Mathematics underlying geometrical modeling:

1. Spline curves (smooth interpolating curves) - used for sketching
2. Rigid-body motions (rotation and translation) - used to move shapes
3. Projective geometry (size variation and lighting) - used to make shapes look like 3D

1. Spline Curves

Problem: * polynomial fits with high order tend to have spurious oscillations

* \( \rightarrow \) problem since \( n^{th} \)-degree polynomial is required to exactly pass through \( n+1 \) points

* polynomial interpolations take a long time for large \( n \)
Spline Approaches (Cubic Spline)

Given data: \( f(x_i) = y_i \) for \( i = 1, 2, \ldots, n \)

Objective: Pass a curve through \( n \) points which is continuous and 1st & 2nd derivatives are continuous.

Advantage:
- Fast
- Avoids spurious oscillations in polynomial fit

Note: \( n+1 \) data points requires \( n \) cubic spline functions of form

\[ f_i(x) = A_{i1} + A_{i2} x + A_{i3} x^2 + A_{i4} x^3, \quad i = 1, \ldots, n \]

- Need to find 4n coefficients \( A_{i1}, A_{i2}, A_{i3}, A_{i4} \)

- Higher-order version of connect-the-dots

\[ \text{①} \]
Smoothing

Restriction: \( f(x) , f'(x), f''(x) \) are continuous over \([x_0, x_n]\)

1.) \( f_i(x_i) = y_i \) for \( i = 1, 2, \ldots, n \) (End Point)
   \( f_{i+1}(x_i) = y_i \) for \( i = 0, 1, \ldots, n-1 \) (Start Point)

2.) \( f'_i(x_i) = f'_{i+1}(x_i) \) for \( i = 1, 2, \ldots, n-1 \) (slope continuity)

3.) \( f''_i(x_i) = f''_{i+1}(x_i) \) for \( i = 1, 2, \ldots, n-1 \) (curvature continuity)

Unknowns: \( 4n \)

Equations: \( n + n + (n-1) + (n-1) = 4n - 2 \)

Additional Restriction: curvature vanishes at first & last data point

\[ f''_1(x_0) = 0 \]

\[ f''_n(x_n) = 0 \]
Define $\Delta x_i = x_i - x_{i-1}$, $i = 1, \ldots, n$

$\Delta y_i = y_i - y_{i-1}$, $i = 1, \ldots, n$

Consider choice:

$$f_i(x) = \frac{f''(x_{i-1}) (x_{i-1} - x)^3}{6 \Delta x_i} + \frac{f''(x_i) (x - x_{i-1})^3}{6 \Delta x_i}$$

$$+ \left( \frac{y_{i-1}}{\Delta x_i} - \frac{f''(x_{i-1}) \Delta x_i}{6} \right) (x_{i-1} - x)$$

$$+ \left( \frac{y_i}{\Delta x_i} - \frac{f''(x_i) \Delta x_i}{6} \right) (x - x_{i-1})$$

* This satisfies Restriction #1 by construction:

$\rightarrow$ at $x = x_i$, we have

$$f_i(x_i) = \frac{f''(x_{i-1}) (0)^3}{6 \Delta x_i} + \frac{f''(x_i) (0)^3}{6 \Delta x_i}$$

$$+ \left( \frac{y_{i-1}}{\Delta x_i} - \frac{f''(x_{i-1}) \Delta x_i}{6} \right) (0) \rightarrow 0$$

$$+ \left( \frac{y_i}{\Delta x_i} - \frac{f''(x_i) \Delta x_i}{6} \right) \Delta x_i = y_i$$

Similarly at $x = x_{i-1}$ $\rightarrow$ $f_i(x_{i-1}) = y_{i-1}$
* This satisfies Restriction 3 by construction:

\[ f''(x_i) = \frac{3.2 \cdot f'(x_{i-1})}{\Delta x_i} + \frac{3.2 \cdot f''(x_i) \Delta x_i}{6 \Delta x_i} = f''(x_i) \]

\[ f'_{i+1}(x_i) = \frac{3.2 \cdot f''(x_i) \Delta x_{i+1}}{6 \Delta x_{i+1}} + \frac{3.2 \cdot f''(x_{i+1}) \Delta x_i}{6 \Delta x_i} = f''(x_i) \]

* To satisfy Restriction 2, we require that the following two equations are equal:

\[ f'_i(x_i) = \frac{1}{2} \Delta x_i f''(x_i) - \left( \frac{y_{i-1}}{\Delta x_i} - f''(x_{i-1}) \Delta x_i \right) + \left( \frac{y_i}{\Delta x_i} - \frac{f''(x_i) \Delta x_i}{6} \right) \]

\[ f'_{i+1}(x_i) = -\frac{1}{2} \Delta x_{i+1} f''(x_i) - \left( \frac{y_i}{\Delta x_{i+1}} - \frac{f''(x_i) \Delta x_{i+1}}{6} \right) + \left( \frac{y_{i+1}}{\Delta x_{i+1}} - \frac{f''(x_{i+1}) \Delta x_i}{6} \right) \]

Equate these two yields

\[ \Delta x_i f''(x_{i-1}) + 2(\Delta x_i + \Delta x_{i+1}) f''(x_i) + \Delta x_{i+1} f''(x_{i+1}) \]

\[ = 6 \left( \frac{\Delta y_i}{\Delta x_i} + \frac{\Delta y_{i+1}}{\Delta x_{i+1}} \right) \quad \text{for } i = 1, \ldots, n-1, \]

- Yields tridiagonal matrix equation for \( f''(x_i) \)
- Use additional restriction \( f''(x_0) = f''(x_n) = 0 \) for \( i = 0 \) and \( n \)
\[ \begin{bmatrix} - \Delta x_1 & 2(\Delta x_1 + \Delta x_2) & \Delta x_2 \\ 0 & - \Delta x_2 & 2(\Delta x_2 + \Delta x_3) \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Delta x_{n-1} \\ 2(\Delta x_{n-1} + \Delta x_n) \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} f''(x_1) \\ f''(x_2) \\ \vdots \\ f''(x_n) \end{bmatrix} \]

\[
\begin{bmatrix}
\frac{\Delta y_1}{\Delta x_1} + \frac{\Delta y_2}{\Delta x_2} \\
\vdots \\
\frac{\Delta y_{n-1}}{\Delta x_{n-1}} + \frac{\Delta y_n}{\Delta x_n}
\end{bmatrix} = 0
\]

- Matrix equation

* Solve for \( f''(x_i) \), \( i = 0, \ldots, n \), using LUD for tridiagonal matrix problems.
* Plug result for \( f''(x_i) \) into above eqn. for \( f_i(x) \).