} \bigg|_{x_2} = C_2 \quad \text{Neumann (Const. heat flux)} \]
Discretize BC using 2nd-order accurate method:

\[ \phi_1 = C_1 \]  

(2)

\[ \frac{\phi_{n+1} - \phi_{n-1}}{2h} = C_2 \]  

(3)

**Problem:** \( \phi_{n+1} \) is outside range of integration

**Resolution:**
- (a) apply (1) at \( i = n \) to get extra eqn for \( \phi_{n+1} \)
- (b) use backward difference with 2nd-order accuracy

Use approach (a) to write

\[ \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - f_n \phi_n = 0 \]

Solve for \( \phi_{n+1} \) from (3) & substitute to write

\[ \phi_{n+1} = \phi_{n-1} + 2h C_2 \]

so we have

\[ \frac{-2\phi_n + 2\phi_{n-1}}{h^2} - f_n \phi_n = -\frac{2C_2}{h} \]  

(4)

3
Equations: \((1), (2), (4)\) = \(n\) total

Unknowns: \(\phi_1, \phi_2, \ldots, \phi_n\) = \(n\) total

Set up matrix equation:

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -(z+h^2f_1) & \cdots & 0 \\
0 & 0 & 1 & \cdots & -(z+h^2f_i) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\vdots \\
\phi_n
\end{pmatrix}
\]

- Tridiagonal matrix
- Solve with tridiagonal form of LU Decomposition
2. Laplace Eqn on a Rectangle

\begin{equation}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\end{equation}

\[ h = x_{i+1} - x_i \]
\[ k = y_{i+1} - y_{i+1} \]

Discretize Laplace eqn

\[ \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{h^2} + \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{k^2} = 0 \]

BC:
\[ \phi_{i,j} = C_2 \quad j = 2,n-1 \]
\[ \phi_{n,j} = C_1 \quad j = 2,n-1 \]
\[ \phi_{i,1} = C_1 \quad i = 1,n \]
\[ \phi_{i,n} = C_2 \quad i = 1,n \]
Matrix structure: 5 bands, outer two separated by zeros

\[ \begin{array}{ccc}
 & m & \\
 m & & m \\
 & i & j \\
 m & & m \\
\end{array} \]

- Solve with iterative method, such as Gauss-Seidel with SOR
3. Heat Eqn on a Circle

**Note:** Must use polar coordinates

(Boundary condition must be imposed on constant coordinate surface)

\[ \phi(R, \theta) = C_1 \]

\[ \frac{\partial \phi}{\partial t} = \alpha \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \right), \quad \phi = \phi(r, t) \]

**BC:**

\[ \frac{\partial \phi}{\partial r} \bigg|_{r \rightarrow 0} = 0 \]

\[ \phi(R, t) = C_2 \]

**IC:**

\[ \phi(r, 0) = C_3 \]

**Solution:** Spatial Center Difference in \( r \)

\[ \frac{1}{h^2} \left( \frac{\partial^n}{\partial r^n} \phi \right) = \frac{1}{h^2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \phi \]

\[ \left. \frac{\partial^2 \phi}{\partial r^2} \right|_i = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}, \quad \left. \frac{\partial \phi}{\partial r} \right|_i = \frac{\phi_{i+1} - \phi_{i-1}}{2h} \]
Problem: \( \frac{1}{r} \frac{\partial \phi}{\partial r} \) blows up at \( r = 0 \) for \( \frac{\partial \phi}{\partial r} \neq 0 \)

Solu: Use L'Hopital's rule to write

\[
\lim_{r \to 0} \frac{\partial \phi}{\partial r} = \frac{\partial^2 \phi}{\partial r^2}
\]

We therefore have

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \bigg|_{r=0} = 2 \frac{\partial^2 \phi}{\partial r^2}
\]

Use 2nd-order forward-difference approximation for \( \frac{\partial^2 \phi}{\partial r^2} \).
\[
\int_{t_i}^{t_{i+1}} \frac{3\phi}{\partial t} \, dt = \alpha \int_{t_i}^{t_{i+1}} \frac{3}{r} \left( \frac{\partial \phi}{\partial r} \right) \, dt
\]

**Explicit:** Evaluate integral with rectangle rule

\[
\phi_{i+1} - \phi_i = \alpha \frac{k}{r} \left( \frac{3\phi}{\partial r} \right)_i + o(k^2)
\]
- solve by stepping forward in time
- global error \(O(k)\), local error \(O(k^2)\)
- Forward-Euler

**Implicit:** Evaluate integral with trapezoidal rule

\[
\phi_{i+1} - \phi_i = \frac{\alpha k}{2r} \left[ \frac{3}{\partial r} \left( \frac{\partial \phi}{\partial r} \right)_i + \frac{3}{\partial r} \left( \frac{\partial \phi}{\partial r} \right)_{i+1} \right] + o(k^2)
\]
- after space discretization, yields matrix equation for \(\phi_{i+1}\)
- solve using iterative method
- global error \(O(k^2)\)
- more stable than explicit method (more on this later, but slower)
- Crank-Nicolson