Example: Thermal advection by a velocity field \( \mathbf{v}(x,y,t) \)

Definitions:

\[
\begin{align*}
\phi &= \phi(x,y,t) & \text{temperature} \\
\rho &= \rho(x,y,t) & \text{density} \\
\mathbf{q} &= \mathbf{q}(x,y,t) & \text{heat flux (positive for energy flow out of CV)} \\
e &= e(x,y,t) & \text{internal energy/ unit volume} \\
c &= c(x,y,t) & \text{specific heat} \\
\mathbf{v} &= \mathbf{v}(x,y,t) & \text{velocity} \\
S &= S(x,y,t) & \text{source term} \\
\end{align*}
\]

Control volume - infinitesimal part of the fluid

Balance equation:

\[
\left\{ \text{Rate of change of energy in CV} \right\} = \left\{ \text{Rate at which energy is carried into CV by velocity} \right\} - \left\{ \text{Rate at which energy is carried out of CV by velocity} \right\} \\
+ \left\{ \text{Rate of diffusion into CV} \right\} - \left\{ \text{Rate of diffusion out of CV} \right\}
\]
\[ \frac{d}{dt} \int_V g e \, dv = - \int_S g e u \cdot n \, da - \int_S g \cdot n \, da \]

Note: \( u \cdot n > 0 \) for flow out of CV
\( g \cdot n > 0 \) for diffusion out of CV

Use divergence theorem:
\[ \int_V \nabla \cdot b \, dv = \int_S b \cdot n \, da \]
for any vector \( b \)

to get
\[ \int_S g e u \cdot n \, da = \int_V \nabla \cdot (g e u) \, dv \]
\[ \int_S g \cdot n \, da = \int_V \nabla \cdot g \, dv \]

Also since \( V \) is fixed in space
\[ \frac{d}{dt} \int_V g e \, dv = \int_V \frac{\partial}{\partial t} (g e) \, dv \]

Combine to write CV energy balance as
\[ \int_V \left[ \frac{\partial}{\partial t} (ge) + \nabla \cdot (geu) + \nabla \cdot g \right] \, dv = 0. \]
Since \( V \) = arbitrary part of the fluid, the integrand must vanish:

\[
\frac{\partial}{\partial t} (ge) + \mathbf{v} \cdot (geu) = -g \cdot \mathbf{e}.
\]  \hspace{1cm} (1)

Repeat above argument for mass conservation

\[
\left\{ \text{Rate of change of mass in CV} \right\} = \left\{ \text{Rate of convection of mass into CV} \right\} - \left\{ \text{Rate of convection of mass out of CV} \right\}
\]

\[
\frac{d}{dt} \int_V g \, dv = -\int_V g \cdot \mathbf{u} \cdot \mathbf{n} \, da
\]

\[
\int_V \frac{\partial}{\partial t} g \, dv = -\int_V \mathbf{v} \cdot (gu) \, dv
\]

\text{So}

\[
\frac{\partial g}{\partial t} + \mathbf{v} \cdot (gu) = 0
\]  \hspace{1cm} (2)

Now expand (1) as

\[
8 \frac{\partial e}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{ue} + e \frac{\partial g}{\partial t} + g \cdot \mathbf{v} \mathbf{e} + e \mathbf{v} \cdot (gu) = -g \cdot \mathbf{e}
\]

\text{vanish due to mass cons. (2)}

(3)
yielding
\[ \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = - \nabla \cdot q. \]  \,(3) 

Recall:

- Fourier law of heat conduction \( q = -k \nabla \phi \)
- Definition of specific heat \( c_p = \frac{\partial e}{\partial \phi} \)

Thus
\[
\nabla \cdot q = -k \nabla \cdot \nabla \phi = -k \nabla^2 \phi
\]
\[
\frac{\partial e}{\partial t} = \frac{\partial e}{\partial \phi} \frac{\partial \phi}{\partial t} = c_p \frac{\partial \phi}{\partial t}
\]
\[
\nabla e = \frac{\partial e}{\partial \phi} \nabla \phi = c_p \nabla \phi.
\]

Substitute to write (3) as
\[
3c_p \left( \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \right) = k \nabla^2 \phi
\]
or
\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = \alpha \nabla^2 \phi \tag{4}
\]

where \( \alpha = \frac{k}{3c_p} \) = thermal diffusivity.

- advection-diffusion equation
Special Cases

1. No convection \((u = 0)\)

\[
\frac{\partial \phi}{\partial t} = \alpha \nabla^2 \phi
\]  
\(\text{heat equation}\)  
\(\text{(5)}\)

2. Steady \((\partial \phi / \partial t = 0)\) and no convection \((u = 0)\)

\[
\nabla^2 \phi = 0
\]  
\(\text{Laplace equation}\)  
\(\text{(6)}\)

1D Forms

\[
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}
\]

\[
\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = 0
\]
Types of 2nd-order PDEs

Consider (PDE for \( \Phi(x,y) \)):

\[ a \Phi_{xx} + b \Phi_{xy} + c \Phi_{yy} + d \Phi_x + e \Phi_y + f \Phi = g \]

Similar to quadratic form

\[ ax^2 + b xy + cy^2 + dx + ey + f = g \]

classifications:

\[ b^2 - 4ac < 0 \] elliptic
\[ b^2 - 4ac = 0 \] parabolic
\[ b^2 - 4ac > 0 \] hyperbolic

Hypertolic eqns. can be decomposed using the method of characteristics into a set two 1st-order PDEs of form

\[ d \Phi_x + e \Phi_y + f \Phi = g \]

called 1st-order hyperbolic equation.
Application to Advection-Diffusion Eqn:

1. Heat equation

\[ \frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \]

\[ q \Delta t = \alpha \]

\[ \beta \phi = 0 \]

\[ c \phi = 0 \]

\[ b^2 - 4ac = 0 \rightarrow \text{PARABOLIC} \]

2. Laplace equation

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]

\[ a = 1 \]

\[ b = 0 \]

\[ c = \phi \]

\[ b^2 - 4ac = -4 < 0 \rightarrow \text{ELLiptic} \]

3. Advection equation

\[ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0 \]

\[ \rightarrow \text{First-order hyperbolic} \]

Advection-diffusion equation can exhibit characteristic of any of these three categories.
well-posedness conditions

1. Parabolic
   \[ \frac{\partial \phi}{\partial t} = c^2 \frac{\partial^2 \phi}{\partial x^2} \]
   - Must be solved forward in time
     Ex: \[ \phi(x,0) = \phi_0(x) \]
     Solve for \( t > 0 \)

2. Elliptic
   \[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]
   - Must have boundary data over entire boundary
     Ex: Find \( \phi(x,y) \) in \( \Omega \).
     Either \( \phi \) or \( \partial \phi / \partial n \) given over entire boundary \( S \) of \( \Omega \).

3. Hyperbolic
   \[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \] (wave equation)
   - Must be initial-value problem in time
     Ex: Find \( \phi(x,t) \) for \( t > 0 \) \& with
     \[ \phi(x,0) = \phi_0(x) \]
     \[ \frac{\partial \phi}{\partial t} (x,0) = \phi_1(x) \]